Backstepping control of an underactuated hyperbolic-parabolic coupled PDE system

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***Abstract*— This paper considers a class of hyperbolic-parabolic Partial Differential Equation (PDE) system with-some interior mixed-coupling terms, a rather unexplored family of systems. Compared with previous literature, the family of systems we explore contains several interior-coupling terms, which makes controller design more challenging. Our goal is to design a boundary controller to exponentially stabilize the coupled system. For that, we propose a controller whose design is based on the backstepping method. Under this controller, we analyse the stability of the closed loop in the H^1 sense. A set of (highly coupled) backstepping kernel equations are derived, and their well-posedness is shown in the appropriate spaces by an infinite induction energy series, which has not been used before in this setting. Moreover, we show the invertibility of transformations by displaying the inverse transformations, as required for closed-loop well-posedness and stability. Finally, a numerical simulation is implemented, and the result illustrates that the control law designed by the backstepping transformation can stabilize the mixed PDE system exponentially.**

***Index Terms*— Backstepping control, hyperbolic-parabolic system, mixed-coupling terms**

I. INTRODUCTION

COUPLED Partial Differential Equation (PDE) systems have recently attracted considerable attention since numerous physical, chemical, and biological processes can be modelled by them; to give some examples, applications range from extreme ultraviolet lithography [1] to thermal heat exchanger tubes [2], biological chemotaxis [3], and others.

Most works only deal with control of coupled hyperbolic-only [4] or parabolic-only [5] PDE systems—sometimes coupled with ODEs as well. For hyperbolic PDEs, backstepping has in recent years established itself as one of the main design approaches for designing controllers or observers.For example, H. Yu et al solve the problem of adaptive output-feedback stabilization for one-dimensional 2×2 hyperbolic partial integro-differential equations (PIDEs) [6]. L. Hu et al yield the stabilization of the heterodirectional linear coupled hyperbolic PDEs by both fullstate and observer-based output feedback [7]. J. Auriol et al design a delay-robust controller [8] and a robust output feedback controller [9] for the two heterodirectional linear coupled hyperbolic PDEs. L. Su et al obtain the stabilization of the hyperbolic PDEs with recirculation in the unactuated channel using a single boundary input [10], and J. Auriol et al implement the two boundary inputs to realize the stabilization of the system [11]. More challenging examples such as $n+1$ coupled linear hyperbolic systems with uncertain boundary parameters are also studied [12]. Beyond the linear cases, quasilinear or nonlinear hyperbolic systems can also be stabilized via the backstepping method. For example, J. M. Coron et al achieve the finite-time stabilization of homogeneous quasilinear hyperbolic systems [13]. A. Hayat get the exponential stability of general 1-D homogeneous quasilinear systems [14]. Besides the homogeneous cases, 1-D inhomogeneous quasilinear hyperbolic systems [15] and 2×2 nonlinear hyperbolic systems [16] are also researched. For parabolic PDEs, backstepping has been extended beyond the 1-D case [17] to coupled systems; first, for linear parabolic PDEs with constant coefficients [18], then spatially-varying coefficients [5], and finally space and time-dependent coefficients [19]. In [20], output feedback stabilization is achieved for coupled reaction-diffusion equations with constant parameters. In addition, output regulation methods have also been used to deal with coupled linear parabolic PIDEs [21], [22].

Besides backstepping, port-Hamiltonian approach [23], [24] and Lyapunov-based method [25], [26] are sometimes suitable for the parabolic or hyperbolic cases.

The mixed system under study is motivated by the physical problem of EUV lithography [27]. EUV lithography includes a liquid metal droplet stream convecting through plasma, which can be modeled with a first-order hyperbolic PDE. The plasma, which influences the transmision of droplet stream, diffuses in space, and therefore can be modeled using a parabolic PDE. Although the structure of the couplings for EUV lithography can vary, all possible couplings amenable to the presented methodology are included, to make the design suitable for other applications.

However, to the best of our knowledge, rather few results are available for mixed hyperbolic-parabolic PDE systems. While obviously Lyapunov-based methods [28] or other general methods such as flatness [29] would in principle be applicable,

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these systems have for the most part been neglected, with a few exceptions [30], [31]. They present quite a challenge, as the physics of the different PDEs are rather dissimilar (transport versus diffusion phenomena), and, particularly, when trying to apply the backstepping method, it is difficult to obtain s-traightforward designs as in the "pure" hyperbolic or parabolic cases. Presently, only a few works have investigated the use of backstepping in mixed PDE systems [32]–[35]. In [32] Krstic firstly explored a linear parabolic PDE with a long input delay, which can be expressed as a coupled hyperbolic-parabolic PDE system, but the coupling only occurs at the boundary of parabolic PDE. In [33] Chen et al. added a Volterra integral source term driven by the parabolic PDE in the domain of the hyperbolic system without considering a direct coupling term. In [34] Ghousein et al. considered the bidirectional couplings within the two distinct PDEs' domains, but the two control inputs are required. Considering input and output delays, in [35] a Fredholm backstepping transformation is utilized to stabilize a parabolic-hyperbolic PDE-PDE-PDE cascade.

The main contribution of this work is addressing systems with boundary-dependent and in-domain direct coupling terms within the hyperbolic system domain, which considerably complicate the controller design. Backstepping transformations are utilized to achieve the control law, which illustrates that the backstepping method has broad adaptability, even for mixed PDEs. Firstly, we propose a controller whose design is based on backstepping method. After that, the (highly coupled) backstepping kernel equations are derived which have not appeared before in the literature, and their well-posedness is shown in the appropriate spaces by an infinite induction energy series, a method also novel compared with other works, at least in this setting; the method requires a novel modification in the target system to enforce the agreement of boundary conditions in the resulting kernel equations. Then, we analyze the stability of target system in the H^1 sense. Moreover, we show the invertibility of transformations by finding the inverse transformations using a successive approximation series and show how they transform H^1 functions into H^1 functions and back; this guarantees closed-loop stability and well-posedness. The paper is organized as follows: Section II presents the structure of the mixed PDE system. Section III gives the design of the boundary controller. Section IV explores the well-posedness of the highly-coupled kernel functions, and, next, Section V analyzes the invertibility of the transformations, the stability of target system, and the well-posedness of closed-loop system. Finally, Section VI validates the effectiveness of the proposed controllers by numerical simulation.

II. Problem Statement

Define, as usual, the L^2 norm of a function f in the $(0,1)$ interval as $\|f\|_{L^2} = \sqrt{\int_0^1 f^2(x)dx}$. The space of functions with finite L^2 norm is called $L^2(0,1)$. Similarly, the H^1 norm and H^2 of a (weakly differentiable) function are denoted as $\|f\|_{H^1} = \sqrt{\|f\|_{L^2}^2 + \|f_x\|_{L^2}^2}$and $\|f\|_{H^2} = \sqrt{\|f\|_{L^2}^2 + \|f_x\|_{L^2}^2 + \|f_{xx}\|_{L^2}^2}$. Note that if a function belongs to H^2, then it also belongs to the H^1 space and L^2 space.

We consider a mixed hyperbolic-parabolic PDE systemwith several interior couplings

$$v_t(x,t) = \epsilon v_{xx}(x,t) + \lambda v(x,t) \tag{1}$$
$$v(0,t) = u(0,t), v(1,t) = 0 \tag{2}$$
$$u_t(x,t) = u_x(x,t) + \mu(x) v(x,t) + g(x) v(0,t) + \int_0^x f(x,y) v(y,t) dy \tag{3}$$
$$u(1,t) = U(t), (x,t) \in [0,1] \times [0,\infty) \tag{4}$$

where $\epsilon, \lambda > 0$, $g \in \mathcal{C}(0,1), \mu \in \mathcal{C}^1(0,1)$ and $f \in \mathcal{C}^1[0,1;\mathcal{C}^1(0,1)]$. Denote the initial conditions as $v_0(0) = v(x,0)$ and $u_0(x) = u(x,0)$. To obtain some insight on (1)–(4), note that $u(x,t)$ has an explicit solution for $t > 1-x$:

$$u = U(t-1+x) + \int_x^1 \int_0^s f(s,y) v(y,t+x-s) dy ds + \int_x^1 [\mu(s) v(s,t+x-s) + g(s) v(0,t+x-s)] ds$$

Thus, setting $x = 0$ in this expression of u and inserting it into (1)–(4) one could obtain a single-variable system, which could be described as a delay control system with rather complex recirculation from the boundary.

Our objective is to design a feedback control law for $U(t)$ so that the system (1)–(4) is well-posed and its origin exponentially stable. We next present our main result.

Theorem 1: *Consider the system (1)–(4) subject to control law*

$$U(t) = \int_0^1 k(1,y) u(y,t) dy + \int_0^1 l(1,y) v(y,t) dy \tag{5}$$

where k(x,y) and l(x,y) are bounded kernel functions obtained from equations given in Section III, specifically(17)–(20), with initial conditions $v_0, u_0 \in H^1(0,1)$ that verify zero-order compatibility conditions, i.e., $v_0(1) = 0, u_0(0) = v_0(0)$ and

$$u_0(1) = \int_0^1 k(1,y) u_0(y) dy + \int_0^1 l(1,y) v_0(y) dy. \tag{6}$$

Then, $u(\cdot,t), v(\cdot,t) \in H^1(0,1)$ for all $t > 0$ and verify the following energy estimate:

$$\|v(\cdot,t)\|_{H^1}^2 + \|u(\cdot,t)\|_{H^1}^2 \le C_1 \mathrm{e}^{-ct} \left(\|v_0\|_{H^1}^2 + \|u_0\|_{H^1}^2 \right), \tag{7}$$

with$C_1 > 1, c > 0$, where c can be chosen as large as desired.

Proof: The proof of Theorem 1 is split between the next three sections. In Section III the kernel equations required for the control law (5) are derived by using the backstepping method. Existence of kernel functions that solve the equations is then shown in Section IV, thus establishing the validity of (5). Finally, the closed-loop system behaviour is demonstrated in Section V, and in particular (7). ■

III. Controller Design

In the past, backstepping has been proven to be effective for mixed PDEs [32]–[34], thus we adapt it for our case. The controller design begins by choosingthe backstepping transformations as

$$\eta(x,t) = v(x,t) - \int_x^1 p(x,y) v(y,t) dy \tag{8}$$

$$\begin{aligned}\omega(x,t) &= u(x,t)-\int_0^x k(x,y)u(y,t)dy \\ &\quad -\int_0^1 l(x,y)v(y,t)dy \end{aligned} \tag{9}$$

which maps the original system (1)–(4) to target system (10)–(13):

$$\begin{aligned}\eta_t(x,t) &= \epsilon\eta_{xx}(x,t)-c\eta(x,t) &(10)\\ \eta(1,t) &= 0,\quad \eta(0,t)=\omega(0,t)+\Delta(t) &(11)\\ \omega_t(x,t) &= \omega_x(x,t) &(12)\\ \omega(1,t) &= 0 &(13)\end{aligned}$$

with the definition of $\Delta(t)$ given at the end of this section in a proposition.

From the original and target systems (1)–(4), (10)–(13), and the transformations (8)–(9), and after a tedious but straightforward procedure, we derive the following kernel equations.

First of all, $p(x,y)$, evolving in the domain $\mathcal{T}'=\{(x,y): 0\le x\le y\le 1\}$ has the following expression

$$\begin{aligned}p_{xx}(x,y)-p_{yy}(x,y) &= \frac{\lambda+c}{\epsilon}p(x,y) &(14)\\ p(x,1) &= 0,\quad p(x,x)=\frac{\lambda+c}{2\epsilon}(x-1) &(15)\end{aligned}$$

A solution to this kernel equation is already known [38], indeed $p(x,y)$ has an analytic solution as follows

$$p(x,y)=-c(1-y)\frac{I_1(\sqrt{c((1-y)^2-(1-x)^2)})}{\sqrt{c((1-y)^2-(1-x)^2)}} \tag{16}$$

where I_1 stands for the first-order modified Bessel function.

The kernel function $k(x,y)$ evolves in $\mathcal{T}=\{(x,y): 0\le y\le x\le 1\}$ as follows

$$\begin{aligned}k_x(x,y) &= -k_y(x,y) &(17)\\ k(x,0) &= \epsilon l_y(x,0)-g(x)+\int_0^x k(x,y)g(y)\,dy &(18)\end{aligned}$$

For $l(x,y)$, with $0\le x,y\le 1$, one has

$$\begin{aligned}l_x(x,y) &= \epsilon l_{yy}(x,y)+\lambda l(x,y)\\ &\quad +h(x-y)\left[k(x,y)\mu(y)-f(x,y)\right]\\ &\quad +h(x-y)\int_y^x k(x,s)f(s,y)ds\\ &\quad -\delta(y-x)\mu(y) &(19)\\ l(x,0) &= 0, l(x,1)=0, l(0,y)=\phi(y) &(20)\end{aligned}$$

where $h(x)$ is the step function satisfying $h(x)=1, x>0$ and $h(x)=0, x\le 0$ and $\delta(x)$ is Dirac's delta function, and ϕ is an $H^1(0,1)$ function verifying $\phi(0)=\phi(1)=0$ that can be chosen arbitrarily; our choice is given in Lemma 1 to help stabilization.

Proposition 1: *Define $\Delta(t)$ in (11) as*

$$\begin{aligned}\Delta(t) &= \int_0^1 D(y)\eta(y,t)dy\\ &\quad +\int_0^1\left[\int_0^1 Q(s,y)D(s)ds\right]\eta(y,t)dy\\ &\quad -\int_0^1\left[\int_y^1 Q(s,y)D(s)ds\right]\eta(y,t)dy \end{aligned}\tag{21}$$

with

$$D(y)=\phi(y)-p(0,y) \tag{22}$$

and $Q(s,y)$ the inverse kernel function of transformation (8), namely,

$$v(x,t)=\eta(x,t)+\int_x^1 Q(x,y)\eta(y,t)dy \tag{23}$$

Then, the last boundary condition of (20) is verified.

Proof: In the derivation of the kernel equations, when verifying the boundary condition at $x=0$ one reaches

$$\Delta(t)=\int_0^1(\phi(y)-p(0,y))v(y,t)dy$$

Using now (22) and the inverse transformation (23), which exists from already known results [38], the result of the proposition is reached, by changing the order of integration. ■

It can be seen how introducing $\Delta(t)$ in (11) helps to solve a potential disagreement in boundary conditions of kernel equations(19)–(20). If we do not introduce Δ, the boundary conditions of kernel function $l(x,y)$ are $l(x,0)=0$ and $l(0,y)=p(0,y)$ and since $p(0,0)\ne 0$, they become incompatible. Now, setting $\phi(0)=0$ enforces compatibility in the boundary conditions. The chosen value of $\phi(y)$ is given in Section V-B.

Regarding the well-posedness of the kernel equations for k and l, the following result holds.

Proposition 2: *There exists a weak solution $k(x,y)$, $l(x,y)$, to the kernel equations (17)–(20) such that $\|k(x,\cdot)\|^2_{L^2(0,x)}+\|l(x,\cdot)\|^2_{H^1(0,1)}\le C_3$, for some positive constant C_3. Thus in particular, $k\in L^2(\mathcal{T})$ and $l\in L^2(0,1;H^1(0,1))$.*

Proof: See Section IV. ■

IV. Well-posedness of Gain Kernel Equations

In thissection, we need to prove the well-posedness of (17)–(20) Our idea is to consider the following sequence of functions. For $m=0$, define l_0 and k_0 as the solution of the following PDEs:

$$\begin{aligned}l_{0,x}(x,y) &= \epsilon l_{0,yy}(x,y)+\lambda l_0(x,y)-h(x-y)f(x,y)\\ &\quad -\delta(y-x)\mu(y) &(24)\\ l_0(x,0) &= 0, l_0(x,1)=0, l_0(0,y)=\phi(0,y) &(25)\\ k_{0,x}(x,y) &= -k_{0,y}(x,y),\quad k_0(x,0)=-g(x) &(26)\end{aligned}$$

whereas for $m=1,2,\cdots,\infty$, the following functions are defined in terms of l_{m-1} and k_{m-1}:

$$\begin{aligned}l_{m,x}(x,y) &= \epsilon l_{m,yy}(x,y)+\lambda l_m(x,y)\\ &\quad +h(x-y)\left[\mu(y)k_{m-1}(x,y)\right]\\ &\quad +h(x-y)\int_y^x k_{m-1}(x,s)f(s,y)ds &(27)\\ l_m(x,0) &= 0, l_m(x,1)=0, l_m(0,y)=0 &(28)\\ k_{m,x}(x,y) &= -k_{m,y}(x,y) &(29)\end{aligned}$$

$$k_m(x,0) = \epsilon l_{m-1,y}(x,0) + \int_0^x k_{m-1}(x,y)g(y)\,dy \quad (30)$$

If (24)–(30) have a solution, then, consider the expression.

$$l(x,y) = \sum_{m=0}^{\infty} l_m(x,y), k(x,y) = \sum_{m=0}^{\infty} k_m(x,y) \quad (31)$$

If the two series in (31) converge (in the appropriate functional spaces), thenby construction it is possible to check they are a solution to (17)–(20),if the dominated convergence theorem can be applied, by using the weak form of the kernel equations (details are skipped for lack of space).

The first step isto find a solution for $l_0(x,y), k_0(x,y)$. Analyzing (24)–(26), $k_0(x,y)$ has an explicit solution, namely,

$$k_0(x,y) = -g(x-y) \quad (32)$$

which means $k_0(x,y)$ belongs to $L^2(\mathcal{T})$. For $l_0(x,y)$, a solution is found based on the method of separation of variables by using a Fourier sine series

$$l_0(x,y) = \sum_{n=0}^{\infty} A_n(x)\sin(n\pi y) \quad (33)$$

After some tedious computation, we indeed obtain

$$\begin{aligned} l_0(x,y) =& -2\sum_{n=0}^{\infty} \exp[(\lambda - \varepsilon n^2\pi^2)x] \\ &\times \left[\int_0^x \frac{\int_0^\tau f(\tau,s)\sin(n\pi s)\,ds + \mu(\tau)\sin(n\pi\tau)}{\exp[(\lambda - \epsilon n^2\pi^2)\tau]} d\tau \right. \\ &\left. - \int_0^1 \phi(\tau)\sin(n\pi\tau)\,d\tau \right] \sin(n\pi y) \end{aligned} \quad (34)$$

The following bound holds for l_0 with $x > 0$:

$$|l_0(x,y)| \le C + \sum_{n=n_1+2}^{\infty} \left[\frac{2\bar{f} + 2\bar{\mu}}{\epsilon n^2\pi^2 - \lambda} + \frac{2\bar{\phi}}{\exp([\epsilon n^2\pi^2 - \lambda]x)}\right] \quad (35)$$

where $\epsilon n_1^2\pi^2 < \lambda < \epsilon(n_1+1)^2\pi^2$,

$$C = \sum_{n=0}^{n_1+1} \exp[(\lambda - \varepsilon n^2\pi^2)x] \left[\frac{(2\bar{f} + 2\bar{\mu})}{\lambda - \epsilon n^2\pi^2} + 2\bar{\phi}\right] \quad (36)$$

and $\bar{f} = \max|f(x,y)|_{x\in[0,1],y\in[0,x]}$, $\bar{\mu} = \max|\mu(x)|_{x\in[0,1]}$, $\bar{\phi} = \max|\phi(x)|_{x\in[0,1]}$. Given the bounds of λ, and since $n^2 - (n_1+1)^2 > (n-(n_1+1))^2$ for $n > n_1+1$, one has that (35) is convergent since

$$\begin{aligned} &\sum_{n=n_1+2}^{\infty} \left[\frac{2\bar{f} + 2\bar{\mu}}{\epsilon n^2\pi^2 - \lambda} + \frac{2\bar{\phi}}{\exp([\epsilon n^2\pi^2 - \lambda]x)}\right] \\ &< \frac{2\bar{f} + 2\bar{\mu}}{\epsilon\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} + 2\bar{\phi}\sum_{k=1}^{\infty} \frac{1}{\exp(\epsilon k^2\pi^2 x)} \end{aligned} \quad (37)$$

for $x > 0$, which converges, as $\exp(cn^2) > cn^2$ for $c > 0$, and thus $\sum_1^{\infty} \frac{1}{\exp(cn^2)} < \frac{1}{c}\sum_1^{\infty}\frac{1}{n^2} < \frac{\pi^2}{6c}$. Therefore, the expression l_0 in (34) makes sense and belongs to L^2 space. From (27), we know $k_1(x,y)$ depends on $l_{0,y}(x,0)$. Differentiate both sides of (34) with respect to y, $l_{0,y}$ has the formal expression

$$\begin{aligned} l_{0y}(x,y) =& -2\sum_{n=0}^{\infty} n\exp[(\lambda - \varepsilon n^2\pi^2)x] \\ &\times \left[\int_0^x \frac{\int_0^\tau f(\tau,s)\sin(n\pi s)\,ds + \mu(\tau)\sin(n\pi\tau)}{\exp[(\lambda - \epsilon n^2\pi^2)\tau]} d\tau \right. \\ &\left. - \int_0^1 \phi(\tau)\sin(n\pi\tau)\,d\tau \right] \cos n\pi y \end{aligned} \quad (38)$$

A very similar bound to (35) can be obtained, and using the fact that $\phi(y)$ is H^1, one obtains that l_{0y} is also bounded and thus an L^2 function. Moreover it can be checked as well that $l_{0,y}(\cdot,0)$ is $L^2(0,1)$ and well-defined. Thus, $l_0 \in L^2(0,1;H^1(0,1))$. Considering now $k_m(x,y), m = 1,2,\cdots$, from (29)–(30) their solutions have the following explicit expression,

$$k_m(x,y) = \epsilon l_{m-1,y}(x-y,0) + \int_0^{x-y} k_{m-1}(x-y,s)g(s)\,ds \quad (39)$$

In particular, for $m = 1$, since both $l_{0,y}(\cdot,0), k_0$ belong to L^2, we know k_1 is well-defined and at least $L^2(\mathcal{T})$. Addressing now $l_1(x,y)$, the structure of the equation in (27)–(28) has become much simpler; it is a reaction-diffusion equation with a forcing term. Using Improved Regularity (Lawrence C. Evans, p382) [36], we immediately obtain a solution l_1 exists and belongs to $L^2(0,1;H^2(0,1))$. Obviously, $l_{1,y}(\cdot,0)$ is then also well-defined and belongs to $L^2(0,1)$. Next, we can iteratively infer that there exist solutions $k_m \in L^2(\mathcal{T}), l_m(x,y) \in L^2(0,1;H^2(0,1)), m = 2,3,\cdots$, all well-defined.Now, we need to check that the series (31) are convergent in that same space of functions; for that, we consider the following sequence of functionals

$$\begin{aligned} V_1(x) &= \int_0^1 \left(l_1^2(x,y) + l_{1,y}^2(x,y)\right) dy \\ &\quad +\theta\int_0^x k_1^2(x,y)\,dy \quad (40) \\ V_m(x) &= V_{m-1} + \int_0^1 l_m^2(x,y)\,dy + \int_0^1 l_{m,y}^2(x,y)\,ds \\ &\quad +\theta\int_0^x k_m^2(x,y)\,dy \quad (41) \end{aligned}$$

where $m = 2,3,\cdots$ and $\theta > 0$.Notice these functionals, at $x = 1$, are equivalent to the norm of the partial sums of (31) in the appropriate space. For $V_1(x)$, differentiating it with respective to x, one has

$$\begin{aligned} \dot{V}_1(x) &= 2\epsilon\int_0^1 l_1 l_{1,yy}(x,y)dy + 2\lambda\int_0^1 l_1^2(x,y)dy \\ &\quad -2\int_0^x \mu(y)l_1(x,y)k_0(x,y)dy \\ &\quad -2\int_0^x l_1(x,y)\left[\int_y^x k_0(x,s)f(s,y)ds\right]dy \\ &\quad +2\int_0^1 l_{1,y}(x,y)l_{1,yx}(x,y)dy \\ &\quad +2\theta\int_0^x k_1(x,y)k_{1x}(x,y) + \theta k_1^2(x,x) \quad (42) \end{aligned}$$

Notice $\int_0^1 l_1 l_{1,yy}(x,y)dy = -\int_0^1 l_{1,y}^2(x,y)$. In addition,

$$\int_0^1 l_{1,y}(x,y)l_{1,yx}(x,y)dy$$

$$\begin{aligned}&= -\epsilon\int_0^1 l_{1,yy}^2(x,y)dy+\lambda\int_0^1 l_{1,y}^2(x,y)dy\\&+\int_0^x \mu(y)l_{1,yy}(x,y)k_0(x,y)dy\\&+\int_0^x l_{1,yy}(x,y)\left[\int_y^x k_0(x,s)f(s,y)ds\right]dy \quad (43)\end{aligned}$$

also, from (29), $2\int_0^x k_1(x,y)k_{1x}(x,y)dy = -k_1^2(x,x)+k_1^2(x,0)$. Applying Young's Inequality, one gets $2\int_0^x l_{1,yy}(x,y)\mu(y)k_0(x,y)dy \leq \beta\|l_{1,yy}(x,\cdot)\|_{L^2}^2+\frac{\bar{\mu}^2}{\beta}\|k(x,\cdot)\|_{L^2(0,x)}^2$, $2\int_0^x l_1(x,y)\int_y^x k_0(x,s)f(s,y)dsdy \leq \|l_1(x,\cdot)\|_{L^2}^2+\bar{f}^2\|k(x,\cdot)\|_{L^2(0,x)}^2$, $k_1^2(x,0) \leq 2\epsilon^2 l_{0,y}^2(x,0)+2\bar{g}^2\|k(x,\cdot)\|_{L^2(0,x)}^2$, and finally , $-2\int_0^x l_{1,yy}(x,y)\int_y^x k_0(x,s)f(s,y)dsdy \leq \beta\|l_{1,yy}(x,\cdot)\|_{L^2}^2+\frac{\bar{f}^2}{\beta}\|k(x,\cdot)\|_{L^2(0,x)}^2$. Using the inequalities in (42) yields:

$$\begin{aligned}\dot{V}_1(x) &\leq 2(\lambda+1)\int_0^1 l_1^2(x,y)dy\\&+2(\lambda-\epsilon)\int_0^1 l_{1,y}^2(x,y)dy\\&-2(\epsilon-\beta)\int_0^1 l_{1,yy}^2(x,y)dy+2\theta\epsilon^2\bar{l}_{0,y}^2\\&+\left(\frac{\bar{f}^2}{\beta}+\frac{\bar{\mu}^2}{\beta}+\bar{\mu}^2+\bar{f}^2+2\theta\bar{g}^2\right)\bar{g}^2 \quad (44)\end{aligned}$$

where $\bar{l}_{0,y} = \max|l_{0,y}(x,0)|_{x\in[0,1]}$, $\bar{g} = \max|g(x)|_{x\in[0,1]}$. We choose $\beta \leq \epsilon$, thus getting $\dot{V}_1(x) \leq aV_1+d$, where $a = \max[2(\lambda+1), 2(\lambda-\epsilon)]$, $d = 2\theta\epsilon^2\bar{l}_{0,y}^2+\left(\frac{\bar{f}^2}{\beta}+\frac{\bar{\mu}^2}{\beta}+\bar{\mu}^2+\bar{f}^2+2\theta\bar{g}^2\right)\bar{g}^2$. For $V_2(x)$

$$\begin{aligned}\dot{V}_2 &\leq \dot{V}_1(x)+2(\lambda+1)\int_0^1 l_2^2(x,y)dy+2\theta\epsilon^2 l_{1,y}^2(x,0)\\&-2(\epsilon-\beta)\int_0^1 l_{2,yy}^2(x,y)dy\\&+\left(\frac{\bar{f}^2}{\beta}+\frac{\bar{\mu}^2}{\beta}+\bar{\mu}^2+\bar{f}^2+2\theta\bar{g}^2\right)\int_0^x k_1^2(x,y)dy\\&+2(\lambda-\epsilon)\int_0^1 l_{2,y}^2(x,y)dy \quad (45)\end{aligned}$$

For $F\in L^2\left([0,1];H^1[0,1]\right)$ the following inequality holds:

$$F^2(x,0)\leq 2\int_0^1 F^2(x,y)dy+2\int_0^1 F_y^2(x,y)dy \quad (46)$$

Applying (46) to $l_{1,y}^2(x,0)$ in (44) yields

$$\begin{aligned}&\dot{V}_2(x)\\ \leq\ &(2\lambda+2)\int_0^1 l_1^2(x,y)dy+2(\lambda-\epsilon+2\theta\epsilon^2)\int_0^1 l_{1,y}^2 dy\\&-2(\epsilon-\beta-2\theta\epsilon^2)\int_0^1 l_{1,yy}^2(x,y)dy\\&+\left(\frac{\bar{f}^2+\bar{\mu}^2}{\beta}+\bar{f}^2+\bar{\mu}^2+2\theta\bar{g}^2\right)\int_0^x k_1^2(x,y)dy\\&+2(\lambda-\epsilon)\int_0^1 l_{2,y}^2(x,y)dy+(2\lambda+2)\int_0^1 l_2^2(x,y)dy\\&-2(\epsilon-\beta)\int_0^1 l_{2,yy}^2(x,y)dy \quad (47)\end{aligned}$$

Then, we set $0<\theta\leq\frac{\epsilon-\beta}{2\epsilon^2}$, thus getting $\dot{V}_2(x)\leq\bar{a}V_2+d$ where $\bar{a}=\max[2\lambda+2, 2(\lambda-\epsilon+2\theta\epsilon^2), \frac{\bar{f}^2+\bar{\mu}^2}{\beta}+\bar{f}^2+\bar{\mu}^2+2\theta\bar{g}^2]$. Notice that since $\bar{a}>a$, for $V_2(x)$ one also has $\dot{V}_2(x)\leq\bar{a}V_2+d$.Iterating for all values of m one reaches $\dot{V}_m(x)\leq\bar{a}V_1+d$. Integrating, $V_m(x)\leq\frac{d}{\bar{a}}(\exp(\bar{a}x)-1)\leq\frac{d}{\bar{a}}(\exp(\bar{a})-1)$, independent of m.

Thus it is clear that $\lim_{m\to\infty}V_m=V_\infty(x)\leq\frac{d}{\bar{a}}(\exp(\bar{a})-1)$ is bounded for all $x\in[0,1]$, and by dominated convergence the series in (31) define valid functions $k\in L^2(\mathcal{T}), l\in L^2(0,1;H^1(0,1))$, and the bound of Proposition 2 is obtained, thus finishing the proof.

V. Stability and well-posedness of closed loop

To prove results of **Theorem** 1, we need to carry out three steps. We start by showing the existence of the inverse transformations, for both transformation (8) and transformation (9), which allows us to recover the original variables from the transformed variables. Then, we display the stability of the target system, as presented in **Lemma** 1. We follow by illustrating that both the two transformations keep the original system and target systems in the same space, as shown in Section. V-C. In terms of **Lemma** 1, **Theorem** 1 is directly constructed.

A. Invertibility of Backstepping Transformations

Corresponding to (8) and (9), we use the following inverse transformations,

$$v(x,t) = \eta(x,t)+\int_x^1 Q(x,y)\eta(y,t)dy \quad (48)$$

$$\begin{aligned}u(x,t) &= \omega(x,t)+\int_0^x R(x,y)\omega(y,t)dy\\&+\int_0^1 S(x,y)\eta(y,t)dy \quad (49)\end{aligned}$$

which can both map target system (10) into original system (1). The existence of these inverse transformations is given in the following result

Proposition 3: *Consider transformations (8)–(9) with the kernel function p given by (16) and the kernel functions k and l verifying the properties given in Proposition 2. Then, the transformations are invertible, with inverse given by (48)–(49) with kernel function Q given by*

$$Q(x,y)=-c(1-y)\frac{J_1(\sqrt{c((1-y)^2-(1-x)^2)})}{\sqrt{c((1-y)^2-(1-x)^2)}} \quad (50)$$

where J_1 stands for the first-order Bessel function, and kernel functions R and S verifying $R\in L^2(\mathcal{T})$ and $S\in L^2(0,1;H^1(0,1))$.

Proof: The existence of (48) is already well-known [38] given (16), taking the form of the (bounded) Bessel function (50). Regarding the invertibility of (49), inserting (48) into (9) and omitting time dependence for simplicity:

$$\omega = u-\int_0^x k(x,y)u(y)dy-\int_0^1 l(x,y)\eta(y)dy$$

$$
\begin{aligned}
&\quad - \int_0^1 l(x,y) \int_y^1 Q(y,s)\eta(s)dsdy \\
&= u - \int_0^x k(x,y)u(y)dy \\
&\quad - \int_0^1 \left(l(x,y) + \int_0^y l(x,s)Q(s,y)ds \right) \eta(y)dy \quad (51)
\end{aligned}
$$

Substituting now (49) in (51):

$$
\begin{aligned}
0 &= \int_0^x R(x,y)\omega(y)dy + \int_0^1 S(x,y)\eta(y)dy \\
&\quad - \int_0^x \left(k(x,y) + \int_y^x k(x,s)R(s,y)ds \right) \omega(y)dy \\
&\quad - \int_0^1 \left(\int_0^x k(x,s)S(s,y)ds \right) \eta(y)dy \\
&\quad - \int_0^1 \left(l(x,y) + \int_0^y l(x,s)Q(s,y)ds \right) \eta(y)dy \quad (52)
\end{aligned}
$$

For (52) to be true, we obtain two integral equation for the kernel functions S and R:

$$
R = k + \int_y^x k(x,s)R(s,y)ds, \quad (53)
$$

$$
S = l + \int_0^y l(x,s)Q(s,y)ds + \int_0^x k(x,s)S(s,y) \quad (54)
$$

These kernel integral equations are solved by successive approximations, and one finally reaches

$$
\int_0^x R^2(x,y)dy \leq \left(\sum_{i=0}^{i=\infty} \frac{\sqrt{x}^i}{\sqrt{i!}} \sqrt{C_3}^{i+1} \right)^2 < \infty \quad (55)
$$

where C_3 is the constant of Proposition 2, and the inequality is deduced from Stirling's formula for large values of the factorial. Thus, $R \in L^2(\mathcal{T})$. A very similar proof demonstrates that both S and S_y in (54) exist and belong to L^2, which is skipped for lack of space. ■

B. Stability of Target System

The stability estimate of target system (10) is based on the following result.

Lemma 1: *Consider the plant (10)–(13) with $\eta_0, \omega_0 \in H^1(0,1)$, $c > 0$, and ϕ defined as*

$$
\phi(y) = \begin{cases} \dfrac{p(0,\tau)}{\tau} y, & 0 \leq y \leq \tau \\ p(0,y), & y > \tau \end{cases} \quad (56)
$$

Then there exists positive $\tau < 1$ and ρ such that

$$
\Phi(t) \leq e^{-ct}\Phi(0) \quad (57)
$$

where $\Phi(t) = W_1(t) + W_2(t) + \xi\left(W_3(t) + W_4(t)\right)$ with

$$
W_1(t) = \frac{1}{2}\int_0^1 \eta^2(x,t)dx, W_2(t) = \frac{1}{2}\int_0^1 \eta_x^2(x,t)dx \quad (58)
$$

$$
W_3(t) = \frac{1}{2}\int_0^1 e^{bx}\omega^2(x,t)dx, \quad (59)
$$

$$
W_4(t) = \frac{1}{2}\int_0^1 e^{bx}\omega_x^2(x,t)dx \quad (60)
$$

for $\xi > \frac{32}{\epsilon}\max\{1, (\epsilon+c)^2\}$ and $b \geq c$.

Proof: Computing $\dot{W}_1(t)$ and integrating by parts

$$
\begin{aligned}
\dot{W}_1 &= -\int_0^1 \epsilon\eta_x^2(x,t)dx - c\int_0^1 \eta^2(x,t)dx \\
&\quad -\epsilon\omega(0,t)\eta_x(0,t) - \epsilon\Delta\eta_x(0,t) \\
&= -2\epsilon W_2 - 2cW_1 - \epsilon\omega(0,t)\eta_x(0,t) - \epsilon\Delta\eta_x(0,t) \quad (61)
\end{aligned}
$$

Similarly, differentiating $W_2(t)$ with respect to t

$$
\begin{aligned}
\dot{W}_2 &= \int_0^1 \eta_x(x,t)\eta_{xt}(x,t)dx \\
&= -\epsilon\int_0^1 \eta_{xx}^2(x,t)dx - c\int_0^1 \eta_x^2(x,t)dx \\
&\quad -\eta_x(0,t)\omega_t(0,t) - \eta_x(0,t)\Delta_t \\
&\quad -c\eta_x(0,t)\omega(0,t) - c\eta_x(0,t)\Delta \\
&= -\epsilon\int_0^1 \eta_{xx}^2(x,t)dx - 2cW_2 - \eta_x(0,t)\omega_t(0,t) \\
&\quad -\eta_x(0,t)\Delta_t - c\eta_x(0,t)\omega(0,t) - c\eta_x(0,t)\Delta \quad (62)
\end{aligned}
$$

Using $\omega_t(0,t) = \omega_x(0,t)$ and Young's inequality,

$$
\begin{aligned}
\dot{W}_1 + \dot{W}_2 &= -\epsilon\int_0^1 \eta_{xx}^2(x,t)dx - 2(\epsilon+c)W_2 - 2cW_1 \\
&\quad -\eta_x(0,t)\left[\chi\omega(0,t) + \chi\Delta + \Delta_t + \omega_x(0,t)\right] \\
&\leq -\epsilon\int_0^1 \eta_{xx}^2(x,t)dx - 2(\epsilon+c)W_2 + \frac{\gamma\eta_x^2(0,t)}{2} \\
&\quad + \frac{8\left[\chi^2\omega^2(0,t) + \chi^2\Delta^2 + \Delta_t^2 + \omega_x^2(0,t)\right]}{\gamma} \\
&\quad -2cW_1 \quad (63)
\end{aligned}
$$

where $\chi = \epsilon + c$. For Δ^2, Δ_t^2, the following result holds, given the chosen $\phi(y)$ in the statement of Lemma 1.

Lemma 2: *For $\eta(\cdot,t) \in H^2(0,1)$, there exists positive scalars K_1 and K_2, not depending on τ, such that*

$$
\Delta^2 \leq K_1\tau W_1(t) \quad (64)
$$

$$
\Delta_t^2 \leq K_2\tau\left(c^2W_1(t) + \frac{\epsilon^2}{2}\int_0^1 \eta_{xx}^2(x,t)dx\right) \quad (65)
$$

Proof: Applying Young's inequality two times to Δ^2 and then repeated used of the Cauchy-Schwarz inequality achieves

$$
\begin{aligned}
\Delta^2 &\leq 4\int_0^\tau \tau D^2(y)\eta^2(y,t)dy \\
&\quad +4\int_0^1 \left[\int_0^\tau \tau Q^2(s,y)D^2(s)ds\right]\eta^2(y,t)dy \\
&\quad +4\int_0^\tau \tau\left[\int_y^\tau (\tau-y)Q^2(s,y)D^2(s)ds\right]\eta^2(y,t)dy \\
&\leq (8\bar{D}^2 + 16\bar{D}^2\bar{Q}^2)\tau W_1(t) \quad (66)
\end{aligned}
$$

with $\bar{D} = \max_{y\in[0,1}|D(y)|$, $\bar{Q} = \max_{\mathcal{T}}|Q(x,y)|$. Similarly,

$$
\Delta_t^2 \leq (16\bar{D}^2 + 32\bar{D}^2\bar{Q}^2)\tau(c^2W_1(t) + \frac{\epsilon^2}{2}\int_0^1 \eta_{yy}^2(y,t)dy)
$$

where (10) was used. The proof is finished by choosing $K_1 \geq 8\bar{D}^2 + 16\bar{D}^2\bar{Q}^2$, $K_2 \geq 16\bar{D}^2 + 32\bar{D}^2\bar{Q}^2$. ■

Applying (46) to $\eta_x^2(0,t)$ in (63) we have

$$
\dot{W}_1(t) + \dot{W}_2(t)
$$

$$\begin{aligned}&\leq -(\epsilon-\frac{4K_2\tau\epsilon^2}{\gamma}-\gamma)\int_0^1\eta_{xx}^2(x,t)dx\\&-\left(2c-\frac{8}{\gamma}(K_1\chi^2+K_2c^2)\tau\right)W_1(t)\\&-2(\epsilon+c-\gamma)W_2(t)+\frac{8}{\gamma}\chi^2\omega^2(0,t)+\frac{8}{\gamma}\omega_x^2(0,t)\end{aligned}$$

Choose τ so $\frac{4K_2\tau\epsilon^2}{\gamma}<\frac{\epsilon}{2}$, $\frac{8}{\gamma}(K_1\chi^2+K_2c^2)\tau<c$, and $\gamma=\frac{\epsilon}{2}$,

$$\begin{aligned}\dot W_1+\dot W_2 \leq& -2(\frac{\epsilon}{2}+c)W_2-cW_1\\&+\frac{16}{\epsilon}\left(\chi^2\omega^2(0,t)+\omega_x^2(0,t)\right)\end{aligned}\tag{67}$$

It is easy to see that $\dot W_3(t)$ and $\dot W_3(t)$ are

$$\dot W_3(t) = -\frac{b}{2}\int_0^1 e^{bx}\omega^2(x,t)dx-\frac{\omega^2(0,t)}{2}\tag{68}$$

$$\dot W_4(t) = -\frac{1}{2}\int_0^1 be^{bx}\omega_x^2(x,t)dx-\frac{\omega_x^2(0,t)}{2}\tag{69}$$

Thus $\dot W_3(t)+\dot W_4(t)=-b(W_3(t)+W_4(t))-\frac{1}{2}[\omega^2(0,t)+\omega_x^2(0,t)]$. Finally,

$$\begin{aligned}\dot\Phi \leq& -2(\frac{\epsilon}{2}+c)W_2(t)-cW_1(t)\\&-(\frac{\xi}{2}-\frac{16}{\epsilon}\chi^2)\omega^2(0,t)-(\frac{\xi}{2}-\frac{16}{\epsilon})\omega_x^2(0,t)\\&-\xi b(W_3(t)+W_4(t))\end{aligned}\tag{70}$$

For any $\xi>\frac{32}{\epsilon}\max\{1,\chi^2\}$, $b\geq c$, one has

$$\dot\Phi \leq -c\{W_1(t)+W_2(t)+\xi(W_3(t)+W_4(t))\}.\tag{71}$$

Hence, we have $\Phi(t)\leq e^{-ct}\Phi(0)$, thus finishing the proof. ■
Since Φ is equivalent to the norm $\|\eta(\cdot,t)\|_{H^1}^2+\|\omega(\cdot,t)\|_{H^1}^2$,

$$\|\eta(\cdot,t)\|_{H^1}^2+\|\omega(\cdot,t)\|_{H^1}^2\leq C_2\mathrm{e}^{-ct}\left(\|\eta_0\|_{H^1}^2+\|\omega_0\|_{H^1}^2\right),\tag{72}$$

which is indeed closer to (7).

C. Well-posedness of Closed loop

Under the conditions of the Theorem 1, the zero-order compatibility conditions for system (10)–(13) are verified. If η_0,ω_0 belong to H^1, from standard results, e.g., Improved Regularity (Lawrence C. Evans, p. 382) [36],$\eta(\cdot,t)$ exists and belongs to $H^2(0,1)$ and similarly there exists $\omega(\cdot,t)\in H^1(0,1)$. Thus, all that remains is to show that transformations (8)–(9) and their inverses (48)–(49) transform H^1 functions into H^1 functions and back, thus establishing a norm equivalence between the norm in (72) and the norm in (7) and finishing the proof. For (8) and (48) the result is obvious, having an analytic kernel function; for (9) and (49) the following result is required.

Lemma 3: *Consider transformations (9) and its inverse (49), for kernel functions verifying the conditions of Propositions 2 and 3, and $u,v\in H^1(0,1)$ verifying (1)–(4). Then it holds*

$$\|\omega\|_{L^2}^2 \leq K_1\left(\|u\|_{L^2}^2+\|v\|_{L^2}^2\right),\tag{73}$$

$$\|u\|_{L^2}^2 \leq K_2\left(\|\eta\|_{L^2}^2+\|\omega\|_{L^2}^2\right),\tag{74}$$

$$\|\omega\|_{H^1}^2 \leq K_3\left(\|u\|_{H^1}^2+\|v\|_{H^1}^2\right),\tag{75}$$

$$\|u\|_{H^1}^2 \leq K_4\left(\|\eta\|_{H^1}^2+\|\omega\|_{H^1}^2\right).\tag{76}$$

Proof: From (9), and omitting time-dependence,

$$\begin{aligned}\|\omega\|_{L^2}^2 =& 3\|u\|_{L^2}^2+3\int_0^1\left(\int_0^x k(x,y)u(y,t)dy\right)^2dx\\&+3\int_0^1\left(\int_0^1 l(x,y)v(y,t)dy\right)^2dx\end{aligned}\tag{77}$$

and (73) follows applying the Cauchy-Schwarz inequality. Equation (74) is similarly proven. To prove (75), differentiating (9), integrating by parts and using the kernel equations:

$$\begin{aligned}\omega_x =& u_x-v(x)\mu(x)-\int_0^x k(x,y)u_y(y)dy\\&+\epsilon\int_0^1 l_y(x,y)v_y(y,t)dy-\int_0^1\lambda l(x,y)v(y)dy\\&+\int_0^x\left[k(x,y)\mu(y)-f(x,y)\right]v(y)dy\\&+\int_0^x\left[\int_y^x k(x,s)f(s,y)ds\right]v(y)dy\end{aligned}\tag{78}$$

and again repeated application of Cauchy-Schwarz leads to the result. Finally, (76) can be found by inverting (78) as in Proposition 3.

VI. Simulation Results

The effectiveness of the control law $U(t)$ is shown through a simple example. We consider plant (1)–(4) with coefficients $\epsilon=1.5$, $\lambda=2$, $\mu=\mathrm{e}^x$, $g(x)=5x$, $f(x,y)=1.5\mathrm{e}^{(1-y)}$. The initial states are set as $v(x,0)=1-x$, $u(x,0)=1-3.3x$ that verify zero-order compatibility conditions, and the adjustment parameter c is chosen as 5. τ is set to 0.01. We solve states of coupled system and the kernel functions $k(x,y)$, $l(x,y)$ with the finite central difference method in Matlab, where the grid size is chosen as $dt=0.0002, dy=0.001$. The gains $l(1,y),k(1,y)$ are calculated, as shown in Fig. 1. When we do not apply a control input (i.e. $U(t)=0$) the states diverge quickly over time (not shown due to lack of space). When we apply the controller (5) to the mixed PDE system, the evolution of $\|v(\cdot,t)\|_{H^1}+\|u(\cdot,t)\|_{H^1}$ is shown in Fig. 2, together with the control law; it decays exponentiallly to zero, which is consistent with the result in Theorem 1.

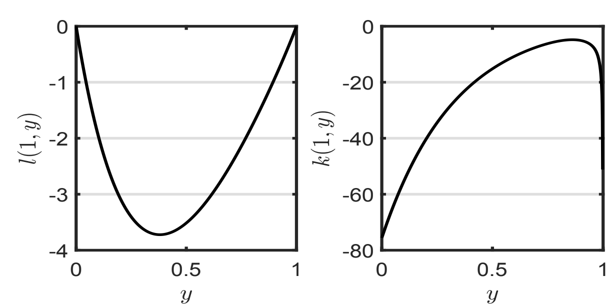

Fig. 1. The gain kernel functions $\boldsymbol{k(1,y),l(1,y)}$

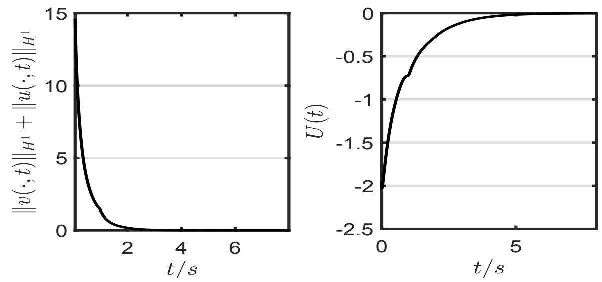

Fig. 2. Evolution of $\|v(\cdot,t)\|_{H^1} + \|u(\cdot,t)\|_{H^1}$ and $U(t)$

VII. Conclusions

This paper presents a boundary feedback control law for a mixed hyperbolic-parabolic PDE system with several interior couplings, extending previous results; in particular, the well-posedness of kernel equations is proven using an infinite induction energy series. Closed-loop well-posedness is analyzed, in the appropriate functional spaces. Finally, numerical simulation is implemented to validate the effectiveness of the proposed controller. Future work includes considering a fully-coupled mixed system with two controls.

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