Backstepping Control of Mixed Hyperbolic-Parabolic PDEs

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Outline

- Mixed hyperbolic-parabolic systems
- Backstepping method
- Solution 1
- Solution 2
- Conclusions

Mixed hyperbolic-parabolic systems

System under consideration

$$
\begin{aligned}
v_t(x,t) &= \varepsilon v_{xx}(x,t) + \lambda v(x,t) \\
v(1,t) &= 0 \\
v(0,t) &= u(0,t) \\
u_t(x,t) &= u_x(x,t) + \mu(x) v(x,t) + g(x) v(0,t) + \int_0^x f(x,y) v(y,t)\, dy \\
u(1,t) &= U(t)
\end{aligned}
$$

For $(x,t) \in [0,1] \times [0,\infty)$. Coefficients verify $\varepsilon > 0$, g, f continuous and differentiable in their respective domains.

Potentially unstable if λ large $\to$ design $U(t)$ to stabilize the equilibrium at the origin.

Mixed hyperbolic-parabolic systems: the challenge

Consider only

$$
\begin{aligned}
v_t(x,t) &= \varepsilon v_{xx}(x,t) + \lambda v(x,t) \\
v(1,t) &= 0 \\
v(0,t) &= u(0,t) \\
u_t(x,t) &= u_x(x,t) + \mu(x) v(x,t) \\
u(1,t) &= U(t)
\end{aligned}
$$

Explicit solution of $u(x,t)$ for $t > 1$ is

$$u = U(t-1+x) + \int_x^1 \mu(s) v(s, t+x-s) ds$$

Thus v subsystem becomes a heat equation with delayed control and a rather complex non-local integral delayed term.

$$
\begin{aligned}
v_t(x,t) &= \varepsilon v_{xx}(x,t) + \lambda v(x,t) \\
v(0,t) &= U(t-1) + \int_0^1 \mu(s) v(s, t-s) ds \\
v(1,t) &= 0
\end{aligned}
$$

Mixed hyperbolic-parabolic systems: previous/related work

- Karafyllis, Iasson, and Miroslav Krstic. “Small-gain stability analysis of certain hyperbolic-parabolic PDE loops,” *Systems & Control Letters*, vol. 118, pp. 52-61, 2018.

- Fu, Qin, et al. “Iterative learning control for a class of mixed hyperbolic-parabolic distributed parameter systems.” *International Journal of Control, Automation and Systems*, vol. 14, no. 6, pp. 1455-1463, 2016.

- M. Krstic, ”Control of an unstable reaction-diffusion PDE with long input delay,” *Systems & Control Letters*, vol. 58, no. 10, pp. 773-782, 2009.

- S. Chen, R. Vazquez and M. Krstic, ”Backstepping control design for a coupled hyperbolic-parabolic mixed class PDE system,” *56th Annual Conference on Decision and Control*, 2017.

- M. Ghousein and E. Witrant, ”Backstepping control for a class of coupled hyperbolic-parabolic PDE systems,” *American Control Conference*, 2020.

- J. Deutscher, Joachim and J. Gabriel. ”Fredholm Backstepping Control of Coupled Linear Parabolic PDEs With Input and Output Delays.” *IEEE Transactions on Automatic Control* , vol. 65, no. 7, pp. 3128-3135, 2019

- And some others (not many).

Mixed hyperbolic-parabolic systems: previous/related work

M. Krstic, ”Control of an unstable reaction-diffusion PDE with long input delay,” *Systems & Control Letters*, vol. 58, no. 10, pp. 773-782, 2009.

$$\begin{aligned} v_t(x,t) &= \varepsilon v_{xx}(x,t) + \lambda v(x,t) \\ v(1,t) &= 0 \\ v(0,t) &= u(0,t) \\ u_t(x,t) &= u_x(x,t) \\ u(1,t) &= U(t) \end{aligned}$$

First paper solving the problem with backstepping. We followed the structure of the backstepping transformation posed in this paper (to be explained later). Interestingly, the kernels equations also become of mixed type.

Mixed hyperbolic-parabolic systems: previous/related work

S. Chen, R. Vazquez and M. Krstic, ”Backstepping control design for a coupled hyperbolic-parabolic mixed class PDE system,” *56th Annual Conference on Decision and Control*, 2017.

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\begin{aligned}
v_t(x,t) &= \varepsilon v_{xx}(x,t) + \lambda v(x,t) \\
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v(0,t) &= u(0,t) \\
u_t(x,t) &= u_x(x,t) + \int_0^x f(x,y)\, v(y,t)\, dy \\
u(1,t) &= U(t)
\end{aligned}
$$

Extension of the previous paper. Kernel equations become more involved/coupled, but the problem is always solvable.

Mixed hyperbolic-parabolic systems: previous/related work

M. Ghousein and E. Witrant, ”Backstepping control for a class of coupled hyperbolic-parabolic PDE systems,” *American Control Conference*, 2020.

$$
\begin{aligned}
v_t(x,t) &= \varepsilon v_{xx}(x,t) + \lambda(x) v(x,t) + \sigma(x) u(x,t) \\
v_x(0,t) &= u(0,t) \\
v(1,t) &= U_1 \\
u_t(x,t) &= u_x(x,t) + \int_0^x f(x,y)\, v(y,t)\, dy \\
u(1,t) &= U_2(t)
\end{aligned}
$$

A bit of a different problem, the coupling is more bidirectional at the price of having more actuations. However, the result hinges on the existence of certain control parameters (a number and a 2-D kernel) verifying certain conditions, which is not guaranteed.

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Backstepping for PDEs

In ODEs, a particular approach to stabilization of dynamic systems with “triangular” structure.

Wildly successful in the area of ODE nonlinear control.

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Wildly successful in the area of ODE nonlinear control.

For PDEs, roughly speaking, backstepping is a *constructive* method that achieves Lyapunov stabilization by **transforming** the system into a stable "target system," which is often achieved by collectively shifting all the eigenvalues in a favorable direction in the complex plane, rather than by assigning individual eigenvalues.

Backstepping allows this task can be achieved in a rather elegant way where the control gains are easy to compute, symbolically, numerically, or even explicitly.

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Backstepping allows this task can be achieved in a rather elegant way where the control gains are easy to compute, symbolically, numerically, and in some cases even explicitly.

sometimes... !

Backstepping for PDEs

1. Identify the undesirable terms in the PDE.

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3. Find the state transformation typically as *identity minus a Volterra operator* (in x).
   Volterra operator = integral operator from 0 up to x or 1 down to x (rather than from 0 to 1).
   A Volterra transformation is “triangular” or “spatially causal.”

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4. Obtain boundary feedback from the backstepping transformation. The transformation alone cannot eliminate the undesirable terms, but the transformation brings them to the boundary, so control can cancel them.

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Gain fcn of boundary controller = kernel of backstepping transformation.

Backstepping kernel satisfies a *linear* PDE.

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Backstepping kernel satisfies a *linear* PDE.

Backstepping is not “one-size-fits-all.” Requires structure-specific effort by designer.

Reward: elegant controller, clear (more or less) closed-loop behavior.

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Target system & transformation number 1

Remember the original system:

$$
\begin{aligned}
v_t(x,t) &= \varepsilon v_{xx}(x,t) + \lambda v(x,t) \\
v(1,t) &= 0 \\
v(0,t) &= u(0,t) \\
u_t(x,t) &= u_x(x,t) + \mu(x) v(x,t) + g(x) v(0,t) + \int_0^x f(x,y) v(y,t)\, dy \\
u(1,t) &= U(t)
\end{aligned}
$$

Target system & transformation number 1

Proposed target system:

$$\begin{aligned}
\eta_t(x,t) &= \varepsilon\eta_{xx}(x,t) - c\eta(x,t) \\
\eta(1,t) &= 0 \\
\eta(0,t) &= \omega(0,t) \\
\omega_t(x,t) &= \omega_x(x,t) + \mu(x)\eta(x,t) \\
\omega(1,t) &= 0
\end{aligned}$$

Transformation:

$$\begin{aligned}
\eta(x,t) &= v(x,t) - \int_x^1 p(x,y)v(y,t)dy \\
\omega(x,t) &= u(x,t) - \int_0^x k(x,y)u(y,t)dy - \int_0^1 l(x,y)v(y,t)dy
\end{aligned}$$

Control:

$$U(t) = \int_0^1 k(1,y)u(y,t)dy + \int_0^1 l(1,y)v(y,t)dy$$

Kernel equations number 1

For $\mathcal{T} = \{(x,y) : 0 \le y \le x \le 1\}$:

$$\begin{aligned} k_x(x,y) &= -k_y(x,y) \\ k(x,0) &= \varepsilon l_y(x,0) - g(x) + \int_0^x k(x,y) g(y)\, dy \end{aligned}$$

For $0 \le x, y \le 1$:

$$\begin{aligned} l_x(x,y) &= \varepsilon l_{yy}(x,y) + \lambda l(x,y) - h(x-y)\left[k(x,y)\mu(y) + f(x,y) - \int_y^x k(x,s) f(s,y) ds\right] \\ l(x,0) &= 0, \quad l(x,1) = 0, \quad l(0,y) = p(0,y) \end{aligned}$$

For $\mathcal{T}' = \{(x,y) : 0 \le x \le y \le 1\}$:

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= \frac{\lambda + c}{\varepsilon} p(x,y) \\ p(x,1) &= 0 \\ p(x,x) &= \frac{\lambda + c}{2\varepsilon}(x-1) \end{aligned}$$

Target system & transformation number 1: main results

Proposition. There exists a solution $k(x,y)$, $l(x,y)$, $p(x,y)$ to the kernel equations such that $k \in L^2[\mathcal{T}]$, $p \in L^2[\mathcal{T}']$and $l \in L^2[0,1;H^1(0,1)]$.

Proof: sketch in next slide.

Theorem. Consider the original system with initial conditions $v_0, u_0 \in H^1(0,1)$ verifying zero-order compatibility conditions, and c in the target system sufficiently large. Then, $u, v \in L^2[0,\infty;H^1(0,1)]$ and they verify the following energy estimate

$$\|v(\cdot,t)\|_{H^1}^2 + \|u(\cdot,t)\|_{H^1}^2 \leq C_1 \exp(-C_2 t)(\|v_0\|_{H^1}^2 + \|u_0\|_{H^1}^2),$$

with $C_1, C_2 > 0$.

Proof: by Lyapunov method, quite obvious from target system.

Sketch of proof of kernel well-posedness

Remember the kernel equations:

$$\begin{aligned}
k_x(x,y) &= -k_y(x,y) \\
k(x,0) &= \varepsilon l_y(x,0) - g(x) + \int_0^x k(x,y) g(y)\, dy \\
l_x(x,y) &= \varepsilon l_{yy}(x,y) + \lambda l(x,y) - h(x-y) \left[k(x,y)\mu(y) + f(x,y) - \int_y^x k(x,s) f(s,y) ds \right] \\
l(x,0) &= 0, \quad l(x,1) = 0, \quad l(0,y) = p(0,y) \\
p_{xx}(x,y) &= p_{yy}(x,y) + \frac{\lambda + c}{\varepsilon} p(x,y) \\
p(x,1) &= 0 \\
p(x,x) &= \frac{\lambda + c}{2\varepsilon}(x-1)
\end{aligned}$$

p is trivially solvable! (typical backstepping kernel)

Idea: for $k(x,y), l(x,y)$, we express their solutions as

$$l(x,y) = \sum_0^\infty l_m(x,y) \qquad k(x,y) = \sum_0^\infty k_m(x,y)$$

Sketch of proof of kernel well-posedness

We find

$$
\begin{aligned}
l_{0,x}(x,y) &= \varepsilon l_{0,yy}(x,y) + \lambda l_0(x,y) - h(x-y) f(x,y) \\
l_0(x,0) &= 0, l_0(x,1) = 0, l_0(0,y) = p(0,y) \\
k_{0,x}(x,y) &= -k_{0,y}(x,y), \\
k_0(x,0) &= -g(x)
\end{aligned}
$$

and for $m = 1, 2, \cdots, \infty$:

$$
\begin{aligned}
l_{m,x}(x,y) &= \varepsilon l_{m,yy}(x,y) + \lambda l_m(x,y) - h(x-y) \left[\mu(y) k_{m-1}(x,y)\right] \\
&\quad + h(x-y) \int_y^x k_{m-1}(x,s) f(s,y) ds \\
l_m(x,0) &= 0, l_m(x,1) = 0, l_m(0,y) = 0 \\
k_{m,x}(x,y) &= -k_{m,y}(x,y) \\
k_m(x,0) &= \varepsilon l_{m-1,y}(x,0) + \int_0^x k_{m-1}(x,y) g(y) dy
\end{aligned}
$$

Sketch of proof of kernel well-posedness

Explicit solution for k kernels:

$$k_0(x,y) = -g(x-y), \quad k_m(x,y) = \varepsilon l_{m-1,y}(x-y,0) + \int_0^{x-y} k_{m-1}(x-y,s) g(s)\, ds$$

Replacing this, l equations become a sequence of (well-posed) reaction diffusion equations (with x acting as "time" and y as "space").

Using an inductive Lyapunov functional, we show that the sum of l kernels is convergent in H^1. Thus proving the result.

A small problem at the boundary that was overlooked

There is a disagreement in the BCs: $l(x,0) = 0$ and $l(0,y) = p(0,y)$ since $p(0,0) \neq 0$, which would invalidate the result. Solution: modify target system BC as

$$\eta(0,t) = \omega(0,t) + \Delta$$

where

$$\Delta = \int_0^\tau D(y)\eta(y,t)dy + \int_0^1 \left[\int_0^\tau Q(s,y)D(s)ds\right]\eta(y,t)dy - \int_0^\tau \left[\int_y^\tau Q(s,y)D(s)ds\right]\eta(y,t)dy$$

with Q the inverse kernel to p (a explicit expression is known) and

$$D(y) = \phi(y) - p(0,y), \phi(y) = \begin{cases} \frac{p(0,\tau)}{\tau} y, 0 \leq y \leq \tau \\ p(0,y), y > \tau \end{cases}$$

and τ is an arbitrarily small positive value.

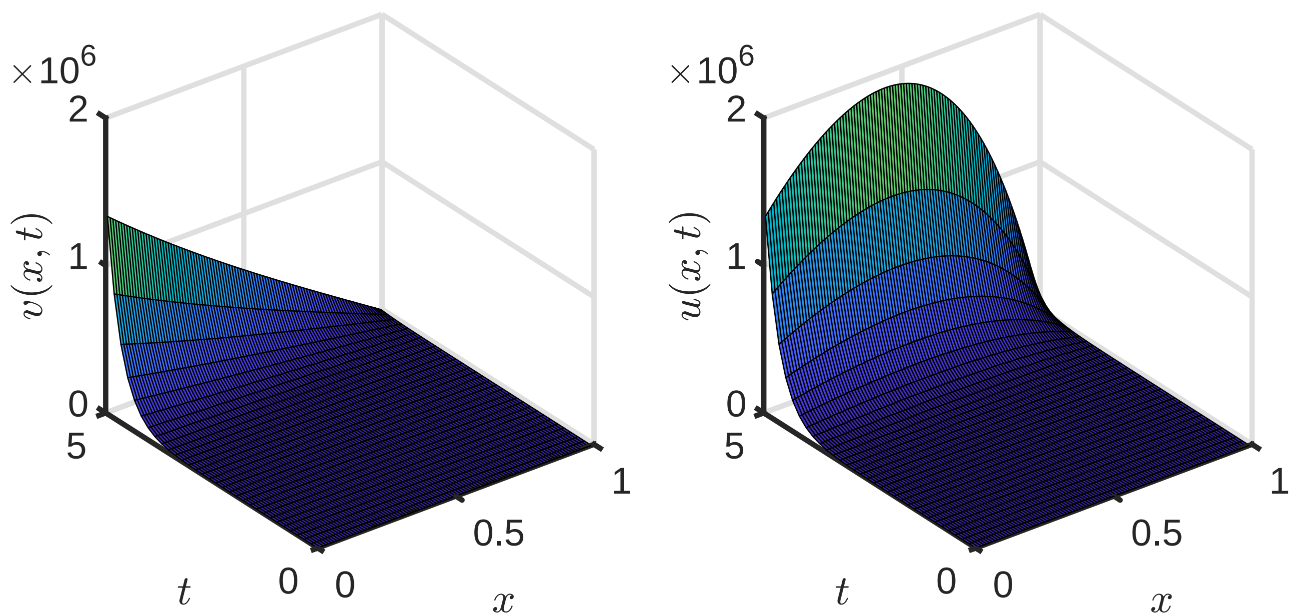
Then $l(0,y) = \phi(y)$ solves the disagreement and there is no interference with stability as the term can be made arbitrarily small.

Simulations

Example:

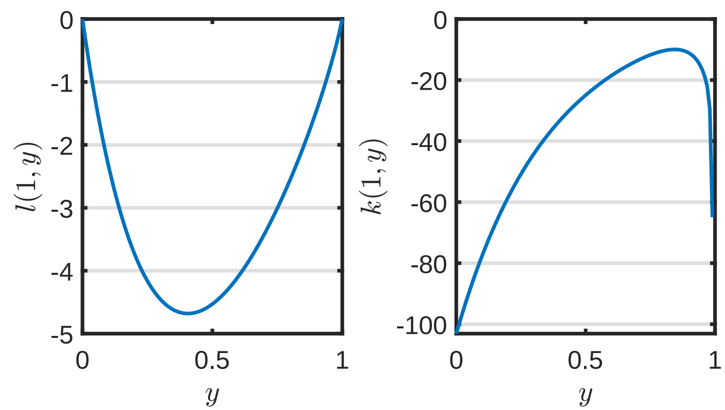
$$v_t(x,t) = 1.5 v_{xx}(x,t) + 2v(x,t)$$
$$u_t(x,t) = u_x(x,t) + \exp(x)\, v(x,t) + 5x \times v(0,t)$$
$$+ \int_0^x 1.5 \exp(1-y)\, v(y,t)\, dy$$
$$v(0,t) = u(0,t),\ v(1,t) = 0,\ u(1,t) = U(t)$$

$v_0 = 0, u_0 = 1$. Open-loop diverges:

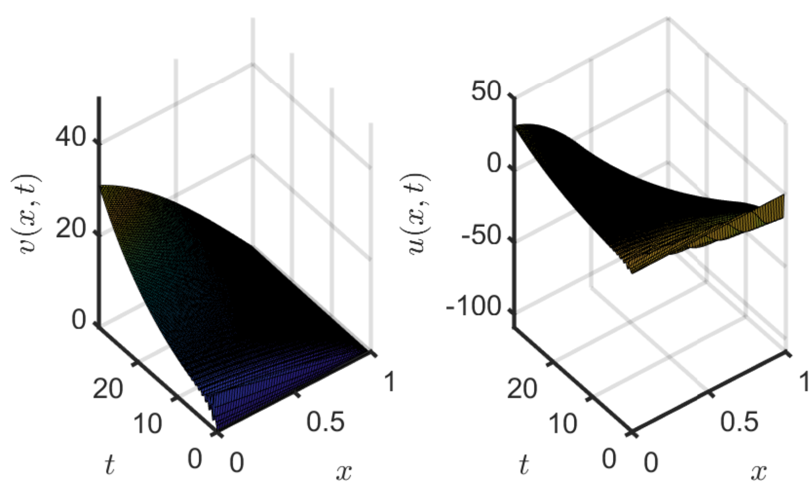


Simulations

Remember that in target system 1 we depend on c chosen large. Kernels for $c = 10$:

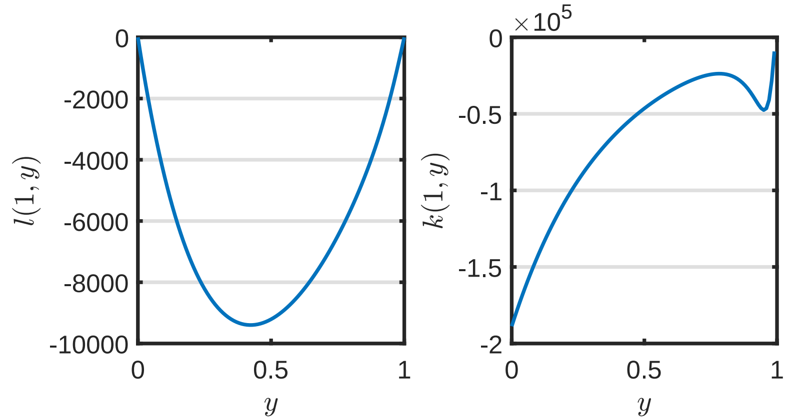


However closed-loop still diverges.

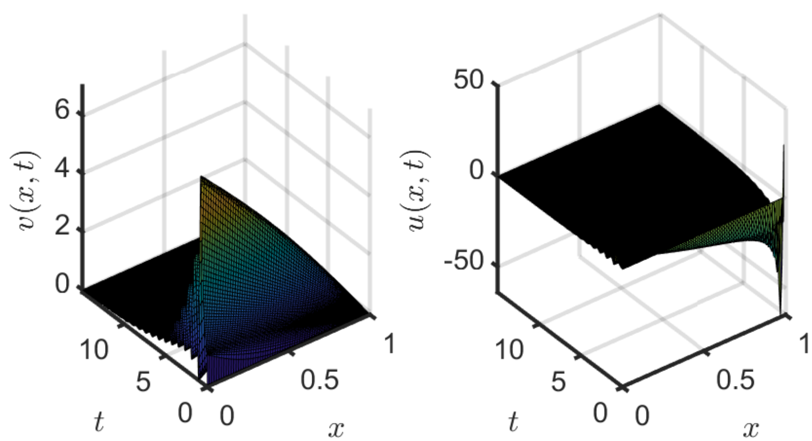


Simulations

Kernels for $c = 200$:



Now closed-loop converges but we paid a steep price: large initial control.



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Target system & transformation number 2

Target system 1 enough for stability but it requires very large c, thus large controls.

Thus a second cleaner system is better:

$$\begin{aligned}
\eta_t(x,t) &= \varepsilon\eta_{xx}(x,t) - c\eta(x,t) \\
\eta(1,t) &= 0 \\
\eta(0,t) &= \omega(0,t) + \Delta \\
\omega_t(x,t) &= \omega_x(x,t) \\
\omega(1,t) &= 0
\end{aligned}$$

Now arbitrary c works! The transformation is the same:

$$\begin{aligned}
\eta(x,t) &= v(x,t) - \int_x^1 p(x,y)v(y,t)dy \\
\omega(x,t) &= u(x,t) - \int_0^x k(x,y)u(y,t)dy - \int_0^1 l(x,y)v(y,t)dy
\end{aligned}$$

Control: $U(t) = \int_0^1 k(1,y)u(y,t)dy + \int_0^1 l(1,y)v(y,t)dy$

Kernel equations number 2

Price to pay: kernel equations are more involved

For $\mathcal{T} = \{(x,y) : 0 \le y \le x \le 1\}$:

$$\begin{aligned} k_x(x,y) &= -k_y(x,y) \\ k(x,0) &= \varepsilon l_y(x,0) - g(x) + \int_0^x k(x,y)g(y)\,dy \end{aligned}$$

For $0 \le x, y \le 1$:

$$\begin{aligned} l_x(x,y) &= \varepsilon l_{yy}(x,y) + \lambda l(x,y) - h(x-y)\left[k(x,y)\mu(y) + f(x,y) - \int_y^x k(x,s)f(s,y)ds\right] \\ &\quad -\delta(y-x)\mu(y) \\ l(x,0) &= 0, \quad l(x,1) = 0, \quad l(0,y) = \phi(y) \end{aligned}$$

For $\mathcal{T}' = \{(x,y) : 0 \le x \le y \le 1\}$:

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= \frac{\lambda + c}{\varepsilon} p(x,y) \\ p(x,1) &= 0 \\ p(x,x) &= \frac{\lambda + c}{2\varepsilon}(x-1) \end{aligned}$$

Target system & transformation number 2: main results

Proposition. There exists a solution $k(x,y)$, $l(x,y)$, $p(x,y)$ to the kernel equations such that $k \in L^2[\mathcal{T}]$, $p \in L^2[\mathcal{T}']$and $l \in L^2[0,1;H^1(0,1)]$.

Proof: same as before but extra care to take care of δ function in kernel equations (a explicit solution is in fact constructed for the first term of the series).

Theorem. Consider the original system with initial conditions $v_0, u_0 \in H^1(0,1)$ verifying zero-order compatibility conditions, and c in the target system positive Then, $u, v \in L^2[0,\infty;H^1(0,1)]$ and they verify the following energy estimate

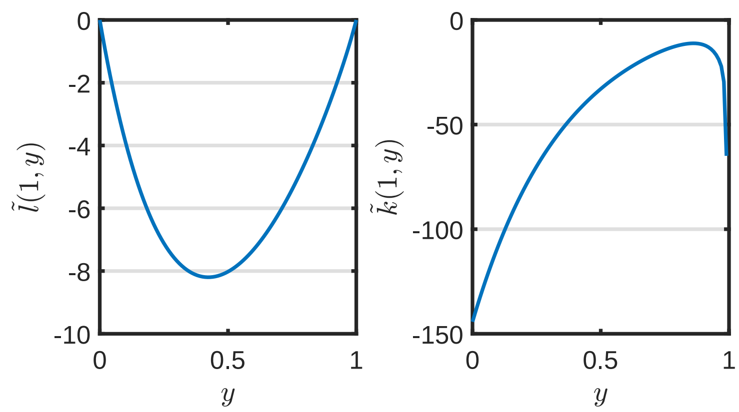
$$\|v(\cdot,t)\|_{H^1}^2 + \|u(\cdot,t)\|_{H^1}^2 \leq C_1 \exp(-C_2 t)(\|v_0\|_{H^1}^2 + \|u_0\|_{H^1}^2),$$

with $C_1, C_2 > 0$.

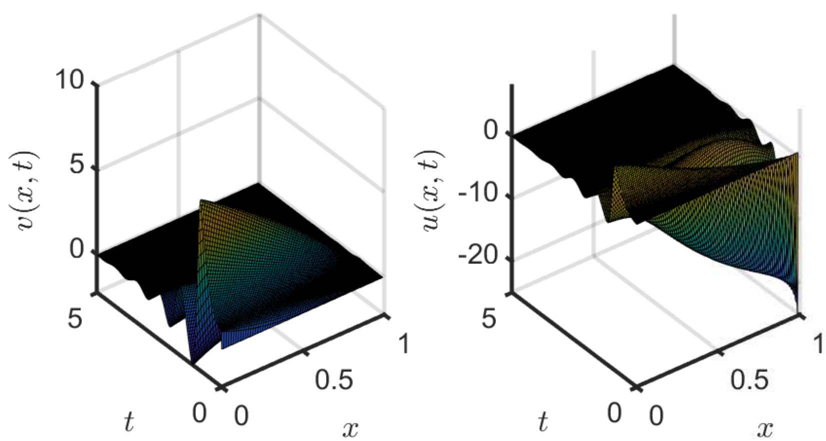
Proof: by Lyapunov method, even easier than before due to clean target system. Rate of convergence can be chosen by increasing c.

Simulations

Kernels for $c = 10$: slightly larger than before.



However, much better behaved system:



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Conclusions

- An unstable mixed hyperbolic-parabolic system has been stabilized by backstepping.

- Two proposed target systems illustrate how a simpler and better-behaved target system results in more involved kernel equations.

- Next step is to consider fully coupled system with 2 controls as in Ghousein and Witrant... which however seems quite difficult to address... related to Fredholm kernels, which are starting to show up in the literature (e.g. Coron and Olive).

Gracias! Questions?

References:

- Guangwei Chen, Rafael Vazquez, Zhitao Liu and Hongye Su, ”Backstepping control of mixed hyperbolic-parabolic PDE system with multiple coupling terms,” submitted to 2021 CDC.

- Guangwei Chen, Rafael Vazquez, Zhitao Liu and Hongye Su, ”Backstepping control of an underactuated hyperbolic-parabolic coupled PDE system,” submitted to IEEE TAC.

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