Nonlinear Control of the Viscous Burgers Equation: Trajectory Generation, Tracking, and Observer Design

Miroslav Krstic[1]
Professor
Department of Mechanical & Aerospace Engineering,
University of California, San Diego,
La Jolla, CA 92093-0411
e-mail: krstic@ucsd.edu

Lionel Magnis
Ecole des Mines de Paris,
60 Boulevard Saint Michel,
75006 Paris, France
e-mail: lionel.magnis@ensmp.fr

Rafael Vazquez
Assistant Professor
Departmento de Ingeniería Aeroespacial,
Universidad de Sevilla,
Avenida, de los Descubrimientos s.n.,
41092 Sevilla, Spain
e-mail: rvazquez@ucsd.edu

In a companion paper we have solved the basic problem of full-state stabilization of unstable "shock-like" equilibrium profiles of the viscous Burgers equation with actuation at the boundaries. In this paper we consider several advanced problems for this nonlinear partial differential equation (PDE) system. We start with the problems of trajectory generation and tracking. Our algorithm is applicable to a large class of functions of time as reference trajectories of the boundary output, though we focus in more detail on the special case of sinusoidal references. Since the Burgers equation is not globally controllable, the reference amplitudes cannot be arbitrarily large. We provide a sufficient condition that characterizes the allowable amplitudes and frequencies, under which the state trajectory is bounded and tracking is achieved. We then consider the problem of output feedback stabilization. We design a nonlinear observer for the Burgers equation that employs only boundary sensing. We employ its state estimates in an output feedback control law, which we prove to be locally stabilizing. The output feedback law is illustrated with numerical simulations of the closed-loop system. [DOI: 10.1115/1.3023128]

1 Introduction

The viscous Burgers equation is considered a basic model of nonlinear convective-diffusive phenomena such as those that arise in Navier–Stokes equations. We study several nonlinear control problems for the system

$$u_t = u_{xx} - u_x u \tag{1}$$

with boundary conditions

$$u_x(0,t) = \omega_0(t), \quad u_x(1,t) = \omega_1(t) \tag{2}$$

where $\omega_0(t)$ and $\omega_1(t)$ are the controls and $u(x,t)$ is the state variable, for $x \in [0,1]$. To save space, we drop the arguments (x,t) whenever the context allows.

In a companion paper [1] we studied the problem of stabilization of a family of stationary solutions called "shock profiles" [2] (or "shocklike" profiles), given by

$$U(x) = -2\sigma \tanh(\sigma(x-1/2)) \tag{3}$$

which are parametrized by $\sigma > 0$, unstable in open loop, i.e., with constant boundary conditions

$$\omega_0 = \omega_1 = -2\sigma^2(1-\tanh^2(\sigma/2)) \leq 0 \tag{4}$$

and not stabilizable (even locally) by simple means, such as the "radiation boundary conditions" (see Refs. [3–9] for the uses of *nonlinear* radiation boundary conditions for *global* asymptotic stabilization of neutrally stable equilibria of the Burgers equation and other PDEs). Let us denote the perturbation variable around the shock profile as $\tilde{u}(x,t)=u(x,t)-U(x)$, and $\tilde{\omega}_0(t)=\omega_0(t)-U'(0)$, $\tilde{\omega}_1(t)=\omega_1(t)-U'(1)$. The Burgers equation written in $\tilde{u}$ is

$$\tilde{u}_t = \tilde{u}_{xx} - U(x)\tilde{u}_x - U'(x)\tilde{u} - \tilde{u}_x\tilde{u} \tag{5}$$

with boundary controls

$$\tilde{u}_x(0,t) = \tilde{\omega}_0(t), \quad \tilde{u}_x(1,t) = \tilde{\omega}_1(t) \tag{6}$$

In Ref. [1] we presented a full-state boundary feedback law for stabilization of the shock-like unstable profiles of the Burgers equation and provided an estimate of the region of attraction for the closed-loop system. This estimate is finite, which is consistent with the fact that the Burgers system is not globally controllable [10].

In this paper we first present results for trajectory generation and tracking for the Burgers equation. Our design is based on a nonlinear spatially-scaled Hopf–Cole style transformation (which is also used in the full-state design [1]) that transforms the system (with the help of one of the two boundary controls) into a linear reaction-diffusion PDE with nonlinear boundary conditions. For the resulting linear PDE, the general trajectory generation problem has been solved in Ref. [11]. However, in this paper we introduce explicit solutions to this problem for a particular class of reference functions of time. While we do not achieve a global result due to the lack of global controllability mentioned above, for the case of tracking a sinusoid in time we give a bound that quantifies the trade-off between the maximum amplitudes and frequencies for which tracking is achieved.

Then we turn our attention to an output-feedback problem. In Ref. [12] output feedback for a marginally stable 2D Burgers equation with in-domain actuation was solved using nonlinear model reduction techniques. Our design of a nonlinear observer (with gains computed using the backstepping observer design method [13]) uses injection of the output estimation error (from the boundaries into the interior of the PDE domain). We combine the observer with the full-state feedback design in Ref. [1]. The resulting output feedback is fully collocated and decentralized, as in the case with "radiation boundary conditions." However, while our feedback at one of the boundaries is static (like the "radiation" feedback), at the other boundary it is dynamic (and nonlinear). We prove the exponential stability of the linearized closed-loop system. The stabilization properties of the observer-based feedback laws are illustrated in simulations.

[1]Corresponding author.

Contributed by the Dynamic Systems, Measurement, and Control Division of ASME for publication in the JOURNAL OF DYNAMIC SYSTEMS, MEASUREMENT, AND CONTROL. Manuscript received January 14, 2008; final manuscript received August 1, 2008; published online February 9, 2009. Assoc. Editor: Fen Wu.

2 Full-State Stabilization of the Shock Profiles of the Burgers Equation

This section summarizes the result in Ref. [1]. In Ref. [1] we designed the following full-state feedback law for achieving stabilization of the shocklike equilibria:

$$\tilde{\omega}_0 = 2\sigma \tanh(\sigma/2)\tilde{u}(0) + \tilde{u}^2(0)/2 \tag{7}$$

$$\begin{aligned}\tilde{\omega}_1 = \frac{\tilde{u}(1)^2}{2} &+ (k(1,1) - 2\sigma\tanh(\sigma/2))\tilde{u}(1) \\ &+ \int_0^1 (k_x(1,y) + \sigma\tanh(\sigma/2)k(1,y)) \\ &\times G(y)\mathrm{e}^{\int_y^1 \tilde{u}(\xi)d\xi}\tilde{u}(y)dy\end{aligned} \tag{8}$$

where

$$G(x) = \frac{\cosh(\sigma(x-1/2))}{\cosh(\sigma/2)} \tag{9}$$

and the kernel $k(x,y)$ in Eq. (8) is computed from

$$\begin{aligned}k_{xx} = k_{yy} &+ \sigma^2[1 - 2\tanh^2(\sigma(y-1/2)) \\ &+ \tanh^2(\sigma(x-1/2))]k + ck\end{aligned} \tag{10}$$

$$k(x,x) = -\frac{\sigma}{2}[\tanh(\sigma(x-1/2)) + \tanh(\sigma/2)] - \frac{cx}{2} \tag{11}$$

$$k_y(x,0) = \sigma\tanh(\sigma/2)k(x,0) \tag{12}$$

which is a linear hyperbolic PDE in the domain $\mathcal{T}=\{(x,y): 0 \leqslant y \leqslant x \leqslant 1\}$. The kernel k can be computed from Eqs. (10)–(12) numerically or symbolically using procedures outlined in Ref. [14]. The constant c in Eqs. (10) and (11) is a design parameter that is chosen positive if $\sigma=0$ and can be set to 0 otherwise.

3 Trajectory Generation

Given systems (1) and (2), with the two inputs $\omega_0(t)$ and $\omega_1(t)$, we consider a trajectory generation problem with $u(0,t)$ as the system's single output. Then, the problem of trajectory generation consists of finding open-loop control input functions $\omega_0^r(t)$, and $\omega_1^r(t)$ to make $u(0,t)$ evolve *exactly* according to a given reference signal $u^r(0,t)$.

We use the invertible transformation

$$v(x,t) = u(x,t)\mathrm{e}^{-1/2\int_0^x u(y,t)dy} \tag{13}$$

$$u(x,t) = \frac{v(x,t)}{1 - \frac{1}{2}\int_0^x v(y,t)dy} \tag{14}$$

which converts the Burgers system (1) into the form

$$v_t(x,t) = v_{xx}(x,t) + v(x,t)\frac{1}{2}\left(\omega_0(t) - \frac{1}{2}v(0,t)^2\right) \tag{15}$$

$$v_x(0,t) = \omega_0(t) - \frac{1}{2}v(0,t)^2 \tag{16}$$

$$\begin{aligned}v_x(1,t) = &\left(1 - \frac{1}{2}\int_0^1 v(y,t)dy\right)\omega_1(t) \\ &- \frac{1}{2}\frac{v(1,t)^2}{1 - \frac{1}{2}\int_0^1 v(y,t)dy}\end{aligned} \tag{17}$$

From Eq. (13) we obtain $v(0,t)=u(0,t)$, hence we get that $v^r(0,t)=u^r(0,t)$. Thus the trajectory generation for the u-system (1) can be approached as a trajectory generation problem for the v-system (15). Then we are looking for functions $v^r(x,t)$, $\omega_0^r(t)$, and $\omega_1^r(t)$ that satisfy Eqs. (15)–(17) and $v^r(0,t)=u^r(0,t)$.

We choose the control at $x=0$ as

$$\omega_0^r(t) = \frac{1}{2}u^r(0,t)^2 \tag{18}$$

which, substituted into Eq. (15), simplifies the nonlinear trajectory generation problem to the trajectory generation problem for the linear heat equation (with nonlinear boundary conditions)

$$v_t^r(x,t) = v_{xx}^r(x,t) \tag{19}$$

$$v_x^r(0,t) = 0 \tag{20}$$

$$\begin{aligned}v_x^r(1,t) = &\left(1 - \frac{1}{2}\int_0^1 v^r(y,t)dy\right)\omega_1^r(t) \\ &- \frac{1}{2}\frac{v^r(1,t)^2}{1 - \frac{1}{2}\int_0^1 v^r(y,t)dy}\end{aligned} \tag{21}$$

A general infinite-series solution for $v^r(x,t)$ in Eqs. (19) and (20) exists for a very broad class of functions of time $u^r(0,t)$ (the Gevrey class), which has been developed in the framework of differential flatness [11]. Furthermore, an explicit solution can be derived for any function $u^r(0,t)$ that can be written as an output of a linear exosystem. For example, if

$$u^r(0,t) = b + a\sin\omega t \tag{22}$$

i.e., we want to track a sinusoid with a bias, then the explicit solution for the reference state is

$$v^r(x,t) = b + a\,\mathrm{Im}\{\cosh(\sqrt{j\omega}x)\mathrm{e}^{j\omega t}\} \tag{23}$$

Once $v^r(x,t)$ is found, the input reference $\omega_1^r(t)$ is computed from Eq. (21) as

$$\omega_1^r(t) = \frac{\frac{1}{2}v^r(1,t)^2 + \left(1 - \frac{1}{2}\int_0^1 v^r(y,t)dy\right)v_x^r(1,t)}{\left(1 - \frac{1}{2}\int_0^1 v^r(y,t)dy\right)^2} \tag{24}$$

Formula (24) requires that both the derivative and the integral of the state trajectory $v^r(x,t)$ be known. In the case of the biased sinusoidal output reference (22) they are easily obtainable as

$$v_x^r(x,t) = a\,\mathrm{Im}\{\sqrt{j\omega}\sinh(\sqrt{j\omega}x)\mathrm{e}^{j\omega t}\} \tag{25}$$

and

$$\int_0^x v_x^r(y,t)dy = a\,\mathrm{Im}\left\{\frac{\sinh(\sqrt{j\omega}x)\mathrm{e}^{j\omega t}}{\sqrt{j\omega}}\right\} + bx \tag{26}$$

It can be shown that the following result holds for the particular case of the output given by Eq. (22).

THEOREM 1. *The following functions*:

$$u^r(x,t) = \frac{b + a\,\mathrm{Im}\{\cosh(\sqrt{j\omega}x)\mathrm{e}^{j\omega t}\}}{1 - \frac{bx}{2} - \frac{a}{2}\mathrm{Im}\left\{\frac{\sinh(\sqrt{j\omega}x)\mathrm{e}^{j\omega t}}{\sqrt{j\omega}}\right\}} \tag{27}$$

$$\omega_0^r(t) = \frac{1}{2}(b + a\sin\omega t)^2 \tag{28}$$

$$\omega_1^r(t) = \frac{1}{\left(1 - \frac{b}{2} - \frac{a}{2}\,\mathrm{Im}\left\{\frac{\sinh(\sqrt{j\omega})}{\sqrt{j\omega}} e^{j\omega t}\right\}\right)^2} \times \left[\frac{1}{2}(b + a\,\mathrm{Im}\{\cosh(\sqrt{j\omega})e^{j\omega t}\})^2 + \left(1 - \frac{b}{2} - \frac{a}{2}\,\mathrm{Im}\left\{\frac{\sinh(\sqrt{j\omega})}{\sqrt{j\omega}} e^{j\omega t}\right\}\right) \times a\,\mathrm{Im}\{\sqrt{j\omega}\sinh(\sqrt{j\omega})e^{j\omega t}\}\right] \quad (29)$$

satisfy the nonlinear PDE

$$u_t^r(x,t) = u_{xx}^r(x,t) - u_x^r(x,t)u^r(x,t) \quad (30)$$

$$u_x^r(0,t) = \omega_0^r(t) \quad (31)$$

$$u_x^r(1,t) = \omega_1^r(t) \quad (32)$$

and, in particular, $u^r(0,t) = b + a \sin \omega t$.

Remark 3.1. Functions (27)–(29) that solve the trajectory generation problem do not scale linearly with the amplitude a or the bias b. Furthermore, for values of a or b that are sufficiently large, the possibility exists of these functions taking infinite values for some (x,t) pairs.

4 Trajectory Tracking

While the open-loop system *might* be stable (in rare cases) around the trajectory found in the trajectory generation problem, usually this is not so. To solve the trajectory tracking problem, we need to find a feedback law that stabilizes the trajectory $u^r(x,t)$ from any initial condition $u(x,0)$ (rather than just generating the desired motion from the special initial condition $u^r(x,0)$).

We introduce the state tracking error $\tilde{v}(x,t) = v(x,t) - v^r(x,t)$. Its *linearization* of Eqs. (15)–(17) around the reference trajectory $v^r(x,t)$ is

$$\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t) - \frac{1}{2}u^r(0,t)v^r(x,t)\tilde{v}(0,t) \quad (33)$$

$$\tilde{v}_x(0,t) = -u^r(0,t)\tilde{v}(0,t) \quad (34)$$

$$\tilde{v}_x(1,t) = -\frac{v^r(1,t)}{1 - \frac{1}{2}\int_0^1 v^r(y,t)dy}\tilde{v}(1,t) - \frac{\frac{1}{2}\int_0^1 \tilde{v}(y,t)dy}{\left(1 - \frac{1}{2}\int_0^1 v^r(y,t)dy\right)^2}\left[-\frac{1}{2}v^r(1,t)^2 + \left(1 - \frac{1}{2}\int_0^1 v^r(y,t)dy\right)v_x^r(1,t)\right] \quad (35)$$

This complicated linear time-varying PDE system in general will not be exponentially stable, thus we need to develop feedback control laws to stabilize the equilibrium $\tilde{v}(x) \equiv 0$.

The nonlinear PDE governing the tracking error $\tilde{v}(x,t)$ is

$$\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t) + v(x,t)\frac{1}{2}\left(\omega_0(t) - \frac{1}{2}u(0,t)^2\right) \quad (36)$$

$$\tilde{v}_x(0,t) = \omega_0(t) - \frac{1}{2}u(0,t)^2 \quad (37)$$

$$\tilde{v}_x(1,t) = \left(1 - \frac{1}{2}\int_0^1 v(y,t)dy\right) \times \left(\omega_1(t) - \frac{1}{2}u(1,t)^2\right) - v_x^r(1,t) \quad (38)$$

First we choose the control $\omega_0(t)$ as the feedback law

$$\omega_0(t) = \frac{1}{2}u(0,t)^2 \quad (39)$$

which changes Eqs. (36) and (37) into $\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t)$, $\tilde{v}_x(0,t) = 0$, while Eq. (38) is unchanged. Next, we choose

$$\omega_1(t) = c_1 u(1,t) + \frac{1}{2}u(1,t)^2 + e^{1/2\int_0^1 u(y,t)dy}(v_x^r(1,t) + c_1 v^r(1,t)) \quad (40)$$

where c_1 is a positive gain. Using Eqs. (13) and (14), it is found that this control law transforms Eq. (38) into $\tilde{v}_x(1,t) = -c_1\tilde{v}(1,t)$. Hence the closed-loop $\tilde{v}$-system is turned into a heat equation with one Neumann and one stabilizing Robin boundary condition

$$\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t) \quad (41)$$

$$\tilde{v}_x(0,t) = 0 \quad (42)$$

$$\tilde{v}_x(1,t) = -c_1\tilde{v}(1,t) \quad (43)$$

Using a Lyapunov estimate as in Ref. [1], we find that the closed-loop $\tilde{v}$ satisfies the following bound

$$\|\tilde{v}(t)\|_{L^2} \leq \|\tilde{v}(0)\|_{L^2} e^{-\tilde{c}t} \quad (44)$$

for some $\tilde{c} > 0$ (whose exact value is not important).

The solution to the plant state $u(x,t)$ is

$$u(x,t) = \frac{v^r(x,t) + \tilde{v}(x,t)}{1 - \frac{1}{2}\int_0^x v^r(y,t)dy - \frac{1}{2}\int_0^x \tilde{v}(y,t)dy} \quad (45)$$

where $\tilde{v}(x,t)$ is the solution of (41)–(43) with initial condition $\tilde{v}_0 = u(x,0)e^{-1/2\int_0^x u(y,0)dy} - v^r(x,0)$. Since $\lim_{t\to\infty}\tilde{v}(x,t) \equiv 0$, we have that $u(x,t)$ converges to

$$u^r(x,t) = \frac{v^r(x,t)}{1 - \frac{1}{2}\int_0^x v^r(y,t)dy} \quad (46)$$

However, the tracking result fails to be global (i.e., to hold for all initial conditions) because solution (45) is only valid if the condition

$$\int_0^x v^r(y,t)dy + \int_0^x \tilde{v}(y,t)dy < 2 \quad (47)$$

is verified for all x and t. This condition holds when $u^r(0,t) \equiv 0$, namely, when $v^r(x,t) \equiv 0$ (which is a basic result of stabilizing u around the origin, which is global), however, it does not necessarily hold in the presence of a nonzero trajectory $v^r(x,t)$.

The following theorem describes the behavior of the closed-loop system with our tracking controller.

THEOREM 2. *Consider system (1) and (2) with control laws (39) and (40), where $v^r(x,t)$ is a solution of the motion planning problem in Sec. 3. Let the reference trajectory $v^r(x,t)$ be bounded for all $x \in [0,1]$, $t \geq 0$, and let $v_x^r(1,t)$ be bounded for all $t \geq 0$. If the following conditions hold*

$$\|v^r(t)\|_{L^2}^2 \le \frac{1}{4}, \quad \forall\, t \ge 0 \tag{48}$$

$$\|u_0\|_{L^2}^2 \le h^{-1}\left(\frac{1}{4}\right) \tag{49}$$

where $h(r)=re^{\sqrt{r}}$, *then the solution* $u(x,t)$ *is bounded for all* $x \in [0,1]$ *and* $t \geq 0$, *the control inputs* $\omega_0(t)$ *and* $\omega_1(t)$ *are bounded for all* $t \geq 0$, *and the function* $\tilde{u}(x,t)=u(x,t)-u^r(x,t)$ *converges to zero exponentially in* t, *for all* $x \in [0,1]$.

Proof. We guarantee the boundedness of $u(x,t)$ by observing that $\tilde{v}(x,t)$ is bounded (as a solution to a heat equation), and by proving that

$$\|v^r(t)\|_{L^2} + \|\alpha(t)\|_{L^2} + \|\beta(t)\|_{L^2} \le \frac{3}{2} \tag{50}$$

where

$$\alpha_t = \alpha_{xx} \tag{51}$$

$$\alpha_x(0,t) = 0 \tag{52}$$

$$\alpha_x(1,t) = -c_1\alpha(1,t) \tag{53}$$

$$\alpha_0 = u_0 e^{-1/2\int_0^x u_0(y)dy} \tag{54}$$

and

$$\beta_t = \beta_{xx} \tag{55}$$

$$\beta_x(0,t) = 0 \tag{56}$$

$$\beta_x(1,t) = -c_1\beta(1,t) \tag{57}$$

$$\beta_0 = v_0^r \tag{58}$$

We first note that $\|\alpha(t)\|_{L^2} \leq \|\alpha_0\|_{L^2}$, $\forall t \geq 0$, and $\|\beta(t)\|_{L^2} \leq \|\beta_0\|_{L^2}$, $\forall t \geq 0$. We also have that $h(\|u_0\|_{L^2}^2) \leq 1/4$ implies that $\|\alpha_0\|_{L^2} \leq 1/2$. Next, we observe that $\|v^r(t)\|_{L^2}^2 \leq \frac{1}{4}$, $\forall t \geq 0$, implies that $\|\beta_0\|_{L^2} \leq 1/2$.

As discussed in Ref. [1], the solution of the error system $\tilde{u}(x,t)$ is defined in the space $H^{2,1}$, which is the space of functions of x and t whose derivatives (from the zeroth) up to xxt have bounded L_2 norms in space-time.

We now apply Theorem 2 to our example with a sinusoidal output (Eq. (22)) with $b=0$ (no bias).

THEOREM 3. *Consider the closed-loop Burgers systems (1) and (2) with the controls*

$$\omega_0(t) = \frac{1}{2}u(0,t)^2 \tag{59}$$

$$\begin{aligned}\omega_1(t) &= -c_1 u(1,t) + \frac{1}{2}u(1,t)^2 \\ &\quad + e^{1/2\int_0^1 u(y,t)dy} a\, \mathrm{Im}\{(\sqrt{j\omega}\sinh(\sqrt{j\omega}x) \\ &\quad + c_1\cosh(\sqrt{j\omega}x))e^{j\omega t}\}\end{aligned} \tag{60}$$

If $\|u_0\|_{L^2}^2 \leq h^{-1}(1/4)$ *and*

$$a \le a_{\max}(\omega) = \frac{1}{8}\sqrt{\frac{2\omega}{\cosh\sqrt{2\omega} - \cos\sqrt{2\omega}}} \tag{61}$$

where $a_{\max}(\omega)$ *is a positive, decreasing function with* $a_{\max}(0) = 1/8$, *then the state and the control inputs remain bounded and the state* $u(x,t)$ *converges to*

$$u^r(x,t) = \frac{a\,\mathrm{Im}\{\cosh(\sqrt{j\omega}x)e^{j\omega t}\}}{1 - \frac{1}{2}a\,\mathrm{Im}\left\{\frac{1}{\sqrt{j\omega}}\sinh(\sqrt{j\omega}x)e^{j\omega t}\right\}} \tag{62}$$

which means, in particular, that $u(0,t)$ *converges to* $u^r(0,t) = a\sin\omega t$.

Proof. All we need to prove is that

$$a\left|\mathrm{Im}\left\{\frac{1}{\sqrt{j\omega}}\sinh(\sqrt{j\omega}x)e^{j\omega t}\right\}\right| \le 1/2 \tag{63}$$

To do this, we use the fact that the absolute value of an imaginary part of a complex number is no greater than the modulus of the complex number, and that the modulus of a product of two complex numbers is no greater than twice the product of the moduli of the two complex numbers. Hence,

$$\begin{aligned}\left|\mathrm{Im}\left\{\frac{1}{\sqrt{j\omega}}\sinh(\sqrt{j\omega}x)e^{j\omega t}\right\}\right| &\le \left|\frac{1}{\sqrt{j\omega}}\sinh(\sqrt{j\omega}x)e^{j\omega t}\right| \\ &\le 2\left|\frac{1}{\sqrt{\omega}}\sinh(\sqrt{j\omega}x)e^{j\omega t}\right| \\ &\le 4\left|\frac{1}{\sqrt{\omega}}\sinh(\sqrt{j\omega}x)\right| \\ &= 4\sqrt{\frac{\cosh(\sqrt{2\omega}x) - \cos(\sqrt{2\omega}x)}{2\omega}}\end{aligned} \tag{64}$$

Taking a derivative, it can be seen that the function in the numerator increases in x. Hence, we get $|\mathrm{Im}\{(1/\sqrt{j\omega})\sinh(\sqrt{j\omega}x)e^{j\omega t}\}| \leq 4\sqrt{(\cosh(\sqrt{2\omega})-\cos(\sqrt{2\omega}))/2\omega}$, which proves Eq. (63).

5 Simulation Example of Trajectory Generation and Tracking

We illustrate the solution to the trajectory tracking problem for the output reference $u^r(0,t)=b+a\sin\omega t$ from Theorem 1. The explicit state trajectory is

$$\begin{aligned}u^r(x,t) &= \left\{b + \frac{a}{2}\left[e^{\sqrt{(\omega/2)}x}\sin\left(\omega t + \sqrt{\frac{\omega}{2}}x\right)\right.\right. \\ &\quad \left.\left.+ e^{-\sqrt{(\omega/2)}x}\sin\left(\omega t - \sqrt{\frac{\omega}{2}}x\right)\right]\right\} \\ &\quad \times\left\{1 - \frac{b}{2}x - \frac{a}{4\sqrt{\omega}}\left[e^{\sqrt{(\omega/2)}x}\sin\left(\omega t - \frac{\pi}{4} + \sqrt{\frac{\omega}{2}}x\right)\right.\right. \\ &\quad \left.\left.+ e^{-\sqrt{(\omega/2)}x}\sin\left(\omega t - \frac{\pi}{4} - \sqrt{\frac{\omega}{2}}x\right)\right]\right\}^{-1}\end{aligned} \tag{65}$$

The presence of the b term in the denominator of Eq. (65) will increase the possibility of a blow-up of the trajectory when $b > 0$, however $b < 0$ will help to keep the denominator away from 0.

Figure 1 shows the plot of the solution to the nonlinear trajectory generation problem. Figure 2 shows the trajectory tracking starting from a zero initial condition for the PDE state, $u(x,0) \equiv 0$, which is different from the "ideal" reference initial condition obtained by setting $t=0$ in Eq. (65). The speed of response can be further improved by increasing the control gain c_1 or by modifying the feedback law (40) to impart an arbitrarily fast decay, using the backstepping design tools in Ref. [14].

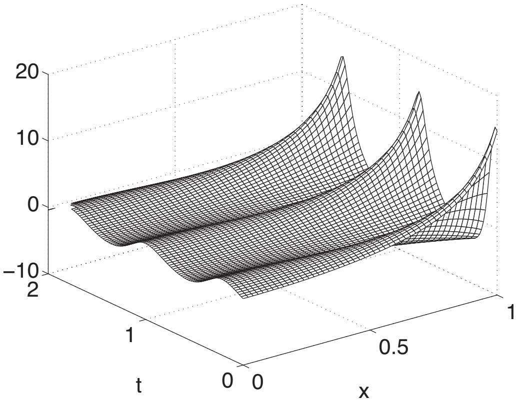

Fig. 1 Solution to the nonlinear trajectory generation problem for a sinusoidal reference with $b=0$, $a=1.2$, $\omega=8$

6 Nonlinear Observer

The control in Sec. 2 requires us to know the state $\tilde{u}(x,t)$ for all $x\in[0,1]$. We now design output feedback laws for $\tilde{\omega}_0(t)$ and $\tilde{\omega}_1(t)$ using the boundary measurement of $\tilde{u}(0,t)$ and $\tilde{u}(1,t)$ only, which is a two-input two-output (TITO) problem. We will design fully collocated (decentralized) feedback laws, i.e., using only the measurement of $\tilde{u}(0,t)$ for the control $\tilde{\omega}_0(t)$, and only the measurement of $\tilde{u}(1,t)$ for $\tilde{\omega}_1(t)$.

Notice that the control law (7) is already an output feedback law requiring only the knowledge of $\tilde{u}(0)$. We saw in Ref. [1] that applying Eq. (7) and the mapping

$$v(x,t)=G(x)\tilde{u}(x,t)\mathrm{e}^{-1/2\int_0^x\tilde{u}(y,t)dy} \tag{66}$$

which has the inverse

$$\tilde{u}(x)=\frac{v(x)/G(x)}{1-\dfrac{1}{2}\displaystyle\int_0^x\frac{v(y)}{G(y)}dy} \tag{67}$$

transforms the plant into the linear system

$$v_t=v_{xx}+\sigma^2\left[\frac{2}{\cosh^2(\sigma(x-1/2))}-1\right]v \tag{68}$$

$$v_x(0)=\sigma\tanh(\sigma/2)v(0) \tag{69}$$

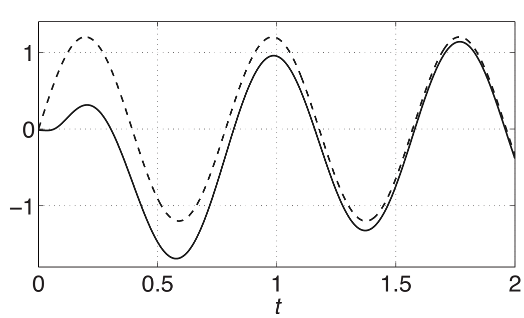

Fig. 2 Trajectory tracking: dashed line: output reference $u^r(0,t)=1.2\sin(8t)$; and solid line: output response $u(0,t)$ with control gain $c_1=5$

$$v_x(1)=\sigma\tanh(\sigma/2)v(1)+\left(1-\frac{1}{2}\int_0^1\frac{v(y)}{G(y)}dy\right)\times\left(\tilde{\omega}_1-\frac{1}{2}\tilde{u}(1)^2\right) \tag{70}$$

Hence the problem reduces to the design of an observer-based feedback controller for Eqs. (68)–(70) using the measurement of $\tilde{u}(1,t)$.

We start with an observation that the boundary condition (70) contains a state of nonlinearity given by the integral term in $v(y)$. Our observer is designed as a copy of the (nonlinear) plant with the injection of the output error

$$\hat{v}_t=\hat{v}_{xx}+\sigma^2\left[\frac{2}{\cosh^2(\sigma(x-1/2))}-1\right]\hat{v}+\left[\left(1-\frac{1}{2}\int_0^1\frac{\hat{v}(y)}{G(y)}dy\right)\tilde{u}(1)-\hat{v}(1)\right]\rho(x) \tag{71}$$

$$\hat{v}_x(0)=\sigma\tanh(\sigma/2)\hat{v}(0) \tag{72}$$

$$\hat{v}_x(1)=(\sigma\tanh(\sigma/2)+\rho_1)\left[\left(1-\frac{1}{2}\int_0^1\frac{\hat{v}(y)}{G(y)}dy\right)\times\tilde{u}(1)-\hat{v}(1)\right]+\left(1-\frac{1}{2}\int_0^1\frac{\hat{v}(y)}{G(y)}dy\right)\times\left(\tilde{\omega}_1-\frac{1}{2}\tilde{u}(1)^2+\sigma\tanh(\sigma/2)\tilde{u}(1)\right) \tag{73}$$

where $\hat{v}(x,t)$ denotes the estimate of the state $v(x,t)$. Notice that, using Eq. (67),

$$\left(1-\frac{1}{2}\int_0^1\frac{v(y)}{G(y)}dy\right)\tilde{u}(1)=v(1)/G(1)=v(1) \tag{74}$$

since $G(1)=1$. Hence the term

$$\left(1-\frac{1}{2}\int_0^1\frac{\hat{v}(y)}{G(y)}dy\right)\tilde{u}(1) \tag{75}$$

appearing in Eqs. (71)–(73), is an estimate of $v(1)$ and is used for output injection. The gains $\rho(x)$ and ρ_1 are determined to ensure the convergence of $\hat{v}$ to v.

7 Design of Output Injection Gains Using Backstepping

To design $\rho(x)$ and ρ_1 we use the backstepping method for observer design [13]. First, we denote the observer error as $e(x,t)=v(x,t)-\hat{v}(x,t)$. Subtracting Eqs. (71)–(73) from Eqs. (68)–(70) we get

$$e_t=e_{xx}+\sigma^2\left[\frac{2}{\cosh^2(\sigma(x-1/2))}-1\right]e-\rho(x)\frac{(\hat{v}(1)+e(1))\dfrac{1}{2}\displaystyle\int_0^1\frac{e(y)}{G(y)}dy}{1-\dfrac{1}{2}\displaystyle\int_0^1\frac{\hat{v}(y)}{G(y)}dy-\dfrac{1}{2}\displaystyle\int_0^1\frac{e(y)}{G(y)}dy}-\rho(x)e(1) \tag{76}$$

$$e_x(0)=\sigma\tanh(\sigma/2)e(0) \tag{77}$$

$$e_x(1) = -(\sigma\tanh(\sigma/2)+\rho_1)e(1) - \frac{1}{2}\int_0^1 \frac{e(y)}{G(y)}dy \times \left[\left(\int_0^1 [k_x(1,y)+\sigma\tanh(\sigma/2)k(1,y)] \times \hat{v}(y)dy\right)\left(1-\frac{1}{2}\int_0^1\frac{\hat{v}(y)}{G(y)}dy\right)^{-1} + \frac{(\rho_1+k(1,1))(\hat{v}(1)+e(1))}{1-\frac{1}{2}\int_0^1\frac{\hat{v}(y)}{G(y)}dy - \frac{1}{2}\int_0^1\frac{e(y)}{G(y)}dy}\right] \quad (78)$$

We now linearize Eqs. (76)–(78) around the origin, obtaining

$$e_t = e_{xx} + \sigma^2\left[\frac{2}{\cosh^2(\sigma(x-1/2))} - 1\right]e - \rho(x)e(1) \quad (79)$$

$$e_x(0) = \sigma\tanh(\sigma/2)e(0) \quad (80)$$

$$e_x(1) = -(\sigma\tanh(\sigma/2)+\rho_1)e(1) \quad (81)$$

We need to design the gains $\rho(x)$ and ρ_1 so that the system (79)–(81) is exponentially stable. The plant (Eqs. (79)–(81)) is a linear 1D reaction-diffusion PDE with Robin boundary conditions, so the backstepping observer design method in Ref. [13] can be applied. We map the state $e(x,t)$ into a new variable $\eta(x,t)$ using the transformation

$$e(x) = \eta(x) - \int_x^1 p(x,y)\eta(y)dy \quad (82)$$

with η verifying the observer error target system

$$\eta_t = \eta_{xx} - [\sigma^2\tanh^2(\sigma(x-1/2))]\eta - c\eta \quad (83)$$

$$\eta_x(0) = \sigma\tanh(\sigma/2)\eta(0) \quad (84)$$

$$\eta_x(1) = -\sigma\tanh(\sigma/2)\eta(1) \quad (85)$$

The system (83)–(85) was shown in Ref. [1] to be exponentially stable in the L^2 norm.

Following the method in Ref. [13], we find the kernel $p(x,y)$ appearing in Eq. (82) from

$$p_{xx} - p_{yy} = -\sigma^2[1-2\tanh^2(\sigma(x-1/2)) + \tanh^2(\sigma(y-1/2))]p - cp \quad (86)$$

$$p(x,x) = -\frac{1}{2}[\sigma\tanh(\sigma(x-1/2)) + \sigma\tanh(\sigma/2) + cx] \quad (87)$$

$$p_x(0,y) = \sigma\tanh(\sigma/2)p(0,y) \quad (88)$$

Once $p(x,y)$ is computed, it is used to compute the output injection gains as follows:

$$\rho(x) = -(p_y(x,1) + \sigma\tanh(\sigma/2)p(x,1)) \quad (89)$$

$$\rho_1 = -p(1,1) \quad (90)$$

Comparing Eqs. (86)–(88) with Eqs. (10)–(12), we deduce that $p(x,y)=k(y,x)$. Hence it is not necessary to solve Eqs. (86)–(88) and we can use the solution for k in Eqs. (89) and (90), obtaining

$$\rho(x) = -(k_x(1,x) + \sigma\tanh(\sigma/2)k(1,x)) \quad (91)$$

$$\rho_1 = -k(1,1) = \sigma\tanh(\sigma/2) + \frac{c}{2} \quad (92)$$

Note that $\rho(x)$ in Eq. (91) is the same function as the control gain in Eq. (8), by just changing the sign. Thus, the output injection gains can be computed from the control kernel.

8 Output Feedback Law

Using control law (7) and the estimate $\hat{v}$ (which is transformed into an estimate of $\tilde{u}$ using Eq. (67)) in control law (8), we obtain our nonlinear output feedback control laws, which are defined as follows

$$\tilde{\omega}_0(t) = 2\sigma\tanh(\sigma/2)\tilde{u}(0,t) + \frac{\tilde{u}^2(0,t)}{2} \quad (93)$$

$$\tilde{\omega}_1(t) = \frac{1}{2}\tilde{u}(1,t)^2 + (-2\sigma\tanh(\sigma/2)+k(1,1))\tilde{u}(1,t) + \left(\int_0^1 [k_x(1,y)+\sigma\tanh(\sigma/2)k(1,y)] \times \hat{v}(y,t)dy\right)\left(1-\frac{1}{2}\int_0^1\frac{\hat{v}(y,t)}{G(y)}dy\right)^{-1} \quad (94)$$

Thus we have obtained a *diagonal* TITO compensator

$$\begin{bmatrix}\tilde{u}(0,t)\\ \tilde{u}(1,t)\end{bmatrix} \mapsto \begin{bmatrix}\tilde{\omega}_0(t)\\ \tilde{\omega}_1(t)\end{bmatrix} \quad (95)$$

with one static channel and one dynamic channel.

If we linearize the compensator, we obtain the following.

For Channel 0,

$$\tilde{\omega}_0 = 2\sigma\tanh(\sigma/2)\tilde{u}(0) \quad (96)$$

For Channel 1,

$$\tilde{\omega}_1 = -(2\sigma\tanh(\sigma/2)+\rho_1)\tilde{u}(1) - \int_0^1 \rho(y)\hat{v}(y)dy \quad (97)$$

$$\hat{v}_t = \hat{v}_{xx} + \sigma^2\left[\frac{2}{\cosh^2(\sigma(x-1/2))} - 1\right]\hat{v} - \rho(x)\hat{v}(1) + \rho(x)\tilde{u}(1) \quad (98)$$

$$\hat{v}_x(0) = \sigma\tanh(\sigma/2)\hat{v}(0) \quad (99)$$

$$\hat{v}_x(1) = -(\sigma\tanh(\sigma/2)+\rho_1)\hat{v}(1) - \int_0^1 \rho(y)\hat{v}(y)dy \quad (100)$$

The dynamic channel (Channel 1) is of relative degree zero (note the throughput term in Eq. (97)) and one can think of it as an infinite dimensional transfer function whose Bode plot is shown in Fig. 3 for $\sigma=15$.

9 Stability of the Linearized Output-Feedback System

THEOREM 4. *Consider the linearized $(\tilde{u},\hat{v})$ system given by $\tilde{u}_t = \tilde{u}_{xx} + 2\sigma\tanh(\sigma(x-1/2))\tilde{u}_x + 2\sigma^2(\cosh(\sigma(x-1/2)))\tilde{u}$ and Eqs. (6) and (96)–(100). The equilibrium $\tilde{u}\equiv\hat{v}\equiv 0$ is exponentially stable in the L^2 norm, i.e., there exists $C_1, c_1>0$ such that $\|\tilde{u}(t)\|_{L^2} + \|\hat{v}(t)\|_{L^2} \leq C_1 e^{-c_1 t}(\|\tilde{u}(0)\|_{L^2} + \|\hat{v}(0)\|_{L^2})$, for all $t\geq 0$.*

Proof. We just outline the proof. It is lengthy but it follows the standard constructions for linear reaction-diffusion systems stabilized by output feedback using the backstepping method (see

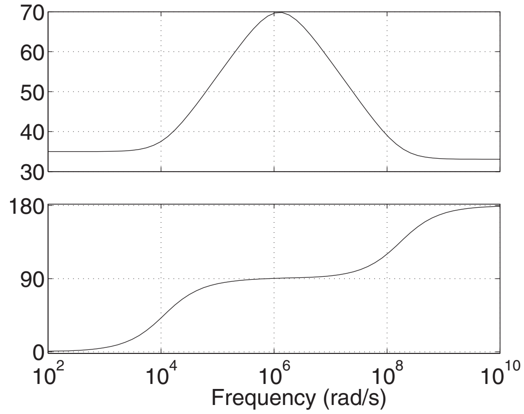

Fig. 3 The Bode plot of the compensator (Eqs. (97)–(100)) for $\sigma=15$

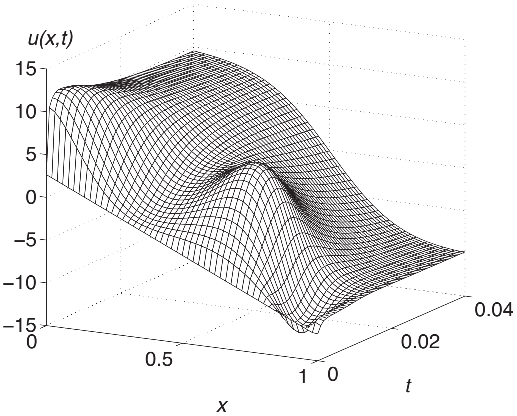

Fig. 5 Convergence of the closed-loop system under the linear output-feedback controller (Eqs. (96)–(100)) for $\sigma=5$

Refs. [13,15,16] for details). It starts by considering the linearized $(e,\hat{v})$ system given by Eqs. (79)–(81) and $\hat{v}_t=\hat{v}_{xx}+\sigma^2[2/\cosh^2(\sigma(x-1/2))-1]\hat{v}+\rho(x)e(1)$, with Eqs. (99) and (100). The key element in the proof is to use the (invertible) controller and observer backstepping transformations, $\hat{w}(x,t)=\hat{v}(x,t)-\int_0^x k(x,y)\hat{v}(y,t)dy$ and Eq. (82), to study the $(\hat{w},\eta)$ system instead of the $(\hat{v},e)$ system. The η system (83)–(85) is an exponentially stable autonomous system, whereas the $\hat{w}$ system (not given here; its autonomous part is identical to Eqs. (83)–(85)) is exponentially stable and driven by η. Using the Lyapunov function $\|\hat{w}\|^2+M\|\eta\|^2$, for sufficiently large M, we obtain exponential stability for the $(\hat{w},\eta)$ system. Using the direct and inverse backstepping transformations we get exponential stability estimates for $(\hat{v},e)$. Using the fact that $v=\hat{v}+e$, we get estimates for $(\hat{v},v)$. Finally applying the origin-preserving transformation $\tilde{u}(x,t)=v(x,t)/G(x)$, we get the estimates for $(\hat{v},\tilde{u})$ stated in Theorem 4.

10 Simulations With Output Feedback

All simulations in this section have been produced using a mixed Cranck–Nicholson/Runge–Kutta three scheme in time and a second order finite difference method in space. The open-loop system (1), (2), and (4) is unstable, as shown in Ref. [1]. A numerical study of the linearized system around the shock profile shows the presence of one positive (though possibly small) eigenvalue for any $\sigma>0$. In Fig. 4 one can see an apparent finite time blow-up of the open-loop system for $\sigma=3$. For the same initial conditions, we show in Fig. 5 the numerical evolution of the system with the linear *observer-based* backstepping controller in Sec. 6. This result is achieved for $c=0$; with $c>0$ the "hump" in the transient can be eliminated (which might be preferable mathematically, though not as interesting visually).

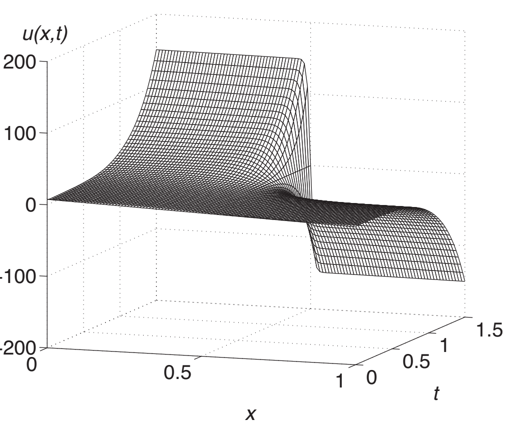

Fig. 4 Simulation of the open-loop system (with constant inputs, Eq. (4)) for $\sigma=3$, and $u_0(x)=U(0)+2+(U(1)-U(0)-4)x$ appears to exhibit a finite-time blow-up

11 Conclusions

We have solved the problems of motion planning, trajectory generation, observer design, and output feedback stabilization for the fully nonlinear viscous Burgers equation. The problem of full-state stabilization was solved in a companion paper [1]. Our results are based on a nonlinear feedback linearizing transformation, which allows us to use trajectory generation for the heat equation [11], the linear backstepping control design method [14], and observer design method [13]. Due to the explicit nature of the transformation and the design methods we use, we were able to derive formulas for the reference trajectories, feedback laws, and observer gains. Since our nonlinear transformation is not globally invertible, our results are not global, which is consistent with the lack of global controllability shown in Ref. [10].

Our result does not apply to the inviscid Burgers equation, which is a completely different problem, with well posedness considerations that may, as time and state evolve, allow one, two, or no boundary conditions/controls.

In our stabilization efforts, such as in Secs. 6–9, we focus on the equilibrium (3). Actually many other monotonically increasing profiles, which are the solutions of the equilibrium problem $u_{xx}-u_x u=0$, are allowed. We consider only the symmetric problem for notational simplicity, conceptual clarity, and due to the fact that this is actually the most difficult equilibrium shape from the stabilization point of view (the "shock," which is the cause of instability, is the furthest away from both boundary actuators).

Dirichlet actuation is crucial for applications in flow control. Unfortunately, the present result does not extend from Neumann to Dirichlet actuation. The actuation at $x=1$ can be changed to Dirichlet, but not the actuation at $x=0$, which has to be of Neumann type. Every single one of our previous designs for backstepping boundary control could be interchangeably implemented through either Dirichlet or Neumann, but not this one.

Interesting problems for future research include a design using only one control input, $\tilde{\omega}_1$, extensions to more general parabolic PDEs with convective nonlinearities, to convective nonlinearities of a more general form, to Burgers equations in higher dimensions, and to other PDEs with convective nonlinearities such as Kuramoto–Sivashinsky and Navier–Stokes.

Acknowledgment

The work was supported by NSF.

References

[1] Krstic, M., Magnis, L., and Vazquez, R., 2008, "Nonlinear Stabilization of Shock-Like Unstable Equilibria in the Viscous Burgers PDE," IEEE Trans. Autom. Control, **53**, pp. 1678–1683.
[2] Engelberg, S., 1998, "The Stability of the Shock Profiles of the Burgers' Equation," Appl. Math. Lett., **11**(5), pp. 97–101.
[3] Balogh, A., and Krstic, M., 2000, "Burgers' Equation With Nonlinear Boundary Feedback: H^1 Stability, Well Posedness, and Simulation," Math. Probl. Eng., **6**, pp. 189–200.
[4] Balogh, A., and Krstic, M., 2000, "Boundary Control of the Korteweg-de Vries-Burgers Equation: Further Results on Stabilization and Numerical Demonstration," IEEE Trans. Autom. Control, **45**, pp. 1739–1745.
[5] Krstic, M., 1999, "On Global Stabilization of Burgers' Equation by Boundary Control," Syst. Control Lett., **37**, pp. 123–141.
[6] Liu, W.-J., and Krstic, M., 2000, "Backstepping Boundary Control of Burgers' Equation With Actuator Dynamics," Syst. Control Lett., **41**, pp. 291–303.
[7] Liu, W.-J., and Krstic, M., 2000, "Stability Enhancement by Boundary Control in the Kuramoto–Sivashinsky Equation," Nonlinear Anal.: Real World Appl., **43**, pp. 485–583.
[8] Liu, W.-J., and Krstic, M., 2001, "Adaptive Control of Burgers' Equation With Unknown Viscosity," Int. J. Adapt. Control Signal Process., **15**, pp. 745–766.
[9] Liu, W.-J., and Krstic, M., 2002, "Global Boundary Stabilization of the Korteweg–de Vries–Burgers Equation," Comput. Appl. Math., **21**, pp. 315–354.
[10] Fernandez-Cara, E., and Guerrero, S., 2007, "Null Controllability of the Burgers' Equation With Distributed Controls," Syst. Control Lett., **56**, pp. 366–372.
[11] Laroche, B., Martin, P., and Rouchon, P., 2000, "Motion Planning for the Heat Equation," Int. J. Robust Nonlinear Control, **10**, pp. 629–643.
[12] Baker, J., Armaou, A., and Christofides, P. D., 2000, "Nonlinear Control of Incompressible Fluid Flow: Application to Burgers' Equation and 2D Channel Flow," J. Math. Anal. Appl., **252**(1), pp. 230–255.
[13] Smyshlyaev, A., and Krstic, M., 2005, "Backstepping Observers for Parabolic PDEs," Syst. Control Lett., **54**, pp. 1953–1971.
[14] Smyshlyaev, A., and Krstic, M., 2004, "Closed Form Boundary State Feedbacks for a Class of Partial Integro-Differential Equations," IEEE Trans. Autom. Control, **49**, pp. 2185–2202.
[15] Krstic, M., Guo, B.-Z., Balogh, A., and Smyshlyaev, A., 2008, "Output-Feedback Stabilization of an Unstable Wave Equation," Automatica, **44**, pp. 63–74.
[16] Vazquez, R., and Krstic, M., 2007, "Explicit Output Feedback Stabilization of a Thermal Convection Loop by Continuous Backstepping and Singular Perturbations," *Proceedings of the 2007 American Control Conference*, pp. 2177–2182.