Singular perturbation stability analysis for a three-time-scale autonomous helicopter

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***Abstract*—This paper presents a singular perturbation stability analysis for a nonlinear three-time-scale autonomous helicopter model in vertical flight. The presented time-scale analysis permits to conduct a stability analysis for the singular perturbed three-time-scale system, and allows to construct a composite Lyapunov function for the resultant closed-loop system using time-scale separation. The methodology here presented provides mathematical expressions for the upper bounds of the singularly perturbed parameters that define the three time scale. Numerical results on the stability analysis are also presented for the studied nonlinear highly coupled helicopter model.**

I. Introduction

The study and demonstration of the asymptotic stability properties of time-scale systems is an important area in the real of singular perturbations. Many studies have been conducted towards demonstrating the asymptotic stability properties of two-time-scale singularly perturbed systems [1]–[3]. These works show the high degree of difficulty required to demonstrate the stability properties of the different time-scale subsystems, and only provide general expressions for the upper bounds, and where these expressions are subject to the satisfaction of the growth requirements inequalities for the different time-scale subsystems, and the selection of arbitrary constants. These complexities are even more evident when dealing with multi-parameter time-scale systems. Several works that appear in the literature approach the multi-parameter asymptotic stability analysis [3]–[6] using all similar methods based on composite stability methods of large scale dynamical systems, but again, without defining precise mathematical upper bounds on the singularly perturbed parameters.

In this paper we approach the problem of multi-parameter asymptotic stability analysis by extending the procedures introduced by Kokotović [3] for the two-time-scale singular perturbation problems, to the three-time-scale singular perturbation problem of an autonomous helicopter on a platform, although the methodology here presented can also extended to more general singular perturbation problems as described by the authors in [7]. The time-scale analysis here presented permits to construct a suitable composite Lyapunov function candidate for an autonomous singularly perturbed system, and analyze the stability of the resulting autonomous subsystems guaranteeing the stability of the equilibrium by providing mathematical expressions that define the upper bounds for the selected singularly perturbed parasitic constants, being this the main contribution of this paper. The presented time-scale analysis methodology has also been used to determine the control laws that guarantee desired closed loop dynamics of the studied helicopter model, which provides an additional tool at the time of conducting the stability analysis.

On the realm of helicopter control, although the use of singular perturbation theory has been employed to simplify the control system structure [8], [9], to our knowledge, the work done by the authors of this paper and of previous publications [7], [10], along with that conducted by Bertrand, Hamel and Piet-Lahanier [11], that presented a stability analysis of a hierarchical controller for an unmanned Aerial Vehicle, are the only investigations that theoretically address stability issues for VTOL UAVs using singular perturbations theory.

This paper is structured as follows: Section II presents the closed loop helicopter model used throughout this paper; Section III presents the three-time-scale analysis here presented, including the time-scale selection; section IV describes the asymptotical stability analysis for the general three time-scale helicopter problem; numerical results for the specific helicopter model studied in this paper are depicted in section V. Conclusions are drawn in Section VI; and finally, Appendix A briefly describes the general two time-scale singular perturbation theory.

II. Closed-Loop Model Definition

The helicopter closed loop model that is used throughout the remainder of this paper is the result of the three time-scale singular perturbation control strategy applied by the authors [7] to a vertical flight helicopter model that has been extensively used as a benchmark in the literature to test a wide variety of control strategies to regulate the vertical position of the helicopter [12]–[15]. The vertical flight helicopter model can be decomposed in three distinct dynamics, in whin the state space vector is given by $\boldsymbol{X} \triangleq [x, \boldsymbol{y}, \boldsymbol{z}]^T$, where x represents angular velocity of the blades, $\boldsymbol{y} \triangleq [y_1, y_2]^T$ represent the state vector for the vertical motion of the helicopter,with y_1 representing the vertical position, and y_2 the vertical velocity, and finally, $\boldsymbol{z} \triangleq [z_1, z_2]^T$ represents the state vector for the collective pitch angle dynamics, where z_1 represents the collective pitch angle of the blades, and z_2 is the collective pitch rate of the blades, recalling that x is the slow variable, $\boldsymbol{y}$ is the fast variable, and $\boldsymbol{z}$ is the ultra-fast variable. The closed-loop helicopter dynamics are expressed in their error

dynamics, denoted with tilde vector notation, yielding

$$\begin{aligned}
\dot{\tilde{x}} &= a_{10}(\tilde{x}+x^*)^2(\sin(\tilde{z}_1+z_1^*)-\sin\tilde{h}_{1_{SS}})-b_3\tilde{x}, && (1)\\
\varepsilon_1\dot{\tilde{y}}_1 &= c_1\tilde{y}_2, && (2)\\
\varepsilon_1\dot{\tilde{y}}_2 &= (\tilde{x}+x^*)^2\left(c_2+c_3(\tilde{z}_1+z_1^*)-\sqrt{c_4+c_5(\tilde{z}_1+z_1^*)}\right)\\
&+ a_9\tilde{y}_2+a_9\tilde{y}_2^2+c_6, && (3)\\
\varepsilon_1\varepsilon_2\dot{\tilde{z}}_1 &= c_7\tilde{z}_2, && (4)\\
\varepsilon_1\varepsilon_2\dot{\tilde{z}}_2 &= a_9(\tilde{z}_1+z_1^*)+c_9\tilde{z}_2+J_2\left[\left(1+\sqrt{s_3\tilde{v}(\tilde{x},\tilde{y})}\right)^2-1\right]. && (5)
\end{aligned}$$

where

$$h_{1_{SS}} = s_2\left[(1+\sqrt{s_3 v_{SS}})^2-1\right], \tag{6}$$

with

$$v_{SS} = -\frac{c_6}{\tilde{x}^2}. \tag{7}$$

and with

$$v(x,\boldsymbol{y}) = -\frac{a_9\tilde{y}_2^2+(a_9+b_2)\,\tilde{y}_2+b_1\tilde{y}+c_6}{\tilde{x}^2}, \tag{8}$$

being the coefficients defined as $c_1 = a_9/a_5$, $c_2 = c_1a_1$, $c_3 = c_1a_2$, $c_4 = c_1^2a_3$, $c_5 = c_1^2a_4$, $c_6 = c_1a_7$, $c_7 = c_{11} = a_9/a_{13}$, $c_8 = c_7a_{14}$, $c_9 = c_7a_{15}$, and $c_{10} = c_7a_{12}$, where the constants a_1 through a_{15} are given by physical properties of the helicopter, and where $s_2 = a_3/a_4$ and $s_3 = 4c_3/c_5$; b_1, b_2, and b_3 represent the *control parameters* that define the desired vertical displacement dynamics of the helicopter, and angular velocity dynamics of the blades. For the X-Cell 50 helicopter being studied in this paper, the constants are defined in Table II in section V. For the stability analysis conducted in this paper, the helicopter model will be treated in their general form, and only numerical results for the specific X-Cell 50 will be address in section V. For further details on the control strategies and the helicopter model refer to [7]

III. THREE-TIME-SCALE ANALYSIS

This section describes the proposed time-scale analysis methodologies presented in this paper, and is divided in two subsections: subsection III-A presents the multi-parameter time-scale decomposition, and subsection III-B provides an intuitive description of a generic three-time-scale system to help understanding the time-scale decomposition.

A. Top-Down and Bottom-Up Time Scale Decomposition

This subsection presents the time-scale methodologies presented in this paper that provide an approach in which, for a specific class of singularly perturbed nonlinear systems, a step-by-step procedure can be employed to select an appropriate composite Lyapunov function for the complete singularly perturbed system, conduct the stability analysis for the closed-loop system, which provides mathematical expressions for the upper bounds of the singularly perturbed parameters, and everything in an all-in-one step-by-step process.

These step-by-step methodologies will be denoted as *Top-Down* (*TD*) and *Bottom-Up* (*BU*) methodologies, and receive their names from the direction in which the singular perturbation parameters are applied, resulting in different time-scales subsystems. Figure 1, shows the *TD* methodology which analyzes the time-scale properties of the Σ_{SFU} system in a descending manner, denoted by the downward arrow, considering first the top singularly perturbed parameter, ε_1, resulting in a simplified two-time-scale problem formed by a one-dimension subsystem, the reduced order slow Σ_S-subsystem, and a two-dimension subsystem, the boundary layer fast Σ_{FU}-subsystem, denoted both by the red dashed boxes; in a second instance, and following the descending direction, the bottom singularly perturbed parameter is applied, ε_2, such that simplifies the second-order Σ_{FU}-subsystem into another two-time-scale problem formed, this time, by two one-dimension subsystems, the Σ_F and Σ_U-subsystems, denoted both by the blue dashed-dotted boxes. A similar methodology is applied in the *Bottom-Up* methodology, but in an ascending manner as seen in Figure 2. Both methodologies are independent but equivalent, and produce the same results, but providing a degree of freedom associated to the direction in which the stretched time-scales are applied, which is chosen depending on the needs by the problem structure. The *TD* and *BU* methodologies are not limited to three-time-scale problems, and can be extended to a more general N-time-scale system as described in [7].

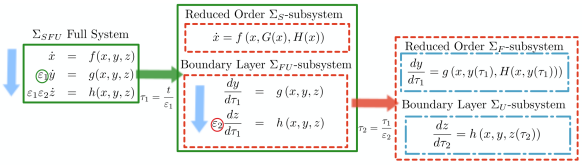


Fig. 1. *Top-Down* methodology

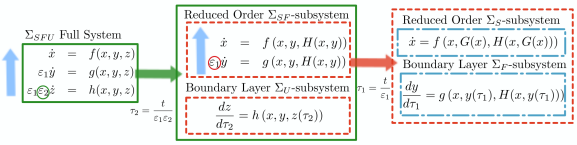


Fig. 2. *Bottom-Up* methodology

B. Intuitive Description of the Three-Time-Scale Decomposition

The understanding of the natural evolution of a generic three-time-scale model, can be achieved by focusing only on above mentioned *BU* sequential methodology, Figure 2, which will help to describe how the ultra-fast, fast and slow variables of a stable system evolve through their own configuration spaces. The generic three-time-scale Σ_{SFU} system can be sequentially decomposed into two different two-time-scale models. The first two-time-scale model considers the time-scale defined by the stretched time scale $\tau_2 = t/(\varepsilon_1\varepsilon_2)$, where the reduced (slow) subsystem, denoted as Σ_{SF}-subsystem, and defined by

$$\begin{aligned}
\dot{x} &= f(x,y,H(x,y)), && (9)\\
\varepsilon_1\dot{y} &= g(x,y,H(x,y)), && (10)
\end{aligned}$$

and where the boundary layer of the Σ_{SFU} system, denoted as Σ_U-subsystem, is given by

$$\frac{\mathrm{d}z}{\mathrm{d}\tau_2} = h(x, y, z(\tau_2)), \tag{11}$$

where in Eq. (11), x and y are treated like fix parameters, and $z(\tau_2)$ evolves on its stretched time scale (τ_2). Note that $H(x, y)$ in Eqns. (9–10), represents the quasi-steady-state of the boundary layer, Eq. (11), when $\varepsilon_2 = 0$, that is $0 = h(x, y, z) \rightarrow z = H(x, y)$. The reduced order Σ_{SF}-subsystem, Eqns. (9–10), can be treated again like a two-time-scale singular perturbation problem by considering the stretched time-scale defined by $\tau_1 = t/\varepsilon_1$, where the reduced Σ_S-subsystem is defined by

$$\dot{x} = f(x, G(x), H(x, G(x))), \tag{12}$$

and where the boundary layer for the Σ_{SF}-subsystem, denoted as Σ_F-subsystem, is now given by

$$\frac{\mathrm{d}y}{\mathrm{d}\tau_1} = g(x, y(\tau_1), H(x, y(\tau_1))), \tag{13}$$

where x is treated like a fix parameters, $H(x, y)$ evolves on its configuration space, and $y(\tau_1)$ evolves on its stretched time-scale (τ_1). Note that $G(x)$ represents the quasi-steady-state of the boundary layer Σ_F-subsystem, Eq. (13), when $\varepsilon_1 = 0$, that is $0 = g(x, y, z) \rightarrow y = G(x)$, and $H(x, G(x))$ represents the quasi-steady-state of the boundary layer Σ_U-subsystem, Eq. (11), when evolving on the manifold of the Σ_F-subsystem.

A better understanding of the evolution of the different time-scales can be achieved by observing Figures III-B–III-B, where the evolution of a generic three-time-scale model is presented. Figure III-B shows the evolution of the ultra-fast variable z as it moves towards the manifold of the boundary layer Σ_U-subsystem, that is, towards the surface that defines the quasi-steady-state equilibrium of the Σ_U-subsystem, given by $h(x, y, z) = 0$, that is $z = H(x, y)$, while x and y behave as fixed parameters; Figure III-B shows the evolution of the fast variable y as it moves on the configuration space of the Σ_U-subsystem, towards the surface that defines the quasi-steady-state equilibrium of Σ_F-subsystem given by $g(x, y, H(x, y)) = 0$, that is $y = G(x)$, while the slow variable x behaves as a fixed parameter, and $z = H(x, G(x))$ evolves on its own manifold; finally, Figure III-B shows the evolution of the slow variable x as it moves in the manifold of the Σ_{FU}-subsystem, towards the quasi-steady state equilibrium of the Σ_S, given by the intersection between the planes $g(x, y, H(x, y)) = 0$ and $h(x, y, z) = 0$.

IV. ASYMPTOTIC STABILITY ANALYSIS

The stability properties of the resulting autonomous helicopter in vertical flight is analyzed in this section, and providing also exact mathematical expressions for the upper bounds of the singularly perturbed parameters ε_1, and ε_2. It is considered that the associated three-time-scale subsystems Σ_S, Σ_F, and Σ_U are each asymptotically stable, and the proposed methodology derives the additional conditions that demonstrate the asymptotic stability properties for the full system by extending the well-known standard asymptotic stability requirements for two-time-scale singular perturbation problems [3] to the three-time-scale problem here described. The following subsections describe the methodology employed to construct the associated Lyapunov functions for each of the subsystems, and the asymptotic stability analysis of the full Σ_{SFU} system, which includes the mathematic expressions that define the bounds for the singular perturbation parameters.

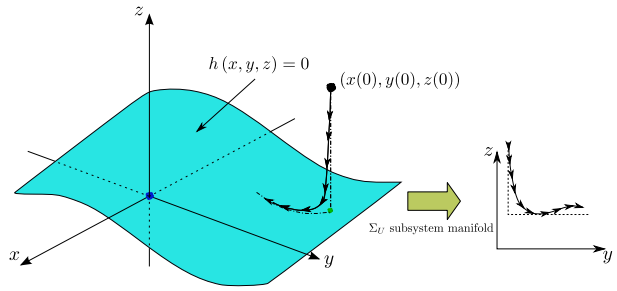


Fig. 3. Movement of the Σ_U-subsystem.

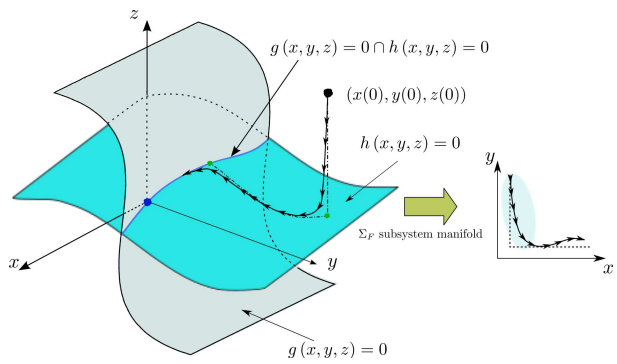


Fig. 4. Movement of the Σ_F-subsystem.

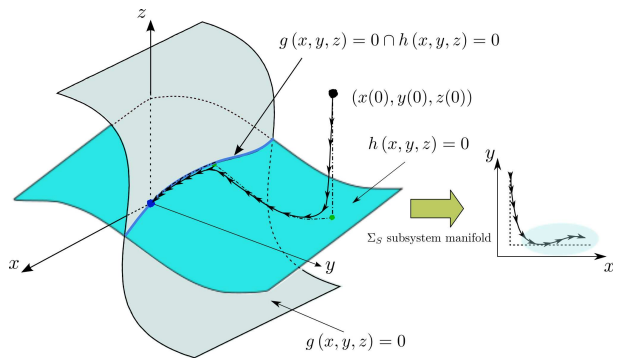


Fig. 5. Movement of the Σ_S-subsystem.

A. Lyapunov Function Candidates

The selection of proper Lyapunov functions that are required to study the asymptotic stability properties of an autonomous system is one of the most challenging issues that a control engineer has to be faced with. For the asymptotic stability analysis of the different multi-parameter time-scale systems being here studied, it is required the existence of Lyapunov functions for each one of the singularly perturbed subsystems.

The fulfillment of certain growth requirements between each of the Lyapunov functions and the use of composite stability methods [3], [16] will ensure the existence of a Lyapunov function for the entire system.

A methodology that provides Lyapunov functions for all three time-scale subsystems is presented. These Lyapunov functions will be used in the stability analysis to obtain a composite Lyapunov function for the complete Σ_{SFU} system. The Lyapunov function for each of the subsystems is obtained by considering the desired dynamics of each of the subsystems, which has been properly selected with the applied control strategy [7], and solving for the associated Lyapunov function for each subsystem. Therefore, after applying the sequential time-scale decomposition, the reduced order Σ_S-subsystem reduces to selected target slow-dynamics of the form

$$\dot{\tilde{x}} = f(\tilde{x}, G(\tilde{x}), H(\tilde{x}, G(\tilde{x}))) = -b_3\tilde{x}, \tag{14}$$

thus we select a natural Lyapunov function candidate of the form

$$V_S(\tilde{x}) = \frac{1}{2}\tilde{x}^T p_s \tilde{x} = \frac{1}{2}p_s\tilde{x}^2, \tag{15}$$

with p_s being a positive constant defined after solving the associated Lyapunov function, and given as $p_s = q_s/(2b_3)$, and where q_s represents a degree of freedom that can be chosen to satisfy the stability requirements. Similarly, the Lyapunov function candidate for the Σ_F-subsystem is obtained considering the reduced Σ_F-subsystem

$$\frac{d\tilde{y}}{d\tau_1} = g(x, y(\tau_1), H(x, y(\tau_1))) = \boldsymbol{A_F}\tilde{\boldsymbol{y}}, \tag{16}$$

being

$$\boldsymbol{A_F} = \begin{bmatrix} 0 & c_1 \\ -b_1 & -b_2 \end{bmatrix}, \tag{17}$$

therefore, we choose a natural Lyapunov function $V_F(\tilde{\boldsymbol{y}})$ of the form

$$V_F(\tilde{\boldsymbol{y}}) = \frac{1}{2}\tilde{\boldsymbol{y}}^T \boldsymbol{P_F}\tilde{\boldsymbol{y}}, \tag{18}$$

where $\boldsymbol{P_F}$ is a 2×2 positive definite matrix that solves the associated Lyapunov equation $\boldsymbol{P_F}\boldsymbol{A_F} + \boldsymbol{A_F}^T\boldsymbol{P_F} + \boldsymbol{Q_F} = 0$, where $\boldsymbol{Q_F}$ and $\boldsymbol{P_F}$ are 2×2 positive definite matrices of the form

$$\boldsymbol{Q_F} = \begin{bmatrix} q_{f_1} & 0 \\ 0 & q_{f_2} \end{bmatrix}, \text{ and } \boldsymbol{P_F} = \begin{bmatrix} p_{f_1} & p_{f_3} \\ p_{f_3} & p_{f_2} \end{bmatrix}, \tag{19}$$

where q_{f_1} and q_{f_2} are degrees of freedom that can be chosen to satisfy the stability analysis, and p_{f_1}, p_{f_2}, and p_{f_3} are the solutions of the associated Lyapunov function. The Lyapunov function for the Σ_F-subsystem becomes

$$V_F(\tilde{\boldsymbol{y}}) = \frac{1}{2}\tilde{\boldsymbol{y}}^T \boldsymbol{P_F}\tilde{\boldsymbol{y}} = \frac{1}{2}p_{f_1}\tilde{y}_1^2 + \frac{1}{2}p_{f_3}\tilde{y}_2^2 + p_{f_2}\tilde{y}_1\tilde{y}_2, \tag{20}$$

with the solutions to the associated Lyapunov equation given as

$$p_{f_1} = \frac{q_{f_1}(b_1c_1 + b_2^2) + b_1^2 q_{f_2}}{b_1b_2c_1},\ p_{f_2} = \frac{q_{f_1}}{2b_1},\ p_{f_3} = \frac{q_{f_1}c_1 + q_{f_2}b_1}{b_1b_2}.$$

Finally, the natural Lyapunov function candidate for the Σ_U-subsystem is obtained by first introducing a change of variables so that the equilibrium of this boundary-layer subsystem is centered at zero, that is, shifting the Σ_U-subsystem equilibrium by $H(\tilde{x}, \tilde{\boldsymbol{y}})$, where $H(\tilde{x}, \tilde{\boldsymbol{y}})$ represents the equilibria of the Σ_U-subsystem. The change of variables is defined as

$$\hat{\boldsymbol{z}} = \tilde{\boldsymbol{z}} - H(\tilde{x}, \tilde{\boldsymbol{y}}) = \begin{bmatrix} \hat{z}_1 & \hat{z}_2 \end{bmatrix}^T, \tag{21}$$

thus rewriting the Σ_U-subsystem in state-space form as

$$\frac{d\tilde{\boldsymbol{z}}}{d\tau_2} = \boldsymbol{A_U}\hat{\boldsymbol{z}}, \tag{22}$$

therefore, we choose a natural Lyapunov function V_U of the form

$$V_U(\tilde{x}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) = V_U(\hat{\boldsymbol{z}}) = \frac{1}{2}\hat{\boldsymbol{z}}^T \boldsymbol{P_U}\hat{\boldsymbol{z}}, \tag{23}$$

where $\boldsymbol{P_U}$ is a positive 2×2 definite matrix that solves the associated Lyapunov equation $\boldsymbol{P_U}\boldsymbol{A_U} + \boldsymbol{A_U}^T\boldsymbol{P_U} + \boldsymbol{Q_U} = 0$, where $\boldsymbol{Q_U}$ and $\boldsymbol{P_U}$ are 2×2 positive definite matrices of the form

$$\boldsymbol{Q_U} = \begin{bmatrix} q_{u_1} & 0 \\ 0 & q_{u_2} \end{bmatrix}, \text{ and } \boldsymbol{P_U} = \begin{bmatrix} p_{u_1} & p_{u_3} \\ p_{u_3} & p_{u_2} \end{bmatrix}. \tag{24}$$

where q_{u_1} and q_{u_2} are degrees of freedom that can be chosen to satisfy the stability analysis, and p_{u_1}, p_{u_2}, and p_{u_3} are the solutions of the associated Lyapunov function. This yields the associated Lyapunov function

$$V_U(\hat{z}) = \frac{1}{2}\hat{z}^T \boldsymbol{P_U}\hat{z} = \frac{1}{2}p_{u_1}\hat{z}_1^2 + \frac{1}{2}p_{u_3}\hat{z}_2^2 + p_{u_2}\hat{z}_1\hat{z}_2, \tag{25}$$

with the solutions to the associated Lyapunov equation given as

$$p_{u_1} = \frac{q_{u_1}\left(c_9^2 - a_9c_7\right) + q_{u_2}a_9^2}{2a_9c_7c_9},\ p_{u_2} = -\frac{q_{u_1}}{2a_9},\ p_{u_3} = \frac{c_7q_{u_1} - q_{u_2}a_9}{2a_9c_9}.$$

It is important for the understanding of the main contribution of the methodology here presented, to note that all three Lyapunov functions posses a degree of freedom in the form of the variables q_s, q_{f_1}, q_{f_2}, q_{u_1}, and q_{u_2}, which will be referred as *stability parameters* from now on. These *stability parameters* will be of great importance when conducting the stability analysis since the proposed time-scale analysis methodology will allow to derive expressions that define the bounds of these *stability parameters* such that will permit to define mathematical upper bounds for the singularly perturbed parameters ε_1 and ε_2 as it will be shown in the following sections.

B. Asymptotic Stability Analysis for Three-Time-Scale Problems

The asymptotic stability analysis of the three-time scale autonomous system, Eqns. (1–5), it is also based on a double application of the standard two-time-scale stability analysis of the Σ_{SFU} full system. The double application of stability analysis is divided in two stages, following the *BU* time-scale decomposition presented in Figure 2. In the first stage, the stability analysis of the Σ_{SF}-subsystem is performed assuming that the Σ_U-subsystem variables evolve in their own configuration space. The analysis of this first stage is performed by ensuring that the previously derived Lyapunov functions for the Σ_S and Σ_F-subsystems, fulfill the growth requirements on $f(\tilde{x}, \tilde{y}, H(\tilde{x}, \tilde{y}))$ and $g(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}}))$ by satisfying certain inequalities. These inequalities, Assumptions A.1 through A.5, are described in detail in Appendix A. For conciseness of the paper, these assumptions are described in detail in Appendix A for the general two-time-scale problem, and can be used to determined the equivalent assumptions for each of the two stability stages, by substituting the analogous system

dynamics which are described in Table I, where for each of the assumptions, there are three rows, where the first corresponds to the inequality presented in Appendix A, the second row corresponds to the equivalent inequality for the Σ_{SF}-*Stability Analysis*, and the third row corresponds to the equivalent inequality for the Σ_{SFU}-*Stability Analysis*. For example, the equivalent of inequality (67) for the Σ_{SF}-subsystem is given by substituting in the functions that appear in the Σ_{SF}-*SA* row in Table I, resulting in

$$\frac{\partial V_S(\tilde{x})}{\partial \tilde{x}} f(\tilde{x}, G(\tilde{x}), H(\tilde{x}, G(\tilde{x}))) \leq -\alpha_1 \psi^2(\tilde{x}). \tag{26}$$

The fulfillment of these growth requirements results in the obtention of a Lyapunov function (V_1) for the singularly perturbed Σ_{SF}-subsystem as a function of a weighted sum of the Lyapunov functions for the Σ_S and Σ_F-subsystems. The results from the stability analysis of the Σ_{SF}-subsystem are used to conduct the stability analysis for the complete Σ_{SFU} system, which, for convenience, is rewritten as

$$\dot{\tilde{\boldsymbol{\chi}}} = F(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}}), \tag{27}$$

$$\varepsilon_1 \varepsilon_2 \dot{\tilde{\boldsymbol{z}}} = h(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}}), \tag{28}$$

with $\tilde{F}(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$ being the augmented system formed by the slow and fast dynamics, and given by

$$F(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}}) \triangleq \begin{bmatrix} f(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}}) \\ g(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}}) \end{bmatrix} = \begin{bmatrix} f(\tilde{x}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) \\ g(\tilde{x}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) \end{bmatrix}, \tag{29}$$

where $\tilde{\boldsymbol{\chi}}$ represents the augmented state vector given by the slow and fast variables, $\tilde{\boldsymbol{\chi}} \triangleq \begin{bmatrix} \tilde{x} & \tilde{\boldsymbol{y}} \end{bmatrix}^T$. Making $\varepsilon_2 = 0$ yields the equilibrium of the boundary layer of the Σ_{SFU} which is given by

$$0 = h(\tilde{\boldsymbol{\chi}}, H(\tilde{\boldsymbol{\chi}})) \rightarrow \tilde{\boldsymbol{z}} = H(\tilde{\boldsymbol{\chi}}). \tag{30}$$

and where the reduced order Σ_{SF}-subsystem is given by substituting Eq. (30) into $F(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}}) \rightarrow F(\tilde{\boldsymbol{\chi}}, H(\tilde{\boldsymbol{\chi}}))$, which is equivalent to the Σ_{SF}-subsystem analyzed in the first stage of the stability analysis, while the boundary layer Σ_U-subsystem is defined in Eq. (22), with $V_U(\hat{\boldsymbol{z}})$ being its associated Lyapunov function. Note that $H(\tilde{\boldsymbol{\chi}}) \triangleq H(\tilde{x}, \tilde{\boldsymbol{y}})$. In a similar analysis to the one conducted in the first stage, the new Lyapunov functions must define these growth requirements for $F(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$ and $h(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$ by satisfying certain inequalities. As a result of the satisfaction of these growth requirements, a new Lyapunov function (V_2) is obtained for the full Σ_{SFU} system. Following sections resume the results for the stability analysis of both stages.

C. Σ_{SF} *Subsystem* Stability Analysis

The asymptotically stability requirements of the Σ_{SF}-subsystem are expressed by the satisfaction of certain growth requirements for the Σ_S and Σ_F-subsystems. These growth requirements are expressed in the form of inequalities, where inequalities (67) and (70) define the growth requirements of the reduced order Σ_S-subsystem, $f(\tilde{x}, G(\tilde{x}), H(\tilde{x}, G(\tilde{x})))$, and the boundary layer Σ_F-subsystems, $g(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}}))$, while inequalities (73) and (75) define the growth requirements of the combined Σ_{SF}-subsystem.

With the analysis of the Σ_S and Σ_F-subsystem, Eqns. (14) and (16) respectively, it is clear that they have a unique and isolated equilibrium at the origin, that is $0 = f(0, 0, H(\tilde{x}, \tilde{\boldsymbol{y}}))$ and $0 = g(0, 0, H(\tilde{x}, \tilde{\boldsymbol{y}}))$, thus fulfilling Assumption A.1 . The fulfillment of Assumption A.2 can be easily checked with the choice of the comparison function $\psi_1(\tilde{x})$ proposed in Table I, where $\tilde{q}_s = q_s/2$ with q_s being the choice of the designer for the associated Σ_S-subsystem Lyapunov function. Assumption A.3 is fulfilled with the choice of the comparison function $\phi_1(\hat{\boldsymbol{y}})$ described in Table I, where $\tilde{Q}_F = Q_F/2$, and with Q_F defined in Eq. (19), and satisfying that $\hat{\boldsymbol{y}} = \tilde{\boldsymbol{y}} - G(\tilde{x}) = 0$ is a stable equilibrium of the boundary layer system. The first interconnection condition, Assumption A.4, is satisfied with the relationship described in Table I, that is selecting $\beta_1 \geq \beta_1^*$, with $\beta_1^* = \beta_1^*(q_s, q_{f_2}, b_1, b_2, b_3, a_*)$ being an elaborated function of the *stability parameters*, the *control parameters*, and the physical parameters of the helicopter model, denoted as a_*, which satisfy inequality (73). Inequality (73) determines the allowed growth of $f(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}}))$ in $\tilde{\boldsymbol{y}}$. Assumption A.5 is satisfied with the relationships described in Table I, where it is only required that $\beta_2 \geq 0$, and $\gamma_1 \geq 0$. Refer to [7] for more details on how the inequalities are satisfied.

The fulfillment of Assumptions A.1 through A.5 determines that the origin is an asymptotically stable equilibrium of the singular perturbed Σ_{SF}-subsystem for sufficiently small ε_1. With the Lyapunov functions $V_S(\tilde{x})$ and $V_F(\tilde{\boldsymbol{y}})$ defined, we obtain a new Lyapunov function candidate defined by a weighted sum of $V_S(\tilde{x})$ and $V_F(\tilde{\boldsymbol{y}})$, and given by

$$V_1(\tilde{x}, \tilde{\boldsymbol{y}}) = (1 - d_1) V_S(\tilde{x}) + d_1 V_F(\tilde{\boldsymbol{y}}),\ d_1 \in (0, 1), \tag{31}$$

where $0 < d_1 < 1$. To explore the freedom that we have choosing the weights, we take d_1 as an unspecified parameter in the interval $(0, 1)$. From the properties of $V_S(\tilde{x})$ and $V_F(\tilde{\boldsymbol{y}})$, and $\| G(\tilde{x}) \| \leq p_1(\| \tilde{x} \|)$, where $p_1(\cdot)$ is a κ function, it follows that $V_1(\tilde{x}, \tilde{\boldsymbol{y}})$ is positive-definite. Computing the time derivative of $V_1(\tilde{x}, \tilde{\boldsymbol{y}})$ along the trajectories of $f(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}}))$ and $g(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}}))$ we obtain

$$\begin{aligned}
\dot{V}_1 &= (1 - d_1) \frac{\partial V_S}{\partial \tilde{x}} f(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}})) + \frac{d_1}{\varepsilon_1} \frac{\partial V_F}{\partial \tilde{\boldsymbol{y}}} g(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}})) \\
&+ d_1 \frac{\partial V_F}{\partial \tilde{x}} f(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}})) \\
&= (1 - d_1) \frac{\partial V_S}{\partial \tilde{x}} f(\tilde{x}, G(\tilde{x}), H(\tilde{x}, \tilde{\boldsymbol{y}})) \\
&+ (1 - d_1) \frac{\partial V_S}{\partial \tilde{x}} \left[f(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}})) - f(\tilde{x}, G(\tilde{x}), H(\tilde{x}, \tilde{\boldsymbol{y}})) \right] \\
&+ \frac{d_1}{\varepsilon_1} \frac{\partial V_F}{\partial \tilde{\boldsymbol{y}}} g(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}})) + d_1 \frac{\partial V_F}{\partial \tilde{x}} f(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}})),
\end{aligned} \tag{32}$$

where using the inequalities in Assumptions A.2 through A.5 we can express Eq. (32) as

$$\begin{aligned}
\dot{V}_1 &\leq -(1 - d_1) \alpha_1 \psi_1^2(\tilde{x}) + (1 - d_1) \beta_1 \psi_1(\tilde{x}) \phi_1(\tilde{\boldsymbol{y}}) \\
&- \frac{d_1}{\varepsilon_1} \alpha_2 \phi_1^2(\tilde{\boldsymbol{y}}) + d_1 \gamma_1 \phi_1^2(\tilde{\boldsymbol{y}}) + d_1 \beta_2 \psi_1(\tilde{x}) \phi_1(\tilde{\boldsymbol{y}}) \\
&= -\begin{bmatrix} \psi_1(\tilde{x}) \\ \phi_1(\tilde{\boldsymbol{y}}) \end{bmatrix}^T \begin{bmatrix} \nu_1 & \nu_2 \\ \nu_2 & \nu_3 \end{bmatrix} \begin{bmatrix} \psi_1(\tilde{x}) \\ \phi_1(\tilde{\boldsymbol{y}}) \end{bmatrix}.
\end{aligned} \tag{33}$$

with

$$\nu_1 = (1-d_1)\alpha_1, \tag{34}$$

$$\nu_2 = -\frac{1}{2}(1-d_1)\beta_1 - \frac{1}{2}d_1\beta_2, \tag{35}$$

$$\nu_3 = d_1\left(\frac{\alpha_2}{\varepsilon_1} - \gamma_1\right). \tag{36}$$

The right hand side of Eq. (33) is a quadratic form in the comparison functions $\psi_1(\tilde{x})$ and $\phi_1(\hat{\boldsymbol{y}})$, where the quadratic form is negative-definite when

$$d_1(1-d_1)\alpha_1\left(\frac{\alpha_2}{\varepsilon_1} - \gamma_1\right) > \frac{1}{4}\left[(1-d_1)\beta_1 + d_1\beta_2\right]^2, \tag{37}$$

where α_1 and α_2 are by definition positive, and we assume that $\beta_1 \geq 0$, $\beta_2 \geq 0$, and $\gamma_1 \geq 0$, thus rewriting Eq. (37) as

$$\varepsilon_1 < \frac{\alpha_1\alpha_2}{\alpha_1\gamma_1 + \frac{1}{4(1-d_1)d_1}\left[(1-d_1)\beta_1 + d_1\beta_2\right]^2} \equiv \varepsilon_{1_d}. \tag{38}$$

Inequality (38) shows that for any choice of d_1, the corresponding $V_1(\tilde{x},\tilde{\boldsymbol{y}})$ is a Lyapunov function for the singular perturbed Σ_{SF}-subsystem for all ε_1 satisfying (38). The maximum value of ε_{1_d} is given by the expressions

$$\varepsilon_1^* = \frac{\alpha_1\alpha_2}{\alpha_1\gamma_1 + \beta_1\beta_2},\ d_1^* = \frac{\beta_1}{\beta_1+\beta_2} \tag{39}$$

If all the growth requirements on the Σ_{SF}-subsystem are satisfied, then the origin is an asymptotically stable equilibrium of the singularly perturbed system Σ_{SF} for all $\varepsilon_1 \in (0, \varepsilon_1^*)$, where ε_1^* is given by Eq. (39). Bounds for the general Σ_{SF}-subsystem *SA* are presented bellow.

1) Bounds of the Stability Parameters *for the Σ_{SF} System:* The satisfaction of $\varepsilon_1 < \varepsilon_1^*$ is achieved by defining the bounds of the *stability parameters*. The freedom on selecting β_2 and γ_1 permits to obtain the upper-bounds on ε_1^* and its d_1^* parameter, such that they match the required parameters that guarantee the asymptotic stability for the full Σ_{SF}-subsystem. This is achieved by selecting γ_1 and β_2 such that desired values of ε_1^* and d_1^* are obtained. The constant γ_1 is selected by solving Eq. (39)

$$\varepsilon_1^* = \frac{\alpha_1\alpha_2}{\alpha_1\gamma_1 + \beta_1\beta_2} \ \rightarrow \gamma_1(\varepsilon_1^\bigstar) = \frac{1}{\alpha_1}\left(\frac{\alpha_1\alpha_2}{\varepsilon_1^\bigstar} - \beta_1\beta_2\right) \tag{40}$$

and where β_2 is defined by solving Eq. (39) yielding

$$d_1^* = \frac{\beta_1}{\beta_1+\beta_2} \ \rightarrow \beta_2(d_1^\bigstar) = \frac{\beta_1}{d_1^\bigstar} - \beta_1, \tag{41}$$

where recall that $d_1^\bigstar$ and $\varepsilon_1^\bigstar$ are the selected values such that satisfy the asymptotic stability properties of the full system, not to be confused with d_1^* and ε_1^* which represents the upper bounds. The power to select $\varepsilon_1^* = \varepsilon_1^\bigstar$, permits to satisfy the fulfillment of the Σ_{SF} *Stability Analysis* by guaranteeing that $\varepsilon_1 < \varepsilon_1^*$.

The percentage contribution of both Lyapunov functions $V_S(\tilde{x})$ and $V_F(\hat{\boldsymbol{y}})$ on the Lyapunov function $V_1(\tilde{x},\tilde{\boldsymbol{y}})$, is defined by the proper selection of the parameter $d_1^\bigstar$, and it is required to complete the second stability analysis, the Σ_{SFU} *Stability Analysis*, to completely define its value. In addition, for safety margin, the $\varepsilon_1^\bigstar$ is selected as $\varepsilon_1^\bigstar = \delta_{\varepsilon_1}\varepsilon_1$, with $\delta_{\varepsilon_1} > 1$.

The power of these results lays in identifying that the Lyapunov functions $V_S(\tilde{x})$ and $V_F(\tilde{x},\tilde{\boldsymbol{y}})$ are functions of the same *stability parameters* q_s, q_{f_1}, and q_{f_2}, as seen in Eqns. (15) and (18) respectively, therefore becoming degrees of freedom. It is also identified, that in order to to guarantee the stability properties of the Σ_{SFU} system, ratios between these *stability parameters* can to be defined. This is achieved by selecting ratios of dependance among the different *stability parameters*, such that all parameters can be written as a function of one of the *stability parameters*. Therefore by selecting $q_{f_1} = \tilde{Q}_{SF}q_s$, and $q_{f_2} = \tilde{Q}_{F_{21}}\tilde{Q}_{SF}q_s$, where both $\tilde{Q}_{SF} = \delta_1 Q_{SF}$ and $\tilde{Q}_{F_{21}} = \delta_1 Q_{F_{21}}$, represent the required ratios to prove the asymptotic stability analysis for the full Σ_{SFU} helicopter system, and with $\delta_1 > 1$ being a constant that increases the ratios for safety margin. This implies, that with the proper selection of $\varepsilon_1^\bigstar$, $d_1^\bigstar$, q_s, and the ratios $\tilde{Q}_{SF}$, and $\tilde{Q}_{F_{21}}$, the asymptotic stability of the Σ_{SF}-subsystem satisfies the requirement $\varepsilon_1 < \varepsilon_1^*$. These results concludes the first step of the asymptotic stability analysis. The following section describes the second step of the asymptotic stability analysis, the Σ_{SFU} *Stability Analysis* for the generic helicopter problem.

D. Σ_{SFU} Stability Analysis

Similarly, the stability analysis of Σ_{SFU} system is performed using the previous stability analysis of the Σ_{SF}-subsystem. The Σ_{SFU} system is treated like a two-time-scale singularly perturbed system, where the Σ_{SF}-subsystem becomes the reduced order system, and the Σ_U-subsystem becomes the boundary layer. The asymptotically stability requirements of the reduced and boundary layer systems are expressed by requiring the existence of Lyapunov functions for both the Σ_{SF} and the Σ_U-subsystems, where the Lyapunov function derived in the Σ_{SF} *Stability Analysis*, $V_1(\tilde{x},\tilde{\boldsymbol{y}})$, becomes the associated Lyapunov function for the reduced order Σ_{SF}-subsystem, while $V_U(\hat{\boldsymbol{z}})$ becomes the associated Lyapunov function for the boundary layer Σ_U-subsystem.

Again, the satisfaction of the growth requirements take the form of inequalities defined in Assumptions A.1 through A.5, where inequalities (67) and (70) define the growth requirements of the Σ_{SF} and Σ_U-subsystems respectively, while (73) and (75) define the growth requirements of the combined $\tilde{F}(\tilde{\boldsymbol{\chi}},\tilde{\boldsymbol{z}})$ and Σ_U-subsystems.

With the analysis of the $F(\tilde{\boldsymbol{\chi}},\tilde{\boldsymbol{z}})$ and Σ_U-subsystems it is clear that they have an isolated equilibrium at the origin, thus fulfilling Assumption A.1 . The fulfillment of Assumption A.2 can be easily checked with the choice of the comparison function $\psi_2(\tilde{\boldsymbol{\chi}})$ proposed in Table I, where $\boldsymbol{\mathcal{R}}$ is a positive definite 3×3 diagonal matrix defined by

$$\boldsymbol{\mathcal{R}} = \begin{bmatrix} \mathcal{R}_1 & 0 & 0 \\ 0 & \mathcal{R}_2 & 0 \\ 0 & 0 & \mathcal{R}_3 \end{bmatrix}, \tag{42}$$

where it can easily be shown that the diagonal entries $\mathcal{R}_i$ $i = 1,2,3$ are expressed as functions of the *stability parameters* q_s, q_{f_1}, q_{f_2}, the *control parameters* b_1, b_2, b_3, the parasitic constants d_1, ε_1, and the physical parameters of the helicopter. It can also be shown that in order to guarantee that $\boldsymbol{\mathcal{R}}$ is positive definite, mathematical expression for both $\tilde{Q}_{SF}$ and

$\tilde{Q}_{F_{21}}$, can be derived, thus allowing to select the proper values of q_s, q_{f_1}, q_{f_2}, b_1, b_2, b_3 and d_1 that ensure the fulfillment of Assumption A.2. Refer to [7] for more details on how the constants are derived.

Assumption A.3 is fulfilled with the choice of $\phi_2(\hat{\boldsymbol{z}})$ described in Table I, with $\tilde{Q}_U = \tilde{Q}_U/2$, defined in Eq. (24), and satisfying that $\hat{\boldsymbol{z}} = \tilde{\boldsymbol{z}} - H(\tilde{\boldsymbol{\chi}}) = 0$ is a stable equilibrium of the boundary layer system. The first interconnection condition, Assumption A.4, is satisfied with the relationships described in Table I, that is selecting $\beta_3 = \delta_{\beta_3}\beta_3^*$. Inequality (73) determines the allowed growth of $F(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$ in $\tilde{\boldsymbol{z}}$. Assumption A.5 is satisfied with the relationships described in Table I, that is selecting $\beta_4 = \delta_{\beta_4}\beta_4^*$, and $\gamma_2 = \delta_{\gamma_2}\gamma_2^*$, with $\delta_{\beta_3} > 1$, $\delta_{\beta_4} > 1$, and $\delta_{\gamma_2} > 1$.

The fulfillment of Assumptions A.1 through A.5 determines that the origin is an asymptotically stable equilibrium of the singular perturbed Σ_{SFU} full system for sufficiently small ε_2. We therefore consider a new Lyapunov function candidate for the Σ_{SFU} system defined by a weighted sum of $V_1(\tilde{\boldsymbol{\chi}})$ and $V_U(\hat{\boldsymbol{z}})$ such

$$V_2(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}}) = (1 - d_2)V_1(\tilde{\boldsymbol{\chi}}) + d_2 V_U(\hat{\boldsymbol{z}}),\ d_2 \in (0, 1), \tag{43}$$

where $0 < d_2 < 1$. To explore the freedom that we have choosing the weights, we take d_2 as an unspecified parameter in $(0, 1)$. From the properties of $V_1(\tilde{\boldsymbol{\chi}})$ and $V_U(\hat{\boldsymbol{z}})$ and $\| H(\tilde{\boldsymbol{\chi}}) \| \leq p_2(\| \tilde{x} \|)$, where $p_2(\cdot)$ is a κ function, it follows that $V_2(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$ is positive-definite. Computing the time derivative of V_2 along the trajectories of $F(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$ and Σ_U in a similar manner as the one conducted in Eqns. (32) and (33) we obtain that

$$\varepsilon_2 < \frac{\alpha_3\alpha_4}{\varepsilon_1\left(\alpha_3\gamma_2 + \dfrac{1}{4(1-d_2)d_2}\left[(1-d_2)\beta_3 + d_2\beta_4\right]^2\right)} \equiv \varepsilon_{2_d}. \tag{44}$$

Inequality (44) shows that for any choice of d_2, the corresponding $V_2(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$ is a Lyapunov function for the singular perturbed system Σ_{SFU} for all ε_2 satisfying Eq. (44). The maximum value of ε_2^* is given by the expressions

$$\varepsilon_2^* = \frac{\alpha_3\alpha_4}{\varepsilon_1\left(\alpha_3\gamma_2 + \beta_3\beta_4\right)},\ d_2^* = \frac{\beta_3}{\beta_3 + \beta_4}. \tag{45}$$

Similarly as in the Σ_{SF} *Stability Analysis*, if all the growth requirements are satisfied, then the origin is an asymptotically stable equilibrium of the singularly perturbed system Σ_{SFU} for all $\varepsilon_2 \in (0, \varepsilon_2^*)$. The following section defines the mathematic expressions for the bounds on the singularly perturbed parameters.

1) Bounds of the Stability Parameters *for the Σ_{SFU} System:* Mathematical expressions for the upper bounds of the Σ_{SFU} system can be obtained by recalling the results from the Σ_{SF} *Stability Analysis*, and recalling that β_3^*, β_4^* and γ_2^* can be defined as

$$\beta_3^* = \mathcal{F}_1\sqrt{\frac{q_s}{q_{u_1}}} = \mathcal{F}_1(\tilde{Q}_{SF}, \tilde{Q}_{F_{21}}, b_1, b_3, d_1, a_*)\sqrt{\frac{q_s}{q_{u_1}}}, \tag{46}$$

$$\beta_4^* = \mathcal{F}_2\sqrt{\frac{q_{u_1}}{q_s}} = \mathcal{F}_2(\tilde{Q}_{SF}, Q_{U_{21}}, b_1, b_2, b_3, d_1, a_*)\sqrt{\frac{q_{u_1}}{q_s}}, \tag{47}$$

$$\gamma_2^* = \mathcal{F}_3(Q_{U_{21}}, b_1, b_2, a_*), \tag{48}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ are elaborated functions of the *stability parameters*, the *control parameters*, and the physical constants of the helicopter (a_*), and where $Q_{U_{21}}$ defines the ratio between the *stability parameters* q_{u_1} and q_{u_2}, as $q_{u_2} = Q_{U_{21}}q_{u_1}$, thus permitting to rewrite Eqns. (45) as

$$\varepsilon_2^* = \frac{\alpha_3\alpha_4\left(\alpha_1\gamma_1(\varepsilon_1^\bigstar) + \beta_1\beta_2(d_1^\bigstar)\right)}{\alpha_1\alpha_2\left(\alpha_3\mathcal{F}_3 + \mathcal{F}_1\sqrt{\frac{q_s}{q_{u_1}}}\mathcal{F}_2\sqrt{\frac{q_{u_1}}{q_s}}\right)}, \tag{49}$$

$$d_2^* = \frac{\mathcal{F}_1\sqrt{\frac{q_s}{q_{u_1}}}}{\mathcal{F}_1\sqrt{\frac{q_s}{q_{u_1}}} + \mathcal{F}_2\sqrt{\frac{q_{u_1}}{q_s}}}. \tag{50}$$

A mathematical expression for $Q_{U_{21}}$ can be obtained such that minimizes $\gamma_2(Q_{U_{21}}, b_1, b_2, a_*)$. With the ratios Q_{SF}, and $Q_{U_{21}}$ defined, the ratio $\tilde{Q}_{F_{21}}$ needs to be bounded, and provide an expression between q_s and q_{u_1}. An expression for $\tilde{Q}_{F_{21}}$ is obtained by analyzing Eq. (49), and recognizing that the upper-bound on ε_2^* only depends on the *stability parameter* ratios $\tilde{Q}_{SF}$, $\tilde{Q}_{F_{21}}$, $Q_{U_{21}}$, and do not depends explicitly in the *stability parameters*, q_s, q_{f_1}, q_{f_2}, q_{u_1} and q_{u_2}, therefore, Eq. (49) can be solved as a function of $\tilde{Q}_{F_{21}}$ by selecting the desired value of $\varepsilon_2^* = \varepsilon_2^\bigstar$, where $\varepsilon_2^\bigstar = \delta_{\varepsilon_2}\varepsilon_2$, with δ_{ε_2} selected as safety margin, and ε_2 being the selected singular perturbation constant that defines the Σ_U time-scale. Noting that only $\mathcal{F}_1$ depends on $\tilde{Q}_{F_{21}}$, Eq. (49) can be rewritten as

$$\mathcal{F}_1 = \frac{\alpha_3\alpha_4\left(\alpha_1\gamma_1(\varepsilon_1^\bigstar) + \beta_1\beta_2(d_1^\bigstar)\right)}{\alpha_1\alpha_2\varepsilon_2^\bigstar\mathcal{F}_2} - \alpha_3\frac{\mathcal{F}_3}{\mathcal{F}_2}, \tag{51}$$

which can be solved for $\tilde{Q}_{F_{21}}$ by solving Eq. (51) resulting in $\tilde{Q}_{F_{21}} = \tilde{Q}_{F_{21}}(\tilde{Q}_{SF}, \tilde{Q}_{U_{21}}, b_1, b_2, b_3, d_1, a_*, \varepsilon_1^\bigstar, \varepsilon_2^\bigstar, d_1^\bigstar)$. The final *stability parameter* ratio, Q_{US} can be obtained by analyzing Eq. (50), and recognizing that, by selecting the value of $d_2^* = d_2^\bigstar$, it can be obtained an expression that determines the missing relation between q_s and q_{u_1}. The selection of $d_2^\bigstar$ defines the percentage contribution of both Lyapunov functions $V_1(\tilde{\boldsymbol{\chi}})$ and $V_U(\hat{\boldsymbol{z}})$ on the Lyapunov function $V_2(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$. Rewriting Eq. (50) results in

$$d_2^* = \frac{\mathcal{F}_1\sqrt{\frac{q_s}{q_{u_1}}}}{\mathcal{F}_1\sqrt{\frac{q_s}{q_{u_1}}} + \mathcal{F}_2\sqrt{\frac{q_{u_1}}{q_s}}}, \rightarrow \frac{1 - d_2^\bigstar}{d_2^\bigstar} = \frac{\mathcal{F}_2\sqrt{\frac{q_{u_1}}{q_s}}}{\mathcal{F}_1\sqrt{\frac{q_s}{q_{u_1}}}} = \frac{\mathcal{F}_2}{\mathcal{F}_1}\frac{q_{u_1}}{q_s}, \tag{52}$$

therefore resulting in

$$Q_{US} = \frac{1 - d_2^\bigstar}{d_2^\bigstar}\frac{\mathcal{F}_1(\tilde{Q}_{SF}, \tilde{Q}_{F_{21}}, b_1, b_3, d_1, a_*)}{\mathcal{F}_2(\tilde{Q}_{SF}, Q_{U_{21}}, b_1, b_2, b_3, d_1, a_*)}, \tag{53}$$

where Eq. (53) completes the relations among all *stability parameters* therefore, providing bounded definitions as a function of the arbitrary parameter q_s for q_{f_1}, q_{f_2}, q_{u_1}, and q_{u_2} as

$$q_{f_1} = \tilde{Q}_{SF}q_s, \tag{54}$$

$$q_{f_2} = \tilde{Q}_{F_{21}}q_{f_1} = \tilde{Q}_{F_{21}}\tilde{Q}_{SF}q_s, \tag{55}$$

$$q_{u_1} = Q_{US}q_s, \tag{56}$$

$$q_{u_2} = Q_{U_{21}}q_{u_1} = Q_{U_{21}}Q_{US}q_s. \tag{57}$$

This powerful result implies the existence of a closed form solution for the proper selection of the *stability parameters* as a function of the arbitrary q_s, which permits to bound the value of ε_1^* in the Σ_{SF}-subsystem and ε_2^* in the Σ_{SFU} full system such that $\varepsilon_1 < \varepsilon_1^*$, and $\varepsilon_2 < \varepsilon_2^*$. This demonstrates that the origin $\tilde{\boldsymbol{\chi}} = 0$, $\tilde{\boldsymbol{z}} = 0$, is an asymptotically stable equilibrium

of the singularly perturbed Σ_{SFU} system for all $\varepsilon_2 \in (0, \varepsilon_2^*)$. Expanding $V_2(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$ it can be seen that the resulting Lyapunov function is also a function of the *stability parameters* and the *control parameters* such

$$V_2(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}}) \quad = \quad (1 - d_2)V_1(\tilde{\boldsymbol{\chi}}) + d_2 V_U(\hat{\boldsymbol{z}}). \tag{58}$$

The proposed stability analysis permits to satisfy the stability properties of the singularly perturbed Σ_{SFU} system by using the *stability parameters*, as long as they satisfy the bounds and the ratios that have been obtained through the fulfillment of the proposed asymptotic stability methodology. Furthermore, it is important to note that at the same time, bounds on the *control parameters*, b_1, b_2, and b_3, are also defined such that guarantee the three-time-scale decomposition. These bounds allow to introduce a degree of freedom on the time responses, for both the slow and fast dynamics, permitting to increase or decrease the time-scale separation between both dynamics, but always satisfying the existence of a three-time-scale separation. This concludes the second step of the asymptotic stability analysis for the generic helicopter problem. Numeric stability results for the X-Cell 50 helicopter model being studied in this paper are presented in section V.

TABLE I
PARAMETERS FOR THE COMPARISON FUNCTIONS AND INEQUALITIES THAT GUARANTEE THE ASYMPTOTIC STABILITY REQUIREMENTS FOR THE HELICOPTER Σ_{SF} SUBSYSTEM.

Assumption A.2			
$\frac{\partial V}{\partial x}$	$f(x, \mathrm{h}(x))$	α_1	$\psi(x)$
$\frac{\partial V_S(\tilde{x})}{\partial \tilde{x}}$	$f(\tilde{x}, G, H)$	$\alpha_1 \leq 1$	$\psi_1(\tilde{x}) = \sqrt{\tilde{q}_s \tilde{x}^2}$
$\left(\frac{\partial V_1(\tilde{\boldsymbol{\chi}})}{\partial \tilde{\boldsymbol{\chi}}}\right)^T$	$F(\tilde{\chi}, H(\tilde{\boldsymbol{\chi}}))$	$\alpha_3 \leq 1$	$\psi_2(\tilde{\boldsymbol{\chi}}) = \left(\tilde{\boldsymbol{\chi}}^T \boldsymbol{\mathcal{R}} \tilde{\boldsymbol{\chi}}\right)^{\frac{1}{2}}$
Assumption A.3			
$\frac{\partial W}{\partial z}$	$g(x, z)$	α_2	$\phi(z - \mathrm{h}(x))$
$\left(\frac{\partial V_F(\tilde{y})}{\partial \tilde{y}}\right)^T$	$g(\tilde{x}, \tilde{\boldsymbol{y}}, H)$	$\alpha_2 \leq 1$	$\phi_1(\hat{\boldsymbol{y}}) = \sqrt{\tilde{\boldsymbol{y}}^T \tilde{\mathbf{Q}}_{\mathbf{F}} \tilde{\boldsymbol{y}}}$
$\left(\frac{\partial V_U(\hat{z})}{\partial \hat{z}}\right)^T$	$h(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$	$\alpha_4 \leq 1$	$\phi_2(\hat{\boldsymbol{z}}) = \left(\hat{\boldsymbol{z}}^T \tilde{\mathbf{Q}}_{\mathbf{U}} \hat{\boldsymbol{z}}\right)^{\frac{1}{2}}$
Assumption A.4			
$\frac{\partial V}{\partial x}$	$f(x, z)$	$f(x, \mathrm{h}(x))$	β_1
$\left(\frac{\partial V_S(\tilde{x})}{\partial \tilde{x}}\right)^T$	$f(\tilde{x}, \tilde{\boldsymbol{y}}, H))$	$f(\tilde{x}, G, H)$	$\beta_1 \geq \beta_1^*$
$\left(\frac{\partial V_1(\tilde{\boldsymbol{\chi}})}{\partial \tilde{\boldsymbol{\chi}}}\right)^T$	$F(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$	$F(\tilde{\boldsymbol{\chi}}, H(\tilde{\boldsymbol{\chi}}))$	$\beta_3 \geq \beta_3^*$
Assumption A.5			
$\frac{\partial W}{\partial x}$	$f(x, z)$	γ_1	β_2
$\left(\frac{\partial V_F(\tilde{y})}{\partial \tilde{x}}\right)^T$	$f(\tilde{x}, \tilde{\boldsymbol{y}}, H(\tilde{x}, \tilde{\boldsymbol{y}}))$	$\gamma_1 \geq 0$	$\beta_2 \geq 0$
$\left(\frac{\partial V_U(\hat{z})}{\partial \tilde{y}}\right)^T$	$F(\tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{z}})$	$\gamma_2 \geq \gamma_2^*$	$\beta_4 \geq \beta_4^*$

V. NUMERICAL RESULTS FOR THE HELICOPTER PROBLEM

This section provides numerical results for the stability analysis on the closed-loop dynamics for the X-Cell 50 helicopter model here studied, where the values of the normalized physical coefficients, a_*, are defined in Table II. The physical restrictions of the proposed model require defining the range of reachable states, and their associated desired final conditions, and are bounded by the physical limits of the state variables and are defined in Table III. See [7] for further details.

This section provides bounds on both ε_1 and ε_2. Recall that ε_1 and ε_2 represent the parasitic constants that define the time-scale behavior among the different subsystems, where the stretched time-scale are given by the time constant $\tau_1 = t/\varepsilon_1$, and $\tau_2 = \tau_1/\varepsilon_2$. Computer simulations supports any time-scale selection, but hardware limitations should drive that time-scale selection by ensuring that dynamics of the different subsystems behave according to their dynamics limits (i.e. vertical velocity of the helicopter, angular velocity of the blades, and collective pitch rate of the servo actuators). Defining bounds on the parasitic parameters, helps to define how small is small enough, and ensures that the available discrete control signals will have sufficient authority to control the fastest systems.

TABLE II
VALUES OF THE NORMALIZED PHYSICAL COEFFICIENTS a_* FOR THE X-CELL 50.

$a_1 = 5.198 \times 10^{-4}$	$a_2 = 1.520 \times 10^{-2}$	$a_3 = 2.702 \times 10^{-7}$
$a_4 = 1.580 \times 10^{-5}$	$a_5 = a_6 = a_{14} - 0.1$	$a_7 = -17.67$
$a_8 = -0.7$	$a_9 = a_{10=} - 0.0028$	$a_{11} = -13.92$
$a_{12} = 434.88$	$a_{13} = -800$	$a_{15} = -65$

TABLE III
LIMITS FOR THE HELICOPTER VARIABLES.

States	$\boldsymbol{\chi_{MIN}}$	$\boldsymbol{\chi_{MAX}}$	$\boldsymbol{\chi^*_{MIN}}$	$\boldsymbol{\chi^*_{MAX}}$
x [rad/s]	74.25	180	81.675	162
y_1 [m]	0	2	0.1	1.9
y_2 [m/s]	-2.8505	5.7009	0	0
z_1 [rad]	0.0175	0.245	0.0935	0.2199
x_5 [rad/s]	-9.52	9.52	0	0

Recalling the physical coefficients a_*, defined in Table II, the parasitic constants for the open loop for the X-Cell 50 helicopter are selected as $\varepsilon_1 = a_9/a_5 = 2.8 \times 10^{-2}$, and $\varepsilon_2 = a_5/a_{13} = 1.25 \times 10^{-4}$, satisfying that $0 < \varepsilon_2 \ll \varepsilon_1 \ll 1$. For the stability analysis here presented the *control parameters* are selected as $b_1 = \varepsilon_1 \omega_n^2 = 0.112$, $b_2 = 2\varepsilon_1 \zeta \omega_n = 0.1008$, $b_3 = 0.5$.

The Σ_{SFU} *Stability Analysis* can be satisfy by requiring $d_1 \in (0.0543, 0.5243)$, therefore, by selecting $d_1^* = d_1^{\bigstar} = 0.5$, the percentage contribution on the Lyapunov function $V_1(\tilde{x}, \tilde{\boldsymbol{y}})$, Eq. (31), is equally distributed for both Lyapunov functions $V_S(\tilde{x})$ and $V_F(\hat{\boldsymbol{y}})$, while by selecting $\delta_{\varepsilon_1} = 1.05$, and $\delta_{\varepsilon_2} = 1.05$, we have $\varepsilon_1^{\bigstar} = 0.02940$, and $\varepsilon_2^{\bigstar} = \varepsilon_2^* = 1.26250 \times 10^{-4}$, thus satisfying $\varepsilon_1 < \varepsilon_1^*$, and $\varepsilon_2 < \varepsilon_2^*$.

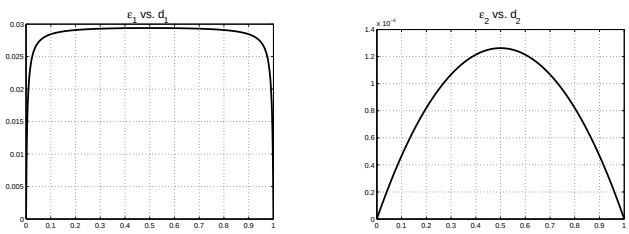
For the selected *control parameters* the *stability parameter* ratios that satisfy the stability analysis of the Σ_{SFU} system are given by $Q_{SF} = 0.259974$, $Q_{F_{21}} = 2.567205$, $Q_{U_{21}} = 0.0041970$, and $Q_{US} = 0.330899$, therefore by selecting $q_s = 0.5$, and recalling Eqns. (55–57), result in $q_{f_1} = 0.129987$, $q_{f_2} = 0.333703$, $q_{u_1} = 0.165449$, and $q_{u_2} = 5.455576$. Therefore, the coefficients that fulfill the growth requirements of the Σ_{SF}-subsystem are given by $\alpha_1 = 0.95$, $\alpha_2 = 0.95$, $\beta_1 = 0.76260$, $\beta_2 = 0.76260$, and $\gamma_1 = 31.7007576$, and the coefficients that fulfill the growth requirements for the Σ_{SFU} system are therefore given by $\alpha_3 = 0.95$, $\alpha_4 = 0.95$, $\beta_3 = 492.14288$, $\beta_4 = 492.14288$, and $\gamma_2 = 11128.12913$.

Figure 6(a) shows the dependance on the right-hand side of Eq. (40) on the unspecified parameter d_1, being able to identify that the maximum value is achieved at the selected $d_1^\star = 0.5$, and given by $\varepsilon_1^\star = 0.02940$, and the dependance on the right-hand side of Eq. (44) on the unspecified parameter d_2 is sketched in Figure 6(b) being able to identify that the maximum value is achieved at the selected $d_1^\star = 0.5$, and given by $\varepsilon_2^\star = 1.26250 \times 10^{-4}$.

VI. CONCLUSIONS

In this paper a stability analysis has been performed taking advantage of the system three-time scales. A methodology for the stability analysis of such type of systems has been presented extending the well-known two-time-scale case. The methodology here presented is not limited to the problem here discussed, but can be suited to more general systems as described in [7].

The proposed time-scale analysis methodology represents an all-in-one tool that permits to construct a valid Lyapunov function based in time-scale separation, and perform the time-scale asymptotic stability analysis of the singularly perturbed system, becoming a really powerful tool for the analysis of the helicopter model in vertical flight here employed, that can also be applied to more general nonlinear singularly perturbed systems.



(a) ε_1 (b) ε_2

Fig. 6. Stability upper bounds on ε_1 and ε_2

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APPENDIX

For completeness of the paper, this section describes the general formulation for the asymptotic stability analysis for the two time-scale singular perturbation problems [3]. It serves as basis for the analysis that was conducted to demonstrate the asymptotic stability of the three-time-scale model that derived in the subsection IV-B. Consider a nonlinear autonomous singular perturbed system of the form:

$$\dot{x} = f(x,z),\ x \in R^n, \tag{59}$$

$$\varepsilon\dot{z} = g(x,z),\ z \in R^m, \tag{60}$$

which has an isolated equilibrium at the origin $(x = 0, z = 0)$. It is assumed throughout the formulation that f and g are smooth to ensure that for specified initial conditions, system (59-60) has a unique solution. The stability of the equilibrium is investigated by examining the reduced (slow) system

$$\dot{x} = f(x, h(x)), \tag{61}$$

where $z = h(x)$ is an associated root of $0 = g(x, z)$, and the boundary-layer (fast) system

$$\frac{dz}{d\tau} = g(x, z(\tau)),\ \tau = \frac{t}{\varepsilon}, \tag{62}$$

where x is treated as a fixed parameter, and ε is the parasitic constant that defines the stretched time-scale of the fast subsystem. Asymptotic stability requirements on the reduced and boundary-layer systems are expressed by requiring the existence of Lyapunov functions for each system that satisfy certain growth condition. The growth requirements of f and g take the form of inequalities satisfied by the Lyapunov functions, which will be addresses later. We assume that the following conditions hold for all

$$(x, z, \varepsilon)\ \in [t_0, \infty) \times B_x \times B_z \times [0, \varepsilon_1], \tag{63}$$

with $B_x \subset R^n$ and $B_z \subset R^m$, and where $B_x \subset R^n$ and$u$ $B_z \subset R^m$ denote closed sets. We add and substract $f(x, h(x))$ to the righthand side of (59) yielding

$$\dot{x} = f(x, h(x)) + f(x, z) - f(x, h(x)), \tag{64}$$

where $f(x, z) - f(x, h(x))$ can be viewed as a perturbation of the reduced system (61). It is natural first to look for a Lyapunov function candidate for (61) and then to consider the effect of the perturbation term $f(x, z) - f(x, h(x))$ [3].

Assumption A.1: Asymptotic Stability of the Origin

The origin ($x = 0$, $z = 0$) is a unique and isolated equilibrium of (59-60), i.e.,

$$0 = f(0, 0),\ \text{and}\ 0 = g(0, 0), \tag{65}$$

moreover, $z = h(x)$ is the unique root of $0 = g(x, z, 0)$ in $B_x \times B_z$, i.e., $0 = g(x, h(x))$ and there exists a class κ function $p(\cdot)$ such that $\| h(x) \| \leq p(\| x \|)$.

The asymptotic stability of the origin is studied using Lyapunov methods. To construct a Lyapunov function candidate for the singular perturbed system (59-60) we consider each of the two equations separately. The growth requirements of f and g take the form of inequalities that must be satisfied by the Lyapunov functions. The growth requirements of the reduced and boundary layer system by separate will be defined in Assumptions A.2 and A.3, while the growth requirements that combine both reduced and boundary layer system requirements, called interconnection conditions, will be defined in Assumptions A.4 and A.5.

Assumption A.2: Reduced System Conditions

There exists a positive-definite an decreasing Lyapunov function candidate $V(x)$ that is,

$$0 < q_1(\| x \|) \leq V(x) \leq q_2(\| x \|), \tag{66}$$

for some class κ function $q_1(\cdot)$ and $q_2(\cdot)$ that satisfies the following inequality,

$$\frac{\partial V}{\partial x} f(x, h(x)) \leq -\alpha_1 \psi^2(x), \tag{67}$$

where $\psi(\cdot)$ is a scalar function of vector arguments which vanishes only when its argument are zero, and satisfying that $x = 0$ is a stable equilibrium of the reduced order system. Condition (67) guarantees that $x = 0$ is an asymptotically stable equilibrium of (61).

Assumption A.3: Boundary-Layer System Conditions

There exists a positive-definite an decreasing Lyapunov function candidate $W(x, z)$ such that for all $(x, z) \in B_x \times B_z$, satisfying

$$0 < q_3(\| z - h(x) \|) \leq W(x, z) \leq q_4(\| z - h(x) \|), \tag{68}$$

for some class κ function $q_3(\cdot)$ and $q_4(\cdot)$, that satisfies,

$$W(x, z) > 0,\ \forall z \neq h(x)\ and\ W(x, h(x)) = 0, \tag{69}$$

and

$$\frac{\partial W}{\partial z} g(x, z) \leq -\alpha_2 \phi^2(z - h(x)),\ \alpha_2 > 0, \tag{70}$$

where $W(x, z)$ is a Lyapunov function of the boundary layer system (62) in which x is treated as a fixed parameter, and $\phi(\cdot)$ is a scalar function of vector arguments which vanishes only when its argument are zero, and satisfying that $z - h(x)$ is a stable equilibrium of the boundary layer system.

Assumption A.4: First Interconnection Condition

$V(x)$ and $W(x, z)$ must satisfy the so called interconnection conditions. The first interconnection condition is obtained by computing the derivative of V along the solution of (64),

$$\dot{V} = \frac{\partial V}{\partial x} f(x, h(x)) + \frac{\partial V}{\partial x} [f(x, z) - f(x, h(x))] \tag{71}$$

$$\leq -\alpha_1 \varphi^2(x) + \frac{\partial V}{\partial x} [f(x, z) - f(x, h(x))], \tag{72}$$

assuming that

$$\frac{\partial V}{\partial x} [f(x, z) - f(x, h(x))] \leq \beta_1 \varphi(x) \phi(z - h(x)). \tag{73}$$

so that

$$\dot{V} \leq -\alpha_1 \varphi^2(x) + \beta_1 \varphi(x) \phi(z - h(x)). \tag{74}$$

Inequality (73) determines the allowed growth of f in z.

Assumption A.5: Second Interconnection Conditions

The second interconnection condition is defined by

$$\frac{\partial W}{\partial x} f(x) \leq \gamma \phi^2(z - h(x)) + \beta_2 \varphi(x) \phi(z - h(x)), \tag{75}$$

where $\psi(\cdot)$ and $\phi(\cdot)$ are scalar functions of vector arguments which vanish only when their arguments are zero, i.e., $\phi(x) = 0$ if and only if $x = 0$. They will be referred to as comparison functions.

With the Lyapunov function $V(x)$ and $W(x, z)$ obtained, a new Lyapunov function $\nu(x, z)$ is considered and defined by the weighted sum of $V(x)$ and $W(x, z)$,

$$\nu(x, z) = (1 - d)V(x) + dW(x, z), \tag{76}$$

for $0 < d < 1$. $\nu(x, z)$ becomes the Lyapunov function candidate for the singular perturbed system (59-60).

Theorem A.1: If $x = 0$ is an asymptotically stable equilibrium of the reduced system (61), $z = h(x)$ is an asymptotically stable equilibrium of the boundary-layer system (62) uniformly in x, that is, the $\varepsilon - \delta$ definition of Lyapunov stability and the convergence $z \to h(x)$ are uniform in x [17], and $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ satisfy growth conditions on the reduced and boundary-layer systems, then the origin is an asymptotically stable equilibrium of the singularly perturbed system (59), for sufficiently small ε [3].