Exploiting Symmetry in Higher-Dimensional PDE Control: A Backstepping Perspective

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Control and Adaptation: Imagine What's Next - Celebrating the 60th Birthday of Miroslav Krstic

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Backstepping for Partial Differential Equations

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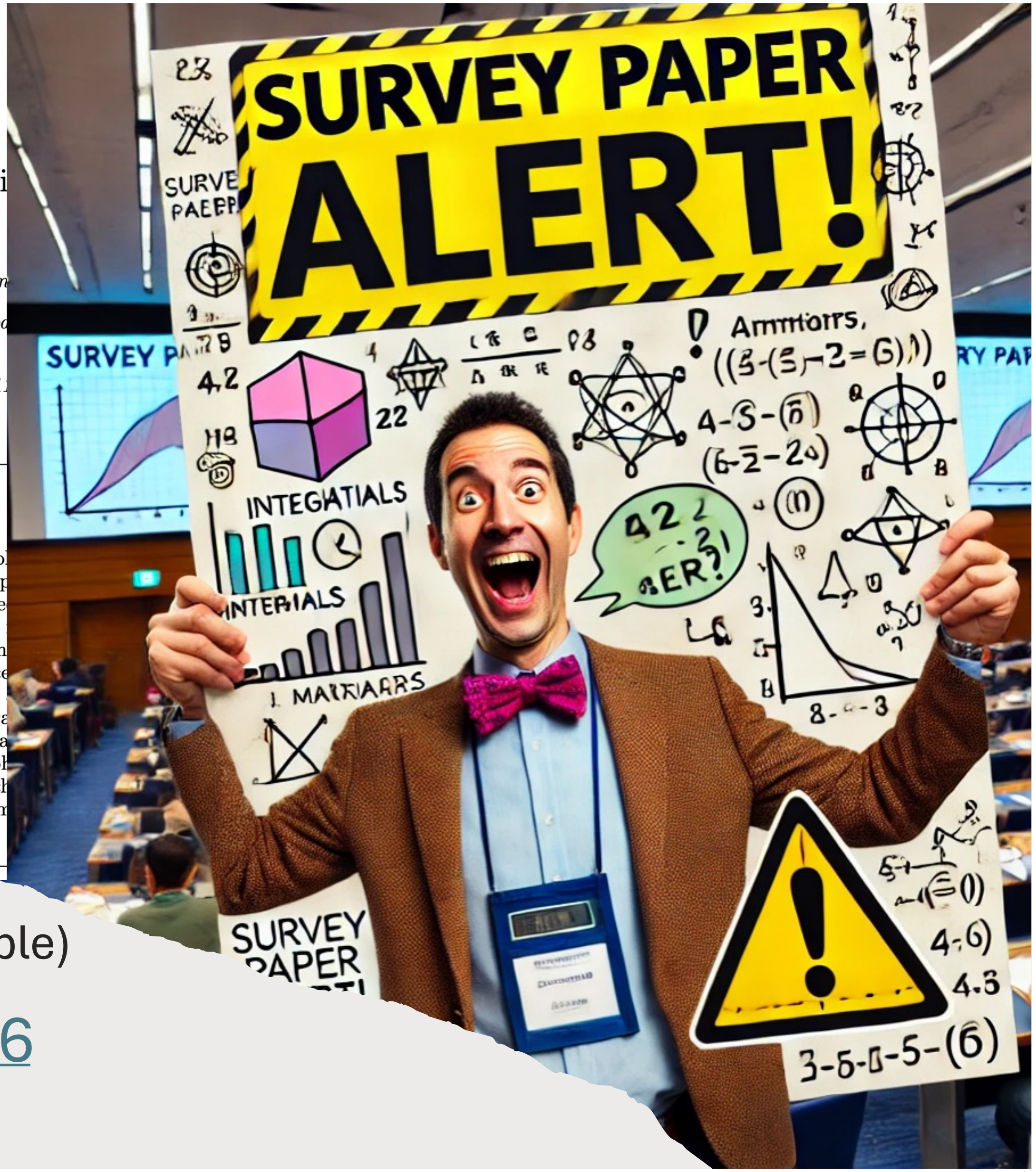
tract

ems modeled by partial differential equations (PDEs) are at least as ubiquitous as systems that are by nature
nsional and modeled by ordinary differential equations (ODEs). And yet, systematic and readily usable methodo
uch a significant portion of real systems, have been historically scarce. Around the year 2000, the backstepping app
DE control began to offer not only a less abstract alternative to PDE control techniques replicating optimal and spe
nment techniques of the 1960s, but also enabled the methodologies of adaptive and nonlinear control, matured
s and 1990s, to be extended from ODEs to PDEs, allowing feedback synthesis for physical and engineering system
ncertain, nonlinear, and infinite-dimensional. The PDE backstepping literature has grown in its nearly a quarter c
velopment to many hundreds of papers and nearly a dozen books. This survey aims to facilitate the entry, for
rcher, into this thriving area of overwhelming size and topical diversity. Designs of controllers and observers, for par
rbolic, and other classes of PDEs, in one and more dimensions (in box and spherical geometries), with nonlinear, ada
led-data, and event-triggered extensions, are covered in the survey. The lifeblood of control are technology and p
survey places a particular emphasis on applications that have motivated the development of the theory and which
fited from the theory and designs: applications involving flows, flexible structures, materials, thermal and chem
nergy (from oil drilling to batteries and magnetic confinement fusions), and vehicles.



New Survey Paper (under review, preprint available)

https://arxiv.org/pdf/2410.15146

~~Exploiting Symmetry in Higher-Dimensional PDE Control: A Backstepping Perspective~~

Now, the true title!

~~Exploiting Symmetry in Higher-Dimensional PDE Control: A Backstepping Perspective~~

Now, the true title!

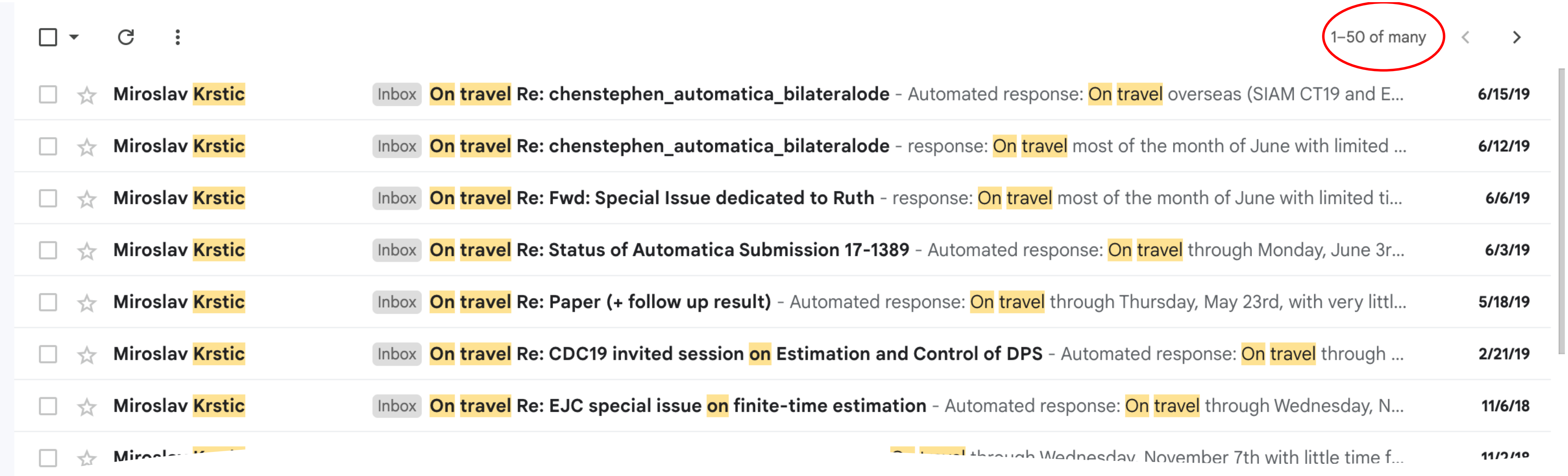
We still don't know how to control the Steinway at the keyboard but know how to control a pizza slice at the crust

But… this requires an explanation…!

Miroslav’s email workflow

1. You spend two hours crafting careful email for Miroslav

(well now… perhaps *a bit less*… thanks to ChatGPT!).



Miroslav's email workflow

1. You spend two hours crafting careful email for Miroslav
2. Immediate answer. **Email server auto-response: ON TRAVEL**

Miroslav’s email workflow

1. You spend two hours crafting careful email for Miroslav

2. Email server auto-response: on travel.

3. 20 minutes later (your mileage may vary), **Miroslav replies with insightful email solving your issues and/or giving great advice, support, etc**

Example. Advice after receiving a **particularly bad review** :

- We have to make reviewers feel respected and feel they have not wasted time and made bad suggestions - even when most of what they have suggested is bad. **Paper text is “compressible” matter**, especially when one puts in the time, which one must, when it comes to the fragile egos of reviewers.

Miroslav Krstic <krstic@... Tue, Feb 12, 2019, 2:54 PM

to Rafael

Thanks, so you are saying that we still don’t know how to control the Steinway at the keyboard but know how to control a pizza slice at the crust?

Miroslav

The origin of the title today...!!

But perhaps let’s blend the old and new title into something a bit more serious and already start!

On controlling pizzas and grand pianos: Exploiting Symmetry in Higher-Dimensional PDE Control

On controlling pizzas and grand pianos: Exploiting Symmetry in Higher-Dimensional PDE Control

1. Quick Overview of 2D and Higher Backstepping
2. Spatial Invariance in PDE Control
3. Square Domains and Fourier Analysis: Breaking and mending Andrey's result
4. The Pizza Control Problem
5. The Grand Piano Control Problem
   - Control at the Keyboard
   - Simultaneous control at the Back: Domain Extension Approach
6. Conclusions

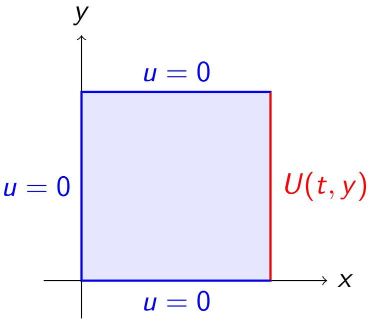
Extension to Higher Dimensions: Key Challenge

- Original backstepping focused on 1D reaction-diffusion equations
- Challenge: extending to higher dimensions
  - Integral transformations become more complex and many possibilities emerge
  - Associated PDEs governing kernels are harder or impossible to solve
  - Need for special properties and symmetries
- Two main approaches have emerged:
  - Exploit specific geometries and simmetries
  - Use spatial invariance for specific configurations

Simplest Result: Andrey's 2D Design in a Square

Consider the 2D reaction-diffusion equation on a unit square:

$$u_t = \epsilon(u_{xx} + u_{yy}) + \lambda u, \quad (x, y) \in [0,1] \times [0,1]$$
$$u(t,0,y) = 0, \quad u(t,1,y) = U(t,y) \quad \text{(control)}$$
$$u(t,x,0) = u(t,x,1) = 0 \quad \text{(Dirichlet boundary conditions)}$$



Solution: Extension of 1D Backstepping

Target System: Extend the same typical stable 1D system to 2D

$$
\begin{aligned}
w_t &= \epsilon(w_{xx} + w_{yy}) - cw \\
w(t,0,y) &= w(t,1,y) = 0 \\
w(t,x,0) &= w(t,x,1) = 0
\end{aligned}
$$

where $c > 0$ ensures stability.

Key Insight: Use the same 1D transformation for each y-slice

$$w(t,x,y) = u(t,x,y) - \int_0^x k(x,\xi)u(t,\xi,y)d\xi$$

Control law: Evaluate transformation at $x = 1$. Control law gets y-dependence from u only.

$$U(t,y) = \int_0^1 k(1,\xi)u(t,\xi,y)d\xi$$

Why Does This Work?

Key Property: y-derivatives commute with the transformation

$$\begin{aligned} w_{yy} &= \partial_{yy}\left(u - \int_0^x k(x,\xi)u(t,\xi,y)d\xi\right) \\ &= u_{yy} - \int_0^x k(x,\xi)u_{yy}(t,\xi,y)d\xi \\ w_t &= \dots u_{yy} \dots - \int_0^x k(x,\xi)u_{yy}(t,\xi,y)d\xi \dots \end{aligned}$$

This commutation property means the design work exactly as in 1D!

- The kernel PDE remains exactly the same as in 1D
- Thus kernel $k(x,\xi)$ identical to 1D case: Modified Bessel function
- However the design is fragile to any y-dependence.

First "true" Higher-Dimensional Results

- Loops and channels: Early successes
  - Thermal convection loops (Vazquez & Krstic, 2006)
  - 2D Navier-Stokes Poiseuille flow (Vazquez & Krstic, 2007)
  - Channel MHD flow (Vazquez & Krstic, 2008)
- Key features:
  - Periodic or infinite domains
  - Exploit special geometric properties
  - Fourier analysis key tool
- Different approach and systematic treatment by Meurer (2012) (another talk).

Progress in Bounded Domains

- General n-dimensional ball, constant coefficients (Vazquez & Krstic, 2016)
- Radially-varying with total angular simmetry, disk domain (Vazquez & Krstic, 2016)
- Extension to sphere (Vazquez & Krstic, 2019)
- Latest: Power series for radially varying reaction (Vazquez et al., 2023)

Key mathematical tools:

- Ultraspherical coordinates
- Ultraspherical harmonics
- Power series methods

On controlling pizzas and grand pianos: Exploiting Symmetry in Higher-Dimensional PDE Control

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Spatial Invariance Approach

- Concept by Bamieh, Paganini & Dahleh (2002)
- Key insight: many distributed parameter systems have symmetries
- For control design:
  - Transform spatial derivatives into algebraic multiplication
  - Convert spatial dependence into parameter dependence
- Result: Original PDE becomes an *ensemble* of simpler systems
- Each member of ensemble easier to control independently

When Can We Use It?

System must be spatially invariant:

- Spatial coordinates form a group
  - Infinite domain: $\mathbb{R}$ (channels)
  - Periodic domain: $\mathcal{S}$ (loops, disks, spheres)
- Actuation/sensing fully distributed over that coordinate
- Dynamics and geometry invariant under translations

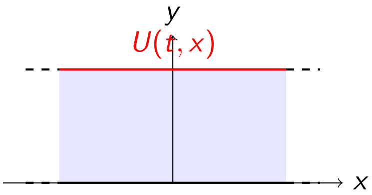
Example: Heat Equation in Semi-Infinite Strip

Consider:

$$u_t = \epsilon(u_{xx} + u_{yy}) + \lambda u$$
$$u_y(t, x, 0) = 0$$
$$u_y(t, x, 1) = U(t, x)$$

on $(x, y) \in (-\infty, \infty) \times [0, 1]$

Key property: Invariant in x (infinite domain)



Power of Fourier Transform

After Fourier transform in x:

$$
\begin{aligned}
u_t &= \epsilon(-4\pi^2 k^2 u + u_{yy}) + \lambda u \\
u_y(t,k,0) &= 0 \\
u_y(t,k,1) &= U(t,k)
\end{aligned}
$$

- Original 2D PDE $\rightarrow$ Family/ensemble of 1D PDEs
- Each parameterized by wavenumber k: continua of PDEs
- x-derivatives become algebraic terms!
- Key property: Higher wavenumbers are more damped ($-4\pi^2 k^2$ term)
- Controllers designed for each k independently up to a certain wave number N
- Design in Fourier space, reconstruct in physical space (inverse Fourier transform)

Control Design in Fourier Space

- Divide wavenumbers in two regions:
  - Controlled: $|k| < N$ (finite number of modes)
  - Uncontrolled: $|k| \geq N$ (naturally damped by $-4\pi^2 k^2$ term)
- Target system for controlled modes:

$$w_t = \epsilon(-4\pi^2 k^2 w + w_{yy}) - cw$$
$$w_y(t, k, 0) = 0, \quad w_y(t, k, 1) = 0$$

Backstepping Solution

Use transformation for each k:

$$w(t,y,k) = u(t,y,k) - \int_0^y \kappa(y,\eta)u(t,\eta,k)d\eta$$

with kernel ($\lambda_0 = \frac{\lambda+c}{\epsilon}$)

$$\kappa(y,\eta) = -\lambda_0\eta\frac{I_1(\sqrt{\lambda_0(y^2-\eta^2)})}{\sqrt{\lambda_0(y^2-\eta^2)}}$$

where I_1 is the first order modified Bessel function of the first kind.
Controller for each mode:

$$U(k) = -\int_0^1 \lambda_0\eta\frac{I_1(\sqrt{\lambda_0(1-\eta^2)})}{\sqrt{\lambda_0(1-\eta^2)}}u(\eta,k)d\eta$$

Reconstruction in Physical Space

For each controlled mode $|k| < N$, apply controller:

$$U(k) = -\int_0^1 \lambda_0 \eta \frac{I_1(\sqrt{\lambda_0(1-\eta^2)})}{\sqrt{\lambda_0(1-\eta^2)}} u(\eta, k) d\eta$$

Final physical control obtained by inverse Fourier transform:

$$\begin{aligned} U(t,x) &= \int_{-\infty}^{\infty} U(k) e^{2\pi ikx} dk = \int_{-N}^{N} U(k) e^{2\pi ikx} dk \\ &= -\int_0^1 \int_{-N}^{N} \lambda_0 \eta \frac{I_1(\sqrt{\lambda_0(1-\eta^2)})}{\sqrt{\lambda_0(1-\eta^2)}} u(\eta, k) e^{2\pi ikx} dk d\eta \end{aligned}$$

Properties:

- Low wavenumbers: actively controlled
- High wavenumbers: naturally damped by diffusion
- Results in spatially-distributed control law
- Exponential stability in L^2 norm

Reconstruction in Physical Space

Substituting Fourier transform of $u(t, \eta, x)$:

$$U(t,x) = -\int_{-N}^{N}\int_{0}^{1} \lambda_0\eta \frac{I_1(\sqrt{\lambda_0(1-\eta^2)})}{\sqrt{\lambda_0(1-\eta^2)}} \left(\int_{-\infty}^{\infty} u(t,\eta,\xi)e^{-2\pi ik\xi}d\xi\right) e^{2\pi ikx} d\eta dk$$
$$= -\int_{0}^{1}\int_{-\infty}^{\infty} \lambda_0\eta \frac{I_1(\sqrt{\lambda_0(1-\eta^2)})}{\sqrt{\lambda_0(1-\eta^2)}} u(t,\eta,\xi) \left(\int_{-N}^{N} e^{2\pi ik(x-\xi)}dk\right) d\xi d\eta$$

After exchanging integrals and computing $\int_{-N}^{N} e^{2\pi ik(x-\xi)}dk$:

$$U(t,x) = -\int_{0}^{1}\int_{-\infty}^{\infty} \lambda_0\eta \frac{I_1(\sqrt{\lambda_0(1-\eta^2)})}{\sqrt{\lambda_0(1-\eta^2)}} [2N\text{sinc}(2\pi N(x-\xi))]\, u(t,\eta,\xi)d\xi d\eta$$

where:

- $\text{sinc}(z) = \frac{\sin(z)}{z}$ is the cardinal sine function
- $2N\text{sinc}(2\pi N(x-\xi))$ is the spatial filtering kernel in x direction
- The kernel splits into two parts: Bessel function part from the backstepping design and Sinc function part from the wavenumber cutoff.

On controlling pizzas and grand pianos: Exploiting Symmetry in Higher-Dimensional PDE Control

Back to the Square Domain: Fourier Sine Series

Consider again the unstable 2D reaction-diffusion:

$$
\begin{aligned}
u_t &= \epsilon(u_{xx} + u_{yy}) + \lambda u, \quad (x, y) \in [0,1] \times [0,1] \\
u(t,0,y) &= 0, \quad u(t,1,y) = U(t,y) \\
u(t,x,0) &= u(t,x,1) = 0
\end{aligned}
$$

Key insight: Use Fourier sine series in y direction

- Dirichlet conditions suggest sine series
- $u(t,x,y) = \sum_{n=1}^{\infty} u_n(t,x) \sin(n\pi y)$
- Each mode satisfies separate 1D equation

Equations for Each Mode

After projection, for each n:

$$\begin{aligned} u_{n,t} &= \epsilon(u_{n,xx} - n^2\pi^2 u_n) + \lambda u_n \\ u_n(t,0) &= 0 \\ u_n(t,1) &= U_n(t) \end{aligned}$$

Properties:

- Each mode is 1D reaction-diffusion
- Natural damping $-\epsilon n^2\pi^2$ increases with mode number
- Same structure as Fourier transform case, but discrete modes
- Can apply same backstepping design for each mode

Backstepping Design for All Modes

For each mode n, transform to target system:

$$\begin{aligned} w_{n,t} &= \epsilon(w_{n,xx} - n^2\pi^2 w_n) - cw_n \\ w_n(t,0) = w_n(t,1) &= 0 \end{aligned}$$

Using backstepping transformation:

$$w_n(t,x) = u_n(t,x) - \int_0^x k_n(x,\xi)u_n(t,\xi)d\xi$$

Key point: Even though n appears, kernel is the same!

$$k_n(x,\xi) = k(x,\xi) = -\lambda_0\xi \frac{I_1(\sqrt{\lambda_0(x^2-\xi^2)})}{\sqrt{\lambda_0(x^2-\xi^2)}}$$

Control Law

Control for each mode:

$$U_n(t) = \int_0^1 k(1,\xi) u_n(t,\xi) d\xi$$

Physical space control:

$$U(t,y) = \sum_{n=1}^{\infty} \left(\int_0^1 k(1,\xi) u_n(t,\xi) d\xi \right) \sin(n\pi y)$$

Properties:

- Same kernel works for all modes!
- Series converges
- No need for mode truncation
- Naturally well-posed in L^2

Final Control Law: Physical Space

Substituting Fourier coefficient definition:

$$u_n(t,\xi) = 2\int_0^1 u(t,\xi,\eta)\sin(n\pi\eta)d\eta$$

Physical space control:

$$U(t,y) = \sum_{n=1}^{\infty}\left(\int_0^1 k(1,\xi)\left(2\int_0^1 u(t,\xi,\eta)\sin(n\pi\eta)d\eta\right)d\xi\right)\sin(n\pi y)$$

Properties:

- Double integral control law
- Series converges
- Well-defined transformation from 2D state to boundary control
- No truncation needed!

Final Control Law: Explicit Form

Let's exchange order of integration and sum:

$$
\begin{aligned}
U(t,y) &= \int_0^1 k(1,\xi)\left(2\int_0^1 u(t,\xi,\eta)\left(\sum_{n=1}^{\infty}\sin(n\pi\eta)\sin(n\pi y)\right)d\eta\right)d\xi \\
&= \int_0^1 k(1,\xi)\left(2\int_0^1 u(t,\xi,\eta)\left(\frac{\delta(y-\eta)}{2}\right)d\eta\right)d\xi
\end{aligned}
$$

(δ function to be understood in the distributional sense rather than a strict pointwise equality). Therefore, final physical control law:

$$
U(t,y) = -\int_0^1 \lambda_0\xi\,\frac{I_1(\sqrt{\lambda_0(1-\xi^2)})}{\sqrt{\lambda_0(1-\xi^2)}}\,u(t,\xi,y)d\xi
$$

Key insight:

- Control is local in y (no integration in y)!
- Only integration in ξ direction needed
- Same kernel as 1D case for each y-slice. We recover Andrey's result. What's the point???

A More Realistic Problem: Finite-Dimensional Control

Instead of distributed control $U(t, y)$, consider finite-dimensional:

$$u_t = \epsilon(u_{xx} + u_{yy}) + \lambda u$$

$$u(t, 0, y) = 0, \quad u(t, 1, y) = \sum_{k=1}^{m} U_k(t)\phi_k(y)$$

$$u(t, x, 0) = u(t, x, 1) = 0$$

where:

- $\{\phi_k(y)\}_{k=1}^{m}$ are shape functions on $[0, 1]$
- Only m control inputs $U_k(t)$
- Cannot directly use previous kernel solution

Strategy: Use Fourier series to understand conditions of controllability

Mode Analysis with Finite Controls

Expand both state and shape functions:

$$u(t, x, y) = \sum_{n=1}^{\infty} u_n(t, x) \sin(n\pi y)$$

$$\phi_k(y) = \sum_{n=1}^{\infty} \phi_{k,n} \sin(n\pi y)$$

For each mode n:

$$u_n(t, 1) = \sum_{k=1}^{m} U_k(t)\phi_{k,n} = g_n(t)$$

Mode-by-Mode Design

Each mode n satisfies:

$$u_{n,t} = \epsilon(u_{n,xx} - n^2\pi^2 u_n) + \lambda u_n$$
$$u_n(t,0) = 0$$
$$u_n(t,1) = \sum_{k=1}^{m} U_k(t)\phi_{k,n} = g_n(t)$$

Key features:

- For $n \leq N$: Design control via backstepping

$$g_n(t) = \int_0^1 k(1,\xi)u_n(t,\xi)d\xi = \sum_{k=1}^{m} U_k(t)\phi_{k,n}$$

- For $n > N$: Natural damping dominates by choosing

$$N \geq \sqrt{\frac{c+\lambda}{\epsilon}}\pi$$

Control Design for First N Modes

For $n \leq N$, kernel solution gives desired boundary values:

$$g_n(t) = \int_0^1 k(1,\xi) u_n(t,\xi) d\xi$$

where $k(1,\xi) = -\lambda_0 \xi \frac{I_1(\sqrt{\lambda_0(1-\xi^2)})}{\sqrt{\lambda_0(1-\xi^2)}}$
Linear system for control:

$$\underbrace{\begin{pmatrix} \phi_{1,1} & \cdots & \phi_{m,1} \\ \vdots & \ddots & \vdots \\ \phi_{1,N} & \cdots & \phi_{m,N} \end{pmatrix}}_{\Phi} \begin{pmatrix} U_1(t) \\ \vdots \\ U_m(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ \vdots \\ g_N(t) \end{pmatrix}$$

rank of matrix Φ of shape function coefficients becomes crucial!

Stability Theorem

Theorem: For the system above, and given c, set N such that $-\epsilon n^2\pi^2 + \lambda < -c$ for $n > N$ (this is $N \geq \sqrt{\frac{c+\lambda}{\epsilon}}\pi$). Then **if matrix Φ of shape function coefficients has rank** N the control law $U_k(t) = (\Phi^\dagger g(t))_k$ achieves for some $K > 0$

$$\|u(t,\cdot,\cdot)\|_{L^2([0,1]^2)} \leq K e^{-ct} \|u(0,\cdot,\cdot)\|_{L^2([0,1]^2)}$$

The proof is easy to complete noticing that the higher modes ($n > N$) exhibit ISS stability w.r.t. to the boundary control (which only involves lower modes and decays to zero).
Note: arbitrary convergence rate no longer achievable. The achievable c will depend on the shape coefficient matrix rank properties. It may also happen that there is no possible value of c: system not stabilizable. Obviously need $m \geq N$ (at least as many controls as modes that need to be controlled.).

Conclusion of the Finite-Dimensional Control Strategy

- Starting from a PDE that was previously straightforward to solve with a full functional control $U(t, y)$, we introduced a finite-dimensional control parameterization.
- Without the Fourier series paradigm, it's unclear how to invert the relationship between $u(t, x, y)$ and $U(t, y)$.
- By resorting to Fourier expansions, we break down the control problem mode-by-mode.
- A finite-dimensional vector of controls $U_k(t)$ is now related to a finite set of critical modes of the PDE, governed by a linear algebraic condition on the chosen support functions.
- This approach reveals the necessity of the Fourier method for tractability and ensures that with appropriate choice of $\{\phi_k\}$, one can systematically achieve control over the system's most influential modes.

On controlling pizzas and grand pianos: Exploiting Symmetry in Higher-Dimensional PDE Control

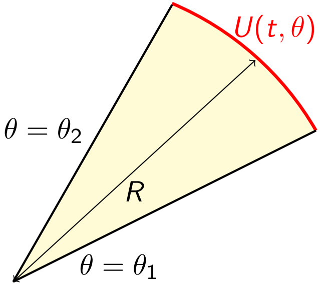
Next Challenge: Control a Pizza! ??

Consider reaction-diffusion on a sector domain:

$$u_t = \epsilon(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) + \lambda u$$

$$u(t, r, \theta_1) = u(t, r, \theta_2) = 0 \quad \text{(no toppings at the edges)}$$

$$u(t, R, \theta) = U(t, \theta) \quad \text{(control at the crust)}$$



Dirichlet on the Sides of the Sector

Since we impose Dirichlet boundary conditions on the sides:

$$u(r, \theta, t) = 0 \quad \text{at } \theta = \theta_1,\ \theta = \theta_2.$$

we now have sine-like eigenfunctions to satisfy the BCs at $\theta = \theta_1, \theta_2$.
Angular Eigenvalue Problem:

$$\frac{d^2\Phi}{d\theta^2} + \lambda^2\Phi = 0, \quad \Phi(\theta_1) = 0,\ \Phi(\theta_2) = 0.$$

The solutions are of the form:

$$\Phi_n(\theta) = \sin\left(\frac{n\pi(\theta - \theta_1)}{\theta_2 - \theta_1}\right), \quad n = 1, 2, \ldots$$

These form an orthonormal basis in $L^2([\theta_1, \theta_2])$ with Dirichlet boundary conditions.

Mode Decomposition in θ

Using eigenfunctions verifying the Dirichlet conditions:

$$u(t,r,\theta) = \sum_{n=1}^{\infty} u_n(t,r) \sin\left(\frac{n\pi(\theta-\theta_1)}{\theta_2-\theta_1}\right)$$

Each mode satisfies:

$$\begin{aligned} u_{n,t} &= \epsilon\left(\frac{1}{r}(r u_{n,r})_r - \frac{n^2\pi^2}{(\theta_2-\theta_1)^2}\frac{u_n}{r^2}\right) + \lambda u_n \\ u_n(t,R) &= U_n(t) \end{aligned}$$

Key observation:

- Angular mode number n appears in denominator r^2
- Higher modes ($n \gg 1$) naturally more damped
- Control design needed only for first N modes

Mode-by-Mode Backstepping Design

Target system for each mode n:

$$w_{n,t} = \epsilon \left(\frac{1}{r}(r w_{n,r})_r - \frac{n^2\pi^2}{(\theta_2 - \theta_1)^2} \frac{w_n}{r^2} \right) - c w_n$$
$$w_n(t, R) = 0$$

Backstepping transformation:

$$w_n(t, r) = u_n(t, r) - \int_0^r k_n(r, \rho) u_n(t, \rho) d\rho$$

Control law for each mode:

$$U_n(t) = \int_0^R k_n(R, \rho) u_n(t, \rho) \rho d\rho$$

Physical Control Reconstruction

Physical control at the crust:

$$U(t,\theta) = \sum_{n=1}^{N} \left(\int_0^R k_n(R,\rho) u_n(t,\rho) d\rho \right) \sin\left(\frac{n\pi(\theta - \theta_1)}{\theta_2 - \theta_1} \right)$$

Properties:

- Only first N modes actively controlled
- Higher modes $(n > N)$ naturally stable when:

$$-\epsilon \frac{n^2\pi^2}{(\theta_2 - \theta_1)^2 R^2} + \lambda < -c$$

Backstepping Design

Kernel k_n must satisfy:

$$\frac{\partial^2 k_n}{\partial r^2} + \frac{1}{r}\ \frac{\partial k_n}{\partial r} - \frac{\partial^2 k_n}{\partial \rho^2} + -\frac{1}{\rho}\frac{\partial k_n}{\partial \rho} - \frac{k_n}{\rho^2} - \alpha_n^2 \left(\frac{1}{r^2} - \frac{1}{\rho^2} \right) k_n(r,\rho) = \frac{\lambda + c}{\epsilon} k_n(r,\rho)$$

$$k_n(r,r) = -\frac{\lambda + c}{2\epsilon r}$$

where $\alpha_n = \frac{n\pi}{\theta_2 - \theta_1}$.
We try the change of variables:[1]

$$k_n(r,\rho) = g_n(r,\rho)\rho \left(\frac{\rho}{r}\right)^{\alpha_n}$$

[1] Inspired by R. Vazquez and M. Krstic, "Boundary Control of Reaction-Diffusion PDEs on Balls in Spaces of Arbitrary Dimensions," ESAIM:Control, Optimization and Calculus of Variations, Vol. 22, No. 4, pp. 1078-1096, 2016

Backstepping Design

We get the equation

$$\begin{aligned}
\partial_{rr} g_n + (1-2\alpha_n)\frac{\partial_r g_n}{r} - \partial_{\rho\rho} g_n - (1+2\alpha_n)\frac{\partial_\rho g_n}{\rho} &= \frac{\lambda+c}{\epsilon} g_n \\
g_n(r,r) &= -\frac{\lambda+c}{2\epsilon}.
\end{aligned}$$

Whose solution is (based on the same ESAIM paper):

$$g_n(r,\rho) = -\frac{\lambda+c}{\epsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda+c}{\epsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda+c}{\epsilon}(r^2-\rho^2)}},$$

with no α_n dependency! Now undoing the change:

$$k_n(r,\rho) = -\frac{\lambda+c}{\epsilon}\rho\left(\frac{\rho}{r}\right)^{\frac{n\pi}{\theta_2-\theta_1}}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda+c}{\epsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda+c}{\epsilon}(r^2-\rho^2)}}$$

Explicit Control Law

Kernel solution:

$$K(r,\rho,\theta,\eta) = 2\sum_{n=1}^{N}\rho\left(\frac{\lambda+c}{\epsilon}\left(\frac{\rho}{r}\right)^{\frac{n\pi}{\theta_2-\theta_1}}\frac{I_1\left[\sqrt{\frac{\lambda+c}{\epsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda+c}{\epsilon}(r^2-\rho^2)}}\sin\left(\frac{n\pi(\eta-\theta_1)}{\theta_2-\theta_1}\right)\sin\left(\frac{n\pi(\theta-\theta_1)}{\theta_2-\theta_1}\right)\right)$$

Full control at the crust:

$$U(t,\theta) = -\int_0^R\int_{\theta_1}^{\theta_2}K(R,\rho,\theta,\eta)u(t,\rho,\eta)\rho d\eta d\rho$$

Properties:

- If choosing first N modes needed with

$$N > \sqrt{\frac{c+\lambda}{\epsilon}}\frac{(\theta_2-\theta_1)R}{\pi}$$

  one gets c decay rate as usual
- Modified Bessel function kernel scaled with angle difference

So, how do you control a pizza at the crust?

Controlling a pizza-shaped domain involves:

- Angular separation of variables and mode expansions.
- Identifying kernel PDEs that are more complex than in simpler geometries.
- Using advanced analytical or numerical methods to solve these kernel PDEs.
- Achieving exponential stability of the PDE state through a carefully constructed boundary control at the crust.

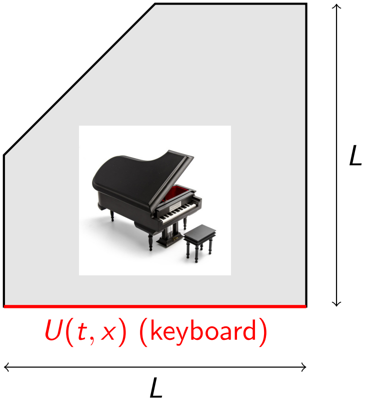
The methodology ensures that, by addressing a finite set of troublesome modes and relying on natural damping of higher modes, one can achieve exponential stabilization. Though the kernel equations are harder, the underlying principle of backstepping remains intact.

We have shown we can control a pizza at its crust!

On controlling pizzas and grand pianos: Exploiting Symmetry in Higher-Dimensional PDE Control

Grand Finale: The Grand Piano! ??

From pizza to piano:



Challenge: Can we control vibrations/temperature from the keyboard?

- Previous domain: square with full control at one edge
- Now: Diagonal cut creates piano shape
- Question: can we cope with this change?

The Mathematical Setup

Consider reaction-diffusion on this domain Ω:

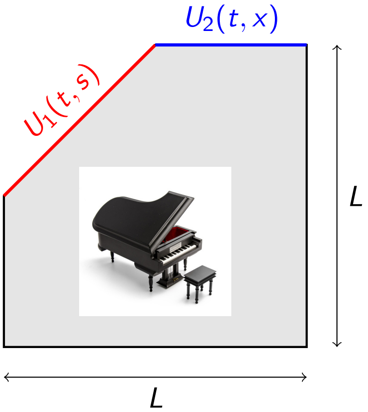
$$u_t = \epsilon(u_{xx} + u_{yy}) + \lambda u \quad \text{in } \Omega$$
$$u = 0 \quad \text{on all edges except keyboard}$$
$$u = U(t, x) \quad \text{on keyboard } (y = 0, 0 < x < L)$$

Key Differences from Square:

- No separation of variables
- Previous technique cannot be used
- Other techniques may work, but we don't know how to pose backstepping transformation

Alternative Strategy: Control from the Back

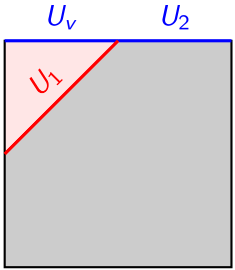
Same piano shape, but control at the back walls:



New Challenge:

- Two control inputs: U_1 on diagonal, U_2 on top
- Different parameterization needed for each boundary

Control Strategy: Extending the Domain



Key Idea:

- Extend piano to full square
- Red triangle: virtual domain
- U_v: virtual control we know (Andrey's) completing U_2.
- U_1: matches solution across diagonal

The Extension Strategy

The trick:

1. Start with Andrey's solution for full square (we know this!)
2. Simulate dynamics in cut-off triangle (red)
   - Known virtual control U_v at top computed partly from real state, partly from simulated triangle.
   - Neumann conditions "measured" from the real state on the triangle.
3. Use value of solution on diagonal to set U_1: this is dynamic feedback.
4. By uniqueness: piano behavior matches square behavior!

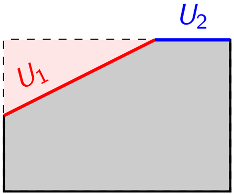
Advantage: Transform hard problem (piano) into known problem (square) + simulation

Key insight: If solution exists, it must be unique. Therefore, if we can make the piano match the square's behavior on the diagonal, we've found our solution!

From Piano to General Domain Extension Method

Key concept: Target Domain (not just Target System)

- Extend non-symmetric domain to simpler one
- Virtual dynamics in extended region
- Match solutions at interface
- Known controller in extended domain



Requirements:

- Enough controls to match at interface
- Known control solution in target domain
- Well-posed virtual problem in extended region

Classical idea: Domain extension methods well-known in mathematics, here adapted for control design

On controlling pizzas and grand pianos: Exploiting Symmetry in Higher-Dimensional PDE Control

1. Quick Overview of 2D and Higher Backstepping
2. Spatial Invariance in PDE Control
3. Square Domains and Fourier Analysis: Breaking and mending Andrey's result
4. The Pizza Control Problem
5. The Grand Piano Control Problem
   - Control at the Keyboard
   - Simultaneous control at the Back: Domain Extension Approach
6. Conclusions

Conclusions

Key Messages:

- Fourier methods and symmetries are powerful tools for PDE control
  - Transform n-D geometries into tractable ensemble 1-D problems
  - Enable mode-by-mode design
  - Bridge infinite and finite-dimensional control at the boundaries
- New explicit solutions obtained for:
  - Semi-infinite strip with cut-off
  - Finite-dimensional control in squares
  - Full pizza control at the crust (exploiting angular symmetry)
- Piano control from the back via domain extension
- New concept: Target domains as natural extension of target systems

Message for Miroslav: These new solutions, from squares to pizzas to pianos, are dedicated to you - I know you love them, specially the explicit solutions!!

Thank You!!

Questions?

Special thanks to Miroslav Krstic
for inspiring this journey from 1D backstepping to pizzas and pianos