Singular perturbation stability analysis for a three-time-scale autonomous helicopter

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Abstract—This paper presents a singular perturbation stability analysis for a nonlinear three-time-scale autonomous helicopter model in vertical flight. The presented time-scale analysis permits to conduct a stability analysis for the singular perturbed three-time-scale system, and allows to construct a composite Lyapunov function for the resultant closed-loop system using time-scale separation. The methodology here presented provides mathematical expressions for the upper bounds of the singularly perturbed parameters that define the three time scale. Numerical results on the stability analysis are also presented for the studied nonlinear highly coupled helicopter model.

I. INTRODUCTION

The study and demonstration of the asymptotic stability properties of time-scale systems is an important area in the real of singular perturbations. Many studies have been conducted towards demonstrating the asymptotic stability properties of two-time-scale singularly perturbed systems [1]–[3]. These works show the high degree of difficulty required to demonstrate the stability properties of the different time-scale subsystems, and only provide general expressions for the upper bounds, and where these expressions are subject to the satisfaction of the growth requirements inequalities for the different time-scale subsystems, and the selection of arbitrary constants. These complexities are even more evident when dealing with multi-parameter time-scale systems. Several works that appear in the literature approach the multi-parameter asymptotic stability analysis [3]–[6] using all similar methods based on composite stability methods of large scale dynamical systems, but again, without defining precise mathematical upper bounds on the singularly perturbed parameters.

In this paper we approach the problem of multi-parameter asymptotic stability analysis by extending the procedures introduced by Kokotović [3] for the two-time-scale singular perturbation problems, to the three-time-scale singular perturbation problem of an autonomous helicopter on a platform, although the methodology here presented can also extended to more general singular perturbation problems as described by the authors in [7]. The time-scale analysis here presented permits to construct a suitable composite Lyapunov function candidate for an autonomous singularly perturbed system, and analyze the stability of the resulting autonomous subsystems guaranteeing the stability of the equilibrium by providing mathematical expressions that define the upper bounds for the selected singularly perturbed parasitic constants, being this the main contribution of this paper. The presented time-scale analysis methodology has also been used to determine the control laws that guarantee desired closed loop dynamics of the studied helicopter model, which provides an additional tool at the time of conducting the stability analysis.

On the realm of helicopter control, although the use of singular perturbation theory has been employed to simplify the control system structure [8], [9], to our knowledge, the work done by the authors of this paper and of previous publications [7], [10], along with that conducted by Bertrand, Hamel and Piet-Lahanier [11], that presented a stability analysis of a hierarchical controller for an unmanned Aerial Vehicle, are the only investigations that theoretically address stability issues for VTOL UAVs using singular perturbations theory.

This paper is structured as follows: Section II presents the closed loop helicopter model used throughout this paper; Section III presents the three-time-scale analysis here presented, including the time-scale selection; section IV describes the asymptotical stability analysis for the general three time-scale helicopter problem; numerical results for the specific helicopter model studied in this paper are depicted in section V. Conclusions are drawn in Section VI; and finally, Appendix A briefly describes the general two time-scale singular perturbation theory.

II. CLOSED-LOOP MODEL DEFINITION

The helicopter closed loop model that is used throughout the remainder of this paper is the result of the three time-scale singular perturbation control strategy applied by the authors [7] to a vertical flight helicopter model that has been extensively used as a benchmark in the literature to test a wide variety of control strategies to regulate the vertical position of the helicopter [12]–[15]. The vertical flight helicopter model can be decomposed in three distinct dynamics, in which the state space vector is given by $X \triangleq [x, y, z]^T$, where $x$ represents angular velocity of the blades, $y \triangleq [y_1, y_2]^T$ represent the state vector for the vertical motion of the helicopter, with $y_1$ representing the vertical position, and $y_2$ the vertical velocity; and finally, $z \triangleq [z_1, z_2]^T$ represents the state vector for the collective pitch angle dynamics, where $z_1$ represents the collective pitch angle of the blades, and $z_2$ is the collective pitch rate of the blades, recalling that $x$ is the slow variable, $y$ is the fast variable, and $z$ is the ultra-fast variable. The closed-loop helicopter dynamics are expressed in their error...
dynamics, denoted with tilde vector notation, yielding
\[ \dot{x} = a_10(\hat{x} + x^2) \left( \sin(\hat{z}_1 + z_1^*) - \sin h_{1,SS} \right) - b_3 \hat{x}, \]
\[ \varepsilon_1 \dot{y}_1 = c_1 y_2, \]
\[ \varepsilon_1 \dot{y}_2 = \left( \hat{x} + x^2 \right)^2 \left( c_2 + c_3(\hat{z}_1 + z_1^*) - \sqrt{c_4 + c_5(\hat{z}_1 + z_1^*)} \right) + a_9 \dot{y}_2 + a_9 \dot{y}_2^* + c_6, \]
\[ \varepsilon_1 \dot{y}_2 = c_7 \dot{z}_2, \]
\[ \varepsilon_1 \dot{y}_2^* = a_9(\hat{z}_1 + z_1^*) + c_9 \dot{z}_2 + J_2 \left( 1 + \sqrt{c_9 \dot{e}(x, y)} \right)^2 - 1 \]
where
\[ h_{1,SS} = s_2 \left( 1 + \sqrt{c_9 \dot{e}(x, y)} \right)^2 - 1, \]
with
\[ v_{SS} = -\frac{c_6}{x^2}, \]
and with
\[ \dot{v}(x, y) = -\frac{a_9 \dot{y}_2^2 + (a_9 + b_2) \dot{y}_2 + b_1 \dot{y} + c_6}{x^2}, \]
being the coefficients defined as \( c_1 = a_9/a_5, c_2 = c_1 a_1, c_3 = c_1 a_2, c_4 = c_1^2 a_3, c_5 = c_1^2 a_4, c_6 = c_1 a_7, c_7 = c_1 = a_9/a_{13}, c_8 = c_7 a_{14}, c_9 = c_7 a_{15}, \) and \( c_{10} = c_7 a_{12}, \) where the constants \( a_1 \) through \( a_{15} \) are given by physical properties of the helicopter, and where \( s_2 = a_9/a_4 \) and \( s_3 = 4c_3/c_5; b_1, b_2, \) and \( b_3 \) represent the control parameters that define the desired vertical displacement dynamics of the helicopter, and angular velocity dynamics of the blades. For the X-Cell 50 helicopter being studied in this paper, the constants are defined in Table V in section V. For the stability analysis conducted in this paper, the helicopter model will be treated in their general form, and only numerical results for the specific X-Cell 50 will be address in section V. For further details on the control strategies and the helicopter model refer to [7].

III. THREE-TIME-SCALE ANALYSIS

This section describes the proposed time-scale analysis methodologies presented in this paper, and is divided in two subsections: subsection III-A presents the multi-parameter time-scale decomposition, and subsection III-B provides an intuitive description of a generic three-time-scale system to help understanding the time-scale decomposition.

A. Top-Down and Bottom-Up Time Scale Decomposition

This subsection presents the time-scale methodologies presented in this paper that provide an approach in which, for a specific class of singularly perturbed nonlinear systems, a step-by-step procedure can be employed to select an appropriate composite Lyapunov function for the complete singularly perturbed system, conduct the stability analysis for the closed-loop system, which provides mathematical expressions for the upper bounds of the singularly perturbed parameters, and everything in an all-in-one step-by-step process.

These step-by-step methodologies will be denoted as Top-Down (TD) and Bottom-Up (BU) methodologies, and receive their names from the direction in which the singular perturbation parameters are applied, resulting in different time-scales subsystems. Figure 1, shows the TD methodology which analyzes the time-scale properties of the \( \Sigma_{SFU} \) system in a descending manner, denoted by the downward arrow, considering first the top singularly perturbed parameter, \( \varepsilon_1, \) resulting in a simplified two-time-scale problem formed by a one-dimension subsystem, the reduced order slow \( \Sigma_S \)-subsystem, and a two-dimension subsystem, the boundary layer fast \( \Sigma_{FU} \)-subsystem, denoted both by the red dashed boxes; in a second instance, and following the descending direction, the bottom singularly perturbed parameter is applied, \( \varepsilon_2, \) such that simplifies the second-order \( \Sigma_{FU} \)-subsystem into another two-time-scale problem formed, this time, by two one-dimension subsystems, the \( \Sigma_F \) and \( \Sigma_U \)-subsystems, denoted both by the blue dashed-dotted boxes. A similar methodology is applied in the Bottom-Up methodology, but in an ascending manner as seen in Figure 2. Both methodologies are independent but equivalent, and produce the same results, but providing a degree of freedom associated to the direction in which the stretched time-scales are applied, which is chosen depending on the needs by the problem structure. The TD and BU methodologies are not limited to three-time-scale problems, and can be extended to a more general N-time-scale problems as described in [7].

\[ \dot{x} = f(x, y, H(x, y)), \]
\[ \varepsilon_1 \dot{y} = g(x, y, H(x, y)), \]
and where the boundary layer of the $\Sigma_{SFU}$ system, denoted as $\Sigma_U$-subsystem, is given by

$$\frac{dx}{d\tau_2} = h(x,y,z(\tau_2)), \quad (11)$$

where in Eq. (11), $x$ and $y$ are treated like fixed parameters, and $z(\tau_2)$ evolves on its stretched time scale ($\tau_2$). Note that $H(x,y)$ in Eqs. (9–10), represents the quasi-steady-state of the boundary layer, Eq. (11), when $\varepsilon_2 = 0$, that is $0 = h(x,y,z) \rightarrow z = H(x,y)$. The reduced order $\Sigma_{SFU}$-subsystem, Eqs. (9–10), can be treated again like a two-time-scale singular perturbation problem by considering the stretched time-scale defined by $\tau_1 = t/\varepsilon_1$, where the reduced $\Sigma_S$-subsystem is defined by

$$\dot{x} = f(x,G(x),H(x,G(x))), \quad (12)$$

and the boundary layer for the $\Sigma_{SF}$-subsystem, denoted as $\Sigma_F$-subsystem, is now given by

$$\frac{dy}{d\tau_1} = g(x,y(\tau_1),H(x,y(\tau_1))), \quad (13)$$

where $x$ is treated like a fixed parameters, $H(x,y)$ evolves on its configuration space, and $y(\tau_1)$ evolves on its stretched time-scale ($\tau_1$). Note that $G(x)$ represents the quasi-steady-state of the boundary layer $\Sigma_F$-subsystem, Eq. (13), when $\varepsilon_1 = 0$, that is $0 = g(x,y,z) \rightarrow y = G(x)$, and $H(x,G(x))$ represents the quasi-steady-state of the boundary layer $\Sigma_U$-subsystem, Eq. (11), when evolving on the manifold of the $\Sigma_F$-subsystem.

A better understanding of the evolution of the different time-scales can be achieved by observing Figures III-B–III-B, where the evolution of a generic three-time-scale model is presented. Figure III-B shows the evolution of the ultra-fast variable $z$ as it moves towards the manifold of the boundary layer $\Sigma_U$-subsystem, that is, towards the surface that defines the quasi-steady-state equilibrium of the $\Sigma_U$-subsystem, given by $h(x,y,z) = 0$, that is $z = H(x,y)$, while $x$ and $y$ behave as fixed parameters; Figure III-B shows the evolution of the fast variable $y$ as it moves on the configuration space of the $\Sigma_F$-subsystem, towards the surface that defines the quasi-steady-state equilibrium of $\Sigma_F$-subsystem given by $g(x,y,H(x,y)) = 0$, that is $y = G(x)$, while the slow variable $x$ behaves as a fixed parameter, and $z = H(x,G(x))$ evolves on its own manifold; finally, Figure III-B shows the evolution of the slow variable $x$ as it moves in the manifold of the $\Sigma_{FU}$-subsystem, towards the quasi-steady state equilibrium of the $\Sigma_S$, given by the intersection between the planes $g(x,y,H(x,y)) = 0$ and $h(x,y,z) = 0$.

IV. ASYMPOTIC STABILITY ANALYSIS

The stability properties of the resulting autonomous helicopter in vertical flight is analyzed in this section, and providing also exact mathematical expressions for the upper bounds of the singularly perturbed parameters $\varepsilon_1$, and $\varepsilon_2$. It is considered that the associated three-time-scale subsystems $\Sigma_S$, $\Sigma_F$, and $\Sigma_U$ are each asymptotically stable, and the proposed methodology derives the additional conditions that demonstrate the asymptotic stability properties for the full system by extending the well-known standard asymptotic stability requirements for two-time-scale singular perturbation problems [3] to the three-time-scale problem here described. The following subsections describe the methodology employed to construct the associated Lyapunov functions for each of the subsystems, and the asymptotic stability analysis of the full $\Sigma_{SFU}$ system, which includes the mathematic expressions that define the bounds for the singular perturbation parameters.

A. Lyapunov Function Candidates

The selection of proper Lyapunov functions that are required to study the asymptotic stability properties of an autonomous system is one of the most challenging issues that a control engineer has to be faced with. For the asymptotic stability analysis of the different multi-parameter time-scale systems being here studied, it is required the existence of Lyapunov functions for each one of the singularly perturbed subsystems.
The fulfillment of certain growth requirements between each of the Lyapunov functions and the use of composite stability methods [3], [16] will ensure the existence of a Lyapunov function for the entire system.

A methodology that provides Lyapunov functions for all three time-scale subsystems is presented. These Lyapunov functions will be used in the stability analysis to obtain a composite Lyapunov function for the complete $\Sigma_{SFU}$ system. The Lyapunov function for each of the subsystems is obtained by considering the desired dynamics of each of the subsystems, which has been properly selected with the applied control strategy [7], and solving for the associated Lyapunov function for each subsystem. Therefore, after applying the sequential time-scale decomposition, the reduced order $\Sigma_{U}$-subsystem reduces to selected target slow-dynamics of the form

$$\dot{x} = f(\tilde{x}, G(\tilde{x}), H(\tilde{x}, G(\tilde{x}))) = -b_3 \tilde{x},$$

(14)

thus we select a natural Lyapunov function candidate of the form

$$V_0(\tilde{x}) = \frac{1}{2} P_s \tilde{x}^2,$$

(15)

with $P_s$ being a positive constant defined after solving the associated Lyapunov function, and given as $P_s = q_s/(2b_3)$, and where $q_s$ represents a degree of freedom that can be chosen to satisfy the stability requirements. Similarly, the Lyapunov function candidate for the $\Sigma_p$-subsystem is obtained considering the reduced $\Sigma_p$-subsystem

$$\frac{d\tilde{y}}{d\tau_1} = g(x, y(\tau_1), H(x, y(\tau_1))) = A_F \tilde{y},$$

(16)

being

$$A_F = \begin{bmatrix} 0 & c_1 \\ -b_1 & -b_2 \end{bmatrix},$$

(17)

therefore, we choose a natural Lyapunov function $V_F(\tilde{y})$ of the form

$$V_F(\tilde{y}) = \frac{1}{2} \tilde{y}^T P_F \tilde{y},$$

(18)

where $P_F$ is a $2 \times 2$ positive definite matrix that solves the associated Lyapunov equation $P_F A_F + A_F^T P_F + Q_F = 0$, where $Q_F$ and $P_F$ are $2 \times 2$ positive definite matrices of the form

$$Q_F = \begin{bmatrix} q_{f_1} & 0 \\ 0 & q_{f_2} \end{bmatrix}, \quad \text{and} \quad P_F = \begin{bmatrix} p_{f_1} & p_{f_2} \\ p_{f_3} & p_{f_4} \end{bmatrix},$$

(19)

where $q_{f_1}$ and $q_{f_2}$ are degrees of freedom that can be chosen to satisfy the stability analysis, and $p_{f_1}$, $p_{f_2}$, $p_{f_3}$, and $p_{f_4}$ are the solutions of the associated Lyapunov function. The Lyapunov function for the $\Sigma_{SFU}$-subsystem becomes

$$V_R(\tilde{y}) = \frac{1}{2} \tilde{y}^T P_F \tilde{y} = \frac{1}{2} p_{f_1} \tilde{y}_1^2 + \frac{1}{2} p_{f_3} \tilde{y}_3^2 + p_{f_2} \tilde{y}_1 \tilde{y}_2,$$

(20)

with the solutions to the associated Lyapunov equation given as

$$p_{f_1} = \frac{q_{f_1}(b_1 c_1 + b_2 b_3)}{b_1 b_2 c_1}, \quad p_{f_2} = \frac{q_{f_2}}{2 b_3}, \quad p_{f_3} = \frac{q_{f_1} c_1 + q_{f_2} b_1}{b_1 b_2}.$$

Finally, the natural Lyapunov function candidate for the $\Sigma_{UL}$-subsystem is obtained by first introducing a change of variables so that the equilibrium of this boundary-layer subsystem is centered at zero, that is, shifting the $\Sigma_{UL}$-subsystem equilibrium by $H(\tilde{x}, \tilde{y})$, where $H(\tilde{x}, \tilde{y})$ represents the equilibria of the $\Sigma_{UL}$-subsystem. The change of variables is defined as

$$\tilde{z} = \tilde{x} - H(\tilde{x}, \tilde{y}) = [\tilde{z}_1 \quad \tilde{z}_2]^T,$$

(21)

thus rewriting the $\Sigma_{UL}$-subsystem in state-space form as

$$\frac{d\tilde{z}}{dt} = A_V \tilde{z},$$

(22)

therefore, we choose a natural Lyapunov function $V_U$ of the form

$$V_U(\tilde{x}, \tilde{y}, \tilde{z}) = V_U(\tilde{z}) = \frac{1}{2} \tilde{z}^T P_U \tilde{z},$$

(23)

where $P_U$ is a $2 \times 2$ definite matrix that solves the associated Lyapunov equation $P_U A_U + A_U^T P_U + Q_U = 0$, where $Q_U$ and $P_U$ are $2 \times 2$ positive definite matrices of the form

$$Q_U = \begin{bmatrix} q_{u_1} & 0 \\ 0 & q_{u_2} \end{bmatrix}, \quad \text{and} \quad P_U = \begin{bmatrix} p_{u_1} & p_{u_3} \\ p_{u_2} & p_{u_4} \end{bmatrix},$$

(24)

where $q_{u_1}$ and $q_{u_2}$ are degrees of freedom that can be chosen to satisfy the stability analysis, and $p_{u_1}$, $p_{u_2}$, $p_{u_3}$, and $p_{u_4}$ are the solutions of the associated Lyapunov function. This yields the associated Lyapunov function

$$V_U(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{1}{2} \tilde{z}^T P_U \tilde{z} = \frac{1}{2} p_{u_1} \tilde{z}_1^2 + \frac{1}{2} p_{u_2} \tilde{z}_2^2 + p_{u_3} \tilde{z}_1 \tilde{z}_2,$$

(25)

with the solutions to the associated Lyapunov equation given as

$$p_{u_1} = \frac{q_{u_1} (c_0^2 - a_0 c_7)}{2 a_3 a_7 c_9}, \quad p_{u_2} = \frac{q_{u_2}}{2 a_3}, \quad p_{u_3} = \frac{c_7 q_{u_1} - q_{u_2} a_0}{2 a_3 a_7 c_9}.$$

It is important for the understanding of the main contribution of the methodology presented, to note that all three Lyapunov functions possess a degree of freedom in the form of the variables $q_s$, $q_{f_1}$, $q_{f_2}$, $q_{u_1}$, and $q_{u_2}$, which will be referred as stability parameters from now on. These stability parameters will be of great importance when conducting the stability analysis since the proposed time-scale analysis methodology will allow to derive expressions that define the bounds of these stability parameters such that will permit to define mathematical upper bounds for the singularly perturbed parameters $\varepsilon_1$ and $\varepsilon_2$ as it will be shown in the following sections.

**B. Asymptotic Stability Analysis for Three-Time-Scale Problems**

The asymptotic stability analysis of the three-time scale autonomous system, Eqns. (1–5), is also based on a double application of the standard two-time-scale stability analysis of the $\Sigma_{SFU}$ full system. The double application of stability analysis is divided in two stages, following the BU time-scale decomposition presented in Figure 2. In the first stage, the stability analysis of the $\Sigma_{SFU}$-subsystem is performed assuming that the $\Sigma_{U}$-subsystem variables evolve in their own configuration space. The analysis of this first stage is performed by ensuring that the previously derived Lyapunov functions for the $\Sigma_{U}$ and $\Sigma_{SFU}$-subsystems, fulfill the growth requirements on $f(\tilde{x}, \tilde{y}, H(\tilde{x}, \tilde{y}))$ and $g(\tilde{x}, \tilde{y}, H(\tilde{x}, \tilde{y}))$ by satisfying certain inequalities. These inequalities, Assumptions A.1 through A.5, are described in detail in Appendix A. For conciseness of the paper, these assumptions are described in detail in Appendix A for the general two-time-scale problem, and can be used to determine the equivalent assumptions for each of the two stability stages, by substituting the analogous system...
dynamics which are described in Table I, where for each of the assumptions, there are three rows, where the first corresponds to the inequality presented in Appendix A, the second row corresponds to the equivalent inequality for the \( \Sigma_{SF} \)-Stability Analysis, and the third row corresponds to the equivalent inequality for the \( \Sigma_{SFU} \)-Stability Analysis. For example, the equivalent of inequality (67) for the \( \Sigma_{SF} \)-subsystem is given by substituting in the functions that appear in the \( \Sigma_{SF} \)-SA row in Table I, resulting in

\[
\frac{\partial V_{S}(\tilde{x})}{\partial \tilde{x}} f(\tilde{x}, G(\tilde{x}), H(\tilde{x}, G(\tilde{x}))) \leq -\alpha_1 \psi^2(\tilde{x}).
\]  

(26)

The fulfillment of these growth requirements results in the obtention of a Lyapunov function \( V_{f} \) for the singularly perturbed \( \Sigma_{SF} \)-subsystem as a function of a weighted sum of the Lyapunov functions for the \( \Sigma_{S} \) and \( \Sigma_{F} \)-subsystems. The results from the stability analysis of the \( \Sigma_{SF} \)-subsystem are used to conduct the stability analysis for the complete \( \Sigma_{SFU} \) system, which, for convenience, is rewritten as

\[
\dot{\chi} = F(\chi, \tilde{x}),
\]

(27)

and where the \( \Sigma_{SF} \)-subsystem is given by

\[
\varepsilon \dot{z} = h(\chi, \tilde{x}),
\]

(28)

with \( F(\chi, \tilde{x}) \) being the augmented system formed by the slow and fast dynamics, and given by

\[
F(\chi, \tilde{x}) \triangleq \left[ \begin{array}{c} f(\chi, \tilde{x}) \\ g(\chi, \tilde{x}) \end{array} \right],
\]

(29)

where \( \chi \) represents the augmented state vector given by the slow and fast variables, \( \chi \triangleq \left[ \begin{array}{c} \tilde{x} \\ \tilde{y} \end{array} \right]^T \). Making \( \varepsilon_2 = 0 \) yields the equilibrium of the boundary layer of the \( \Sigma_{SFU} \) which is given by

\[
0 = h(\chi, H(\chi)) \rightarrow \dot{z} = H(\chi).
\]

(30)

With the analysis of the \( \Sigma_{S} \) and \( \Sigma_{F} \)-subsystems, Eqns. (14) and (16) respectively, it is clear that they have a unique and isolated equilibrium at the origin, that is \( 0 = f(0, 0, H(x, y)) \) and \( 0 = g(0, 0, H(x, y)) \), thus fulfilling Assumption A.1. The fulfillment of Assumption A.2 can be easily checked with the choice of the comparison function \( \psi_{1}(\tilde{x}) \) proposed in Table I, where \( q_3 = q_2/2 \) with \( q_2 \) being the choice of the designer for the associated \( \Sigma_{S} \)-subsystem Lyapunov function. Assumption A.3 is fulfilled with the choice of the comparison function \( \phi_{1}(\tilde{y}) \) described in Table I, where \( Q_{F} = Q_{F2} \), and with \( Q_{F} \) defined in Eq. (19), and satisfying that \( \tilde{y} = \tilde{y} - G(\tilde{x}) = 0 \) is a stable equilibrium of the boundary layer system. The first interconnection condition, Assumption A.4, is satisfied with the relationship described in Table I, that is selecting \( \beta_1 \geq \beta_1^* \), with \( \beta_1^* = \beta_1^*(q_3, q_2, b_1, b_2, b_3, a_\alpha) \) being an elaborated function of the stability parameters, the control parameters, and the physical parameters of the helicopter model, denoted as \( a_\alpha \), which satisfy inequality (73). Inequality (73) determines the allowed growth of \( f(\tilde{x}, \tilde{y}, H(\tilde{x}, \tilde{y})) \) in \( \tilde{y} \). Assumption A.5 is satisfied with the relationships described in Table I, where it is only required that \( \beta_2 \geq 0 \), and \( \gamma_1 \geq 0 \). Refer to [7] for more details on how the inequalities are satisfied.

The fulfillment of Assumptions A.1 through A.5 determines that the origin is an asymptotically stable equilibrium of the singularly perturbed \( \Sigma_{SF} \)-subsystem for sufficiently small \( \varepsilon_1 \). With the Lyapunov functions \( V_{S}(\tilde{x}) \) and \( V_{F}(\tilde{y}) \) defined, we obtain a new Lyapunov function candidate defined by a weighted sum of \( V_{S}(\tilde{x}) \) and \( V_{F}(\tilde{y}) \), and given by

\[
V_{1}(\tilde{x}, \tilde{y}) = (1 - d_1) V_{S}(\tilde{x}) + d_1 V_{F}(\tilde{y}), \quad d_1 \in (0,1),
\]

(31)

where \( 0 < d_1 < 1 \). To explore the freedom that we have in choosing the weights, we take \( d_1 \) as an unspecified parameter in the interval \((0,1)\). From the properties of \( V_{S}(\tilde{x}) \) and \( V_{F}(\tilde{y}) \), and \( \| G(\tilde{x}) \| \leq p_1(\| \tilde{x} \|) \), where \( p_1(\cdot) \) is a positive-definite function, it follows that \( V_{1}(\tilde{x}, \tilde{y}) \) is positive-definite. Computing the time derivative of \( V_{1}(\tilde{x}, \tilde{y}) \) along the trajectories of \( f(\tilde{x}, \tilde{y}, H(\tilde{x}, \tilde{y})) \) and \( g(\tilde{x}, \tilde{y}, H(\tilde{x}, \tilde{y})) \), we obtain

\[
\dot{V}_{1} = (1 - d_1) \frac{\partial V_{S}}{\partial \tilde{x}} f(\tilde{x}, \tilde{y}, H(\tilde{x}, \tilde{y})) + \frac{d_1}{\varepsilon_1} \frac{\partial V_{F}}{\partial \tilde{y}} g(\tilde{x}, \tilde{y}, H(\tilde{x}, \tilde{y}))
\]

\[
+ d_1 \frac{\partial V_{F}}{\partial \tilde{y}} f(\tilde{x}, \tilde{y}, H(\tilde{x}, \tilde{y}))
\]

\[
= (1 - d_1) \frac{\partial V_{S}}{\partial \tilde{x}} f(\tilde{x}, G(\tilde{x}), H(\tilde{x}, \tilde{y}))
\]

\[
+ (1 - d_1) \frac{\partial V_{F}}{\partial \tilde{y}} [f(\tilde{x}, \tilde{y}, H(\tilde{x}, \tilde{y})) - f(\tilde{x}, G(\tilde{x}), H(\tilde{x}, \tilde{y}))]
\]

\[
+ d_1 \frac{\partial V_{F}}{\partial \tilde{y}} g(\tilde{x}, \tilde{y}, H(\tilde{x}, \tilde{y})),
\]

(32)

where using the inequalities in Assumptions A.2 through A.5 we can express Eq. (32) as

\[
\dot{V}_{1} \leq - (1 - d_1) \alpha_1 \psi_{1}^2(\tilde{x}) + (1 - d_1) \beta_1 \psi_{1}(\tilde{x}) \phi_{1}(\tilde{y})
\]

\[
- \frac{d_1}{\varepsilon_1} \alpha_2 \phi_{1}^2(\tilde{y}) + d_1 \gamma_1 \phi_{1}^2(\tilde{y}) + d_1 \beta_2 \psi_{1}(\tilde{x}) \phi_{1}(\tilde{y})
\]

\[
= - \psi_{1}(\tilde{x})^T \left[ \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \right] \phi_{1}(\tilde{y}) \psi_{1}(\tilde{x}).
\]

(33)
with

\[ v_1 = \frac{1}{1 - d_1} \alpha_1, \]
\[ v_2 = \frac{1}{2(1 - d_1)} \beta_1 - \frac{1}{2} d_1 \beta_2, \]
\[ v_3 = d_1 \left( \frac{\alpha_2}{\varepsilon_1} - \gamma_1 \right). \]

The right hand side of Eq. (33) is a quadratic form in the comparison functions \( \psi_1(\hat{x}) \) and \( \phi_1(\hat{y}) \), where the quadratic form is negative-definite when

\[ d_1(1 - d_1) \alpha_1 \left( \frac{\alpha_2}{\varepsilon_1} - \gamma_1 \right) > \frac{1}{4} \left( \left( 1 - d_1 \right) \beta_1 + d_1 \beta_2 \right)^2, \]

where \( \alpha_1 \) and \( \alpha_2 \) are by definition positive, and we assume that \( \beta_1 \geq 0, \beta_2 \geq 0, \) and \( \gamma_1 \geq 0, \) thus rewriting Eq. (37) as

\[ \varepsilon_1 < \frac{\alpha_1 \alpha_2}{\alpha_1 \gamma_1 + \frac{1}{4(1 - d_1)} \beta_1 + d_1 \beta_2 \beta_2} \equiv \varepsilon_{1d}. \]

Inequality (38) shows that for any choice of \( d_1 \), the corresponding \( V_1(\hat{x}, \hat{y}) \) is a Lyapunov function for the singular perturbed \( \Sigma_{SF} \)-subsystem for all \( \varepsilon_1 \) satisfying (38). The maximum value of \( \varepsilon_{1d} \) is given by the expressions

\[ \varepsilon_1^* = \frac{\alpha_1 \alpha_2}{\alpha_1 \gamma_1 + \frac{1}{4(1 - d_1)} \beta_1 + d_1 \beta_2 \beta_2} d_1^* = \frac{\beta_1}{\beta_1 + \beta_2} \] (39)

If all the growth requirements on the \( \Sigma_{SF} \)-subsystem are satisfied, then the origin is an asymptotically stable equilibrium of the singularly perturbed system \( \Sigma_{SF} \) for all \( \varepsilon_1 \in (0, \varepsilon_1^*), \) where \( \varepsilon_1^* \) is given by Eq. (39). Bounds for the general \( \Sigma_{SF} \)-subsystem \( \Sigma \) are presented below.

1) Bounds of the Stability Parameters for the \( \Sigma_{SF} \) System:

The satisfaction of \( \varepsilon_1 < \varepsilon_1^* \) is achieved by defining the bounds of the stability parameters. The freedom on selecting \( \beta_2 \) and \( \gamma_1 \) permits to obtain the upper-bounds on \( \varepsilon_1^* \) and its \( d_1^* \) parameter, such that they match the required parameters that guarantee the asymptotic stability for the full \( \Sigma_{SF} \)-subsystem. This is achieved by selecting \( \gamma_1 \) and \( \beta_2 \) such that desired values of \( \varepsilon_1^* \) and \( d_1^* \) are obtained. The constant \( \gamma_1 \) is selected by solving Eq. (39)

\[ \varepsilon_1^* = \frac{\alpha_1 \alpha_2}{\alpha_1 \gamma_1 + \frac{1}{4(1 - d_1)} \beta_1 + d_1 \beta_2 \beta_2} \rightarrow \gamma_1(\varepsilon_1^*) = \frac{1}{\alpha_1} \left( \frac{\alpha_1 \alpha_2}{\alpha_1 \varepsilon_1^*} - \beta_1 \beta_2 \right) \]

and where \( \beta_2 \) is defined by solving Eq. (39) yielding

\[ d_1^* = \beta_1 \beta_2 \rightarrow \beta_1(d_1^*) = \frac{\beta_1}{d_1^*} \beta_1, \]

where recall that \( d_1^* \) and \( \varepsilon_1^* \) are the selected values such that satisfy the asymptotic stability properties of the full system, not to be confused with \( d_1^* \) and \( \varepsilon_1^* \) which represent the upper bounds. The power to select \( \varepsilon_1^* = \varepsilon_1^* \), permits to satisfy the fulfillment of the \( \Sigma_{SF} \) Stability Analysis by guaranteeing that \( \varepsilon_1 < \varepsilon_1^* \).

The percentage contribution of both Lyapunov functions \( V_S(\hat{x}) \) and \( V_F(\hat{x}, \hat{y}) \) on the Lyapunov function \( V_1(\hat{x}, \hat{y}) \), is defined by the proper selection of the parameter \( d_1^* \), and it is required to complete the second stability analysis, the \( \Sigma_{SFU} \) Stability Analysis, to completely define its value. In addition, for safety margin, the \( \varepsilon_1^* \) is selected as \( \varepsilon_1^* = \delta_2 \varepsilon_1, \) with \( \delta_2 > 1 \).

The power of these results lays in identifying that the Lyapunov functions \( V_S(\hat{x}) \) and \( V_F(\hat{x}, \hat{y}) \) are functions of the same stability parameters \( q_s, q_{f1}, \) and \( q_{f2}, \) as seen in Eqns. (15) and (18) respectively, therefore becoming degrees of freedom. It is also identified, that in order to to guarantee the stability properties of the \( \Sigma_{SFU} \) system, ratios between these stability parameters can to be defined. This is achieved by selecting ratios of dependance among the different stability parameters, such that all parameters can be written as a function of one of the stability parameters. Therefore by selecting \( q_{f1} = \bar{Q}_{SF} q_{s}, \) and \( q_{f2} = \bar{Q}_{F_s} \bar{Q}_{SF} q_{s}, \) where both \( \bar{Q}_{SF} = d_1 \bar{Q}_{SF} \) and \( \bar{Q}_{F_s} = d_1 \bar{Q}_{F_s}, \) represent the required ratios to prove the asymptotic stability analysis for the full \( \Sigma_{SFU} \) helicopter system, and with \( d_1 > 1 \) being a constant that increases the ratios for safety margin. This implies, that with the proper selection of \( \varepsilon_1^*, d_1^*, q_s, \) and the ratios \( \bar{Q}_{SF}, \) and \( \bar{Q}_{F_s}, \) the asymptotic stability of the \( \Sigma_{SF} \)-subsystem satisfies the requirement \( \varepsilon_1 < \varepsilon_1^* \). These results concludes the first step of the asymptotic stability analysis. The following section describes the second step of the asymptotic stability analysis, the \( \Sigma_{SFU} \) Stability Analysis for the generic helicopter problem.

D. \( \Sigma_{SFU} \) Stability Analysis

Similarly, the stability analysis of \( \Sigma_{SFU} \) system is performed using the previous stability analysis of the \( \Sigma_{SF} \)-subsystem. The \( \Sigma_{SFU} \) system is treated like a two-time-scale singularly perturbed system, where the \( \Sigma_{SF} \)-subsystem becomes the reduced order system, and the \( \Sigma_{U} \)-subsystem becomes the boundary layer. The asymptotically stability requirements of the reduced and boundary layer systems are expressed by requiring the existence of Lyapunov functions for both the \( \Sigma_{SF} \) and the \( \Sigma_{U} \)-subsystems, where the Lyapunov function derived in the \( \Sigma_{SF} \) Stability Analysis, \( V_1(\hat{x}, \hat{y}) \), becomes the associated Lyapunov function for the reduced order \( \Sigma_{SFU} \)-subsystem, while \( V_1(\hat{z}) \) becomes the associated Lyapunov function for the boundary layer \( \Sigma_{U} \)-subsystem.

Again, the satisfaction of the growth requirements take the form of inequalities defined in Assumptions A.1 through A.5, where inequalities (67) and (70) define the growth requirements of the \( \Sigma_{SF} \) and \( \Sigma_{U} \)-subsystems respectively, while (73) and (75) define the growth requirements of the combined \( \bar{F}(\hat{x}, \hat{z}) \) and \( \Sigma_{U} \)-subsystems.

With the analysis of the \( F(\hat{x}, \hat{z}) \) and \( \Sigma_{U} \)-subsystems it is clear that they have an isolated equilibrium at the origin, thus fulfilling Assumption A.1. The fulfillment of Assumption A.2 can be easily checked with the choice of the comparison function \( \psi_2(\hat{x}) \) proposed in Table I, where \( \mathcal{R} \) is a positive definite \( 3 \times 3 \) diagonal matrix defined by

\[ \mathcal{R} = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{bmatrix}, \]

where it can easily be shown that the diagonal entries \( R_i, i = 1, 2, 3 \) are expressed as functions of the stability parameters \( q_s, q_{f1}, q_{f2}, \) the control parameters \( b_1, b_2, b_3, \) the parasitic constants \( d_1, \varepsilon_1, \) and the physical parameters of the helicopter. It can also be shown that in order to guarantee that \( \mathcal{R} \) is positive definite, mathematical expression for both \( \bar{Q}_{SF} \) and
We can derive $\dot{Q}_{F_2}$, thus allowing to select the proper values of $q_s$, $q_f_1$, $q_f_2$, $b_1$, $b_2$, $b_3$ and $d_1$ that ensure the fulfillment of Assumption A.2. Refer to [7] for more details on how the constants are derived.

Assumption A.3 is fulfilled with the choice of $\phi_2(\tilde{z})$ described in Table I, with $\dot{Q}_{U} = \dot{Q}_{U}/2$, defined in Eq. (24), and satisfying that $\tilde{z} = \tilde{z} - H(\tilde{X}) = 0$ is a stable equilibrium of the boundary layer system. The first interconnection condition, Assumption A.4, is satisfied with the relationships described in Table I, that is selecting $\beta_1 = \delta_3\beta_4^*$. Inequality (73) determines the allowed growth of $F(\tilde{X}, \tilde{z})$ in $\tilde{z}$. Assumption A.5 is satisfied with the relationships described in Table I, that is selecting $\beta_1 = \delta_3\beta_4^*$, and $\gamma_2 = \delta_2\gamma_4^*$, with $\delta_\beta > 1$, $\delta_\beta > 1$, and $\delta_\gamma > 1$.

The fulfillment of Assumptions A.1 through A.5 determines that the origin is an asymptotically stable equilibrium of the singular perturbed $\Sigma_{SFU}$ full system for sufficiently small $\varepsilon_2$. We therefore consider a new Lyapunov function candidate for the $\Sigma_{SFU}$ system defined by a weighted sum of $V_1(\tilde{X})$ and $V_2(\tilde{X})$ such

$$V_2(\tilde{X}, \tilde{z}) = (1 - d_2)\tilde{V}_1(\tilde{X}) + d_2V_2(\tilde{X}), \quad d_2 \in (0,1),$$

where $0 < d_2 < 1$. To explore the freedom that we have in choosing the weights, we take $d_2$ as an unspecified parameter in (0, 1). From the properties of $V_1(\tilde{X})$ and $\tilde{V}_2(\tilde{X})$ and $H(\tilde{X})$, $\parallel H(\tilde{X}) \parallel \leq \rho_2(\parallel \tilde{X} \parallel)$, where $p_2(\cdot)$ is a $\kappa$ function, it follows that $V_2(\tilde{X}, \tilde{z})$ is positive-definite. Computing the time derivative of $V_2$ along the trajectories of $F(\tilde{X}, \tilde{z})$ and $\Sigma_U$ in a similar manner as the one conducted in Eqs. (32) and (33) we obtain that

$$\varepsilon_2 < \frac{\alpha_3\alpha_4}{\varepsilon_1 (\alpha_3\gamma_2 + \frac{1}{4(1-d_2)d_2}(1-d_2)\delta_5 + d_2\delta_4^2)} \equiv \varepsilon_{2\delta},$$

Inequality (44) shows that for any choice of $d_2$, the corresponding $V_2(\tilde{X}, \tilde{z})$ is a Lyapunov function for the singular perturbed system $\Sigma_{SFU}$ for all $\varepsilon_2$ satisfying Eq. (44). The maximum value of $\varepsilon_2^*$ is given by the expressions

$$\varepsilon_2^* = \frac{\alpha_3\alpha_4}{\varepsilon_1 (\alpha_3\gamma_2 + \beta_3\gamma_4^2)} = \frac{\beta_4}{\beta_3 + \beta_4}.$$  

Similarly as in the $\Sigma_{SF}$ Stability Analysis, if all the growth requirements are satisfied, then the origin is an asymptotically stable equilibrium of the singularly perturbed system $\Sigma_{SFU}$ for all $\varepsilon_2 \in (0, \varepsilon^*_2)$. The following section defines the mathematical expressions for the bounds on the singularly perturbed parameters.

1) Bounds of the Stability Parameters for the $\Sigma_{SFU}$ System: Mathematical expressions for the upper bounds of the $\Sigma_{SFU}$ system can be obtained by recalling the results from the $\Sigma_{SF}$ Stability Analysis, and recalling that $\beta_3^*$, $\beta_4^*$ and $\gamma_2^*$ can be defined as

$$\beta_3^* = F_1\frac{\sqrt{q_s}}{q_1}, \quad F_1 = F_1(Q_{SF}, Q_{F_2}, b_1, b_2, d_1, a_*)\sqrt{\frac{q_s}{q_1}};$$

$$\beta_4^* = F_2\frac{\sqrt{q_s}}{q_2}, \quad F_2 = F_2(Q_{SF}, Q_{U_2}, b_1, b_2, d_1, a_*)\sqrt{\frac{q_s}{q_2}};$$

$$\gamma_2^* = F_3(\tilde{Q}_{U_2}, b_1, b_2, a_*),$$

where $F_1$, $F_2$, and $F_3$ are elaborated functions of the stability parameters, the control parameters, and the physical constants of the helicopter ($a_*$), and where $\tilde{Q}_{U_2}$ defines the ratio between the stability parameters $q_s$ and $q_{U_2}$ as $q_{U_2} = Q_{U_2}q_s$, thus permitting to rewrite Eqs. (45) as

$$\varepsilon_2^* = \frac{\alpha_3\alpha_4}{\varepsilon_1 (\alpha_3\gamma_2 + \beta_3\beta_4^*)} = \frac{\alpha_1\alpha_2}{\alpha_1\alpha_2(3\beta_3 + 2\beta_4^*)} = \frac{\alpha_1\alpha_2}{\alpha_1\alpha_2(3\beta_3 + 2\beta_4^*)},$$

which can be solved for $F_{U_2}$, by solving Eq. (51) resulting in $F_{U_2} = F_{U_2}(Q_{SF}, Q_{F_2}, b_1, b_2, d_1, a_*, \varepsilon_1^*, \varepsilon^*_2, \varepsilon^*_3)$. The final stability parameter ratio, $Q_{US}$ can be obtained by analyzing Eq. (50), and recognizing that, by selecting the value of $\varepsilon_2^* = \varepsilon_{2\delta}$, it can be obtained an expression that determines the missing relation between $q_s$ and $q_{U_2}$. The selection of $d_2^*$ defines the percentage contribution of both Lyapunov functions $V_1(\tilde{X})$ and $V_2(\tilde{X})$ on the Lyapunov function $V_2(\tilde{X}, \tilde{z})$. Rewriting Eq. (50) results in

$$d_2^* = \frac{\alpha_3\alpha_4}{\varepsilon_1 (\alpha_3\gamma_2 + \beta_3\gamma_4^2)} = \frac{\beta_4}{\beta_3 + \beta_4}.$$  

Similarly as in the $\Sigma_{SF}$ Stability Analysis, if all the growth requirements are satisfied, then the origin is an asymptotically stable equilibrium of the singularly perturbed system $\Sigma_{SFU}$ for all $\varepsilon_2 \in (0, \varepsilon^*_2)$. The following section defines the mathematical expressions for the bounds on the singularly perturbed parameters.

Thus this powerful result implies the existence of a closed form solution for the proper selection of the stability parameters as a function of the arbitrary $q_s$, which permits to bound the value of $\varepsilon_2^*$ in the $\Sigma_{SF}$-subsystem and $\varepsilon_2^*$ in the $\Sigma_{SFU}$ full system such that $\varepsilon_1 < \varepsilon_2^*$, and $\varepsilon_2 < \varepsilon_2^*$. This demonstrates that the origin $\tilde{X} = 0, \tilde{z} = 0$, is an asymptotically stable equilibrium.
The proposed stability analysis permits to satisfy the stability properties of the singularly perturbed $\Sigma_{SFU}$ system by using the stability parameters, as long as they satisfy the bounds and the ratios that have been obtained through the fulfillment of the proposed asymptotic stability methodology. Furthermore, it is important to note that at the same time, bounds on the control parameters, $b_1$, $b_2$, and $b_3$, are also defined such that guarantee the three-time-scale decomposition. These bounds allow to introduce a degree of freedom on the time responses, for both the slow and fast dynamics, permitting to increase or decrease the time-scale separation between both dynamics, but always satisfying the existence of a three-time-scale separation. This concludes the second step of the asymptotic stability analysis for the generic helicopter problem. Numerical stability results for the X-Cell 50 helicopter model being studied in this paper are presented in section V.

**TABLE I**

<table>
<thead>
<tr>
<th>Assumption A.2</th>
<th>$\frac{\partial}{\partial \varepsilon} f(x, h(x))$</th>
<th>$\alpha_1$</th>
<th>$\psi(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial}{\partial \varepsilon} f(\tilde{x}, G, H)$</td>
<td>$\alpha_1 \leq 1$</td>
<td>$\psi_2(\tilde{x}) = \sqrt{\tilde{q}_2^2}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{\partial}{\partial \varepsilon} F(\tilde{x}, H(\tilde{x}))$</td>
<td>$\alpha_3 \leq 1$</td>
<td>$\psi_3(\tilde{x}) = (\tilde{q}_3^T \mathbf{Q}_3 \tilde{x})^\frac{1}{2}$</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE II**

<table>
<thead>
<tr>
<th>Constants</th>
<th>$a_1 = 5.198 \times 10^{-4}$</th>
<th>$a_2 = 1.520 \times 10^{-4}$</th>
<th>$a_3 = 2.702 \times 10^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_2 = 1.25 \times 10^{-4}$</td>
<td>$a_4 = 1.580 \times 10^{-8}$</td>
<td>$a_5 = a_4 = a_14 = a_15 = 0.1$</td>
<td></td>
</tr>
<tr>
<td>$a_0 = 0.1$</td>
<td>$a_7 = -17.67$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_9 = 520$</td>
<td>$a_9 = 0.0028$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_{12} = 334.88$</td>
<td>$a_{11} = -13.92$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_{13} = -800$</td>
<td>$a_{15} = -60$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE III**

<table>
<thead>
<tr>
<th>Constraints</th>
<th>$X_{MIN}$</th>
<th>$X_{MAX}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>States</td>
<td>$\chi_{MIN}$</td>
<td>$\chi_{MAX}$</td>
</tr>
<tr>
<td>$x$ (rad/s)</td>
<td>74.25</td>
<td>180</td>
</tr>
<tr>
<td>$y$ (rad/s)</td>
<td>180</td>
<td>81.67</td>
</tr>
<tr>
<td>$z$ (rad/s)</td>
<td>162</td>
<td>0</td>
</tr>
</tbody>
</table>

**V. NUMERICAL RESULTS FOR THE HELICOPTER PROBLEM**

This section provides numerical results for the stability analysis on the closed-loop dynamics for the X-Cell 50 helicopter model here studied, where the values of the normalized physical coefficients, $a_\ast$, are defined in Table II. The physical restrictions of the proposed model require defining the range of reachable states, and their associated desired final conditions, and are bounded by the physical limits of the state variables and are defined in Table III. See [7] for further details.

This section provides bounds on both $\varepsilon_1$ and $\varepsilon_2$. Recall that $\varepsilon_1$ and $\varepsilon_2$ represent the parasitic constants that define the time-scale behavior among the different subsystems, where the stretched time-scale are given by the time constant $\tau_1 = \varepsilon_1 / \varepsilon_2$, and $\tau_2 = \varepsilon_1 / \varepsilon_2$. Computer simulations supports any time-scale selection, but hardware limitations should drive that time-scale selection by ensuring that dynamics of the different subsystems behave according to their dynamics limits (i.e. vertical velocity of the helicopter, angular velocity of the blades, and collective pitch rate of the servo actuators). Defining bounds on the parasitic parameters, helps to define how small is small enough, and ensures that the available discrete control signals will have sufficient authority to control the fastest systems.

**Recalling the physical coefficients $a_\ast$, defined in Table II,** the parasitic constants for the open loop for the X-Cell 50 helicopter are selected as $\varepsilon_1 = a_9 / a_5 = 2.8 \times 10^{-2}$, and $\varepsilon_2 = a_5 / a_13 = 1.25 \times 10^{-4}$, satisfying that $0 < \varepsilon_2 \ll \varepsilon_1 < 1$. For the stability analysis here presented the control parameters are selected as $b_1 = \varepsilon_1 \omega_n^2 = 0.112$, $b_2 = 2 \varepsilon_1 \omega_n = 0.1008$, $b_3 = 0.5$.

The $\Sigma_{SFU}$ Stability Analysis can be satisfy by requiring $d_1 \in (0.0543, 0.5243)$, therefore, by selecting $d_1^* = d_1^* = 0.5$, the percentage contribution on the Lyapunov function $V_1(\tilde{x}, \tilde{y})$, Eq. (31), is equally distributed for both Lyapunov functions $V_2(\tilde{z})$ and $V_2(\tilde{y})$, while by selecting $\delta_1 = 1.05$, and $\delta_2 = 1.05$, we have $\varepsilon_2^* = 0.02940$, and $\varepsilon_3^* = \varepsilon_3^* = 1.26250 \times 10^{-4}$, thus satisfying $\varepsilon_1 < \varepsilon_2^*$, and $\varepsilon_2 < \varepsilon_2^*$. 
For the selected control parameters the stability parameters that satisfy the stability analysis of the $\Sigma_{SFU}$ system are given by $Q_{SF} = 0.259974$, $Q_{EF} = 2.567205$, $Q_{U1} = 0.0041970$, and $Q_{US} = 0.330899$, therefore by selecting $q_s = 0.5$, and recalling Eqs. (55–57), result in $q_{f1} = 0.129987$, $f_{q1} = 0.333703$, $q_{U1} = 0.165449$, and $q_{U2} = 5.455576$. Therefore, the coefficients that fulfill the growth requirements of the $\Sigma_{SFU}$-subsystem are given by $\alpha_1 = 0.95$, $\alpha_2 = 0.95$, $\beta_1 = 0.76260$, $\beta_2 = 0.76260$, and $\gamma_1 = 31.7007576$, and the coefficients that fulfill the growth requirements for the $\Sigma_{SFU}$ system are therefore given by $\alpha_3 = 0.95$, $\alpha_4 = 0.95$, $\beta_3 = 492.14288$, $\beta_4 = 492.14288$, and $\gamma_2 = 11128.12913$.

Figure 6(a) shows the dependence on the right-hand side of Eq. (40) on the unspecified parameter $d_1$, being able to identify that the maximum value is achieved at the selected $d_1^\star = 0.5$, and given by $\varepsilon_1^\star = 0.02940$, and the dependence on the right-hand side of Eq. (44) on the unspecified parameter $d_2$ is sketched in Figure 6(b) being able to identify that the maximum value is achieved at the selected $d_1^\star = 0.5$, and given by $\varepsilon_2^\star = 1.20250 \times 10^{-4}$.

VI. CONCLUSIONS

In this paper a stability analysis has been performed taking advantage of the system three-time scales. A methodology for the stability analysis of such type of systems has been presented extending the well-known two-time-scale case. The methodology here presented is not limited to the problem here discussed, but can be suited to more general systems as described in [7].

The proposed time-scale analysis methodology represents an all-in-one tool that permits to construct a valid Lyapunov function based in time-scale separation, and perform the time-scale asymptotic stability analysis of the singularly perturbed system, becoming a really powerful tool for the analysis of the helicopter model in vertical flight here employed, that can also be applied to more general nonlinear singularly perturbed systems.

![Figure 6](image)

(a) $\varepsilon_1$  
(b) $\varepsilon_2$

Fig. 6. Stability upper bounds on $\varepsilon_1$ and $\varepsilon_2$

ACKNOWLEDGMENT

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REFERENCES


APPENDIX

For completeness of the paper, this section describes the general formulation for the asymptotic stability analysis for the two time-scale singular perturbation problems [3]. It serves as basis for the analysis that was conducted to demonstrate the asymptotic stability of the three-time-scale model that derived in the subsection IV-B.

Consider a nonlinear autonomous singular perturbed system of the form:

$$
\dot{x} = f(x, z), \quad x \in \mathbb{R}^n, \tag{59}
$$

$$
\varepsilon \dot{z} = g(x, z), \quad z \in \mathbb{R}^m, \tag{60}
$$

which has an isolated equilibrium at the origin $(x = 0, z = 0)$. It is assumed throughout the formulation that $f$ and $g$ are smooth to ensure that for specified initial conditions, system (59–60) has a unique solution. The stability of the equilibrium is investigated by examining the reduced (slow) system

$$
\dot{\tilde{x}} = f(x, h(x)), \tag{61}
$$
where \( z = h(x) \) is an associated root of \( 0 = g(x, z) \), and the boundary-layer (fast) system
\[
\frac{dx}{d\tau} = g(x, z(\tau)), \quad \tau = \frac{t}{\varepsilon}
\]  
where \( x \) is treated as a fixed parameter, and \( \varepsilon \) is the parasitic constant that defines the stretched time-scale of the fast subsystem. Asymptotic stability requirements on the reduced and boundary-layer systems are expressed by requiring the existence of Lyapunov functions for each system that satisfy certain growth condition. The growth requirements of \( f \) and \( g \) take the form of inequalities satisfied by the Lyapunov functions, which will be addressed later. We assume that the following conditions hold for all
\[
(x, z, \varepsilon) \in [0, \infty) \times B_x \times B_x \times [0, \varepsilon_1],
\]  
with \( B_x \subset \mathbb{R}^n \) and \( B_z \subset \mathbb{R}^m \), and where \( B_z \subset \mathbb{R}^n \) and \( B_z \subset \mathbb{R}^m \) denote closed sets. We add and subtract \( f(x, h(x)) \) to the right-hand side of (59) yielding
\[
\hat{x} = f(x, h(x)) + f(x, z) - f(x, h(x)),
\]  
where \( f(x, z) - f(x, h(x)) \) can be viewed as a perturbation of the reduced system (61). It is natural first to look for a Lyapunov function candidate for (61) and then to consider the effect of the perturbation term \( f(x, z) - f(x, h(x)) \) [3].

**Assumption A.1:** Asymptotic Stability of the Origin

The origin \((x = 0, z = 0)\) is a unique and isolated equilibrium of (59-60), i.e.,
\[
0 = f(0, 0), \quad 0 = g(0, 0),
\]  
where \( x = h(x) \) is the unique root of \( 0 = g(x, z, 0) \) in \( B_x \times B_z \), i.e., \( 0 = g(x, h(x)) \) and there exists a class \( \kappa \) function \( p(\cdot) \) such that \( \| h(x) \| \leq p(\| x \|) \).

The asymptotic stability of the origin is studied using Lyapunov methods. To construct a Lyapunov function candidate for the singular perturbed system (59-60) we consider each of the two equations separately. The growth requirements of \( f \) and \( g \) take the form of inequalities that must be satisfied by the Lyapunov functions. The growth requirements of the reduced and boundary layer system by separate will be defined in Assumptions A.2 and A.3, while the growth requirements that combine both reduced and boundary layer system requirements, called interconnection conditions, will be defined in Assumptions A.4 and A.5.

**Assumption A.2:** Reduced System Conditions

There exists a positive-definite an decreasing Lyapunov function candidate \( V(x) \) that is,
\[
0 < q_1(\| x \|) \leq V(x) \leq q_2(\| x \|),
\]  
for some class \( \kappa \) function \( q_1(\cdot) \) and \( q_2(\cdot) \) that satisfies the following inequality,
\[
\frac{\partial V}{\partial x} f(x, h(x)) \leq -\alpha_1\psi^2(x),
\]  
where \( \psi(\cdot) \) is a scalar function of vector arguments which vanishes only when its argument are zero, and satisfying that \( x = 0 \) is a stable equilibrium of the reduced order system. Condition (67) guarantees that \( x = 0 \) is an asymptotically stable equilibrium of (61).

**Assumption A.3:** Boundary-Layer System Conditions

There exists a positive-definite an decreasing Lyapunov function candidate \( W(x, z) \) such that for all \( (x, z) \in B_x \times B_z \), satisfying
\[
0 < q_3(\| z - h(x) \|) \leq W(x, z) \leq q_4(\| z - h(x) \|),
\]  
for some class \( \kappa \) function \( q_3(\cdot) \) and \( q_4(\cdot) \), that satisfies,
\[
W(x, z) > 0, \forall z \neq h(x) \text{ and } W(x, h(x)) = 0,
\]  
and
\[
\frac{\partial W}{\partial z} g(x, z) \leq -\alpha_2\psi^2(z - h(x)), \quad \alpha_2 > 0,
\]  
where \( W(x, z) \) is a Lyapunov function of the boundary layer system (62) in which \( x \) is treated as a fixed parameter, and \( \psi(\cdot) \) is a scalar function of vector arguments which vanishes only when its argument are zero, and satisfying that \( z - h(x) \) is a stable equilibrium of the boundary layer system.

**Assumption A.4:** First Interconnection Condition

\( V(x) \) and \( W(x, z) \) must satisfy the so called interconnection conditions. The first interconnection condition is obtained by computing the derivative of \( V \) along the solution of (64),
\[
\dot{V} = \frac{\partial V}{\partial x} f(x, h(x)) + \frac{\partial V}{\partial x} [f(x, z) - f(x, h(x))]
\]  
\[
\leq -\alpha_1\psi^2(x) + \frac{\partial V}{\partial x} [f(x, z) - f(x, h(x))],
\]  
assuming that
\[
\frac{\partial V}{\partial x} [f(x, z) - f(x, h(x))] \leq \beta_1\psi(z - h(x)).
\]  
so that
\[
\dot{V} \leq -\alpha_1\psi^2(x) + \beta_1\psi(z - h(x)).
\]  
Inequality (73) determines the allowed growth of \( f \) in \( z \).

**Assumption A.5:** Second Interconnection Conditions

The second interconnection condition is defined by
\[
\frac{\partial W}{\partial z} f(x, z) \leq \gamma\psi^2(z - h(x)) + \beta_2\psi(z - h(x)),
\]  
where \( \psi(\cdot) \) and \( \phi(\cdot) \) are scalar functions of vector arguments which vanish only when their arguments are zero, i.e., \( \phi(x) = 0 \) if and only if \( x = 0 \). They will be referred to as comparison functions.

With the Lyapunov function \( V(x) \) and \( W(x, z) \) obtained, a new Lyapunov function \( \nu(x, z) \) is considered and defined by the weighted sum of \( V(x) \) and \( W(x, z) \),
\[
\nu(x, z) = (1 - d) V(x) + d W(x, z),
\]  
for \( 0 < d < 1 \). \( \nu(x, z) \) becomes the Lyapunov function candidate for the singular perturbed system (59-60).

**Theorem A.1:** If \( x = 0 \) is an asymptotically stable equilibrium of the reduced system (61), \( z = h(x) \) is an asymptotically stable equilibrium of the boundary-layer system (62) uniformly in \( x \), that is, the \( \varepsilon - \delta \) definition of Lyapunov stability and the convergence \( z \to h(x) \) are uniform in \( x \) [17], and \( f(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) satisfy growth conditions on the reduced and boundary-layer systems, then the origin is an asymptotically stable equilibrium of the singularly perturbed system (59), for sufficiently small \( \varepsilon [3] \).