A PDE Control Approach to Stabilization and Estimation of Thermoacoustic Instabilities in a Rijke Tube

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Pieter J. Rijke
(Professor at Leiden University)

“Notice of a new way to set into oscillation the air contained in a tube with both ends open”, Annalen der Physik und Chemie, vol. 107, pp. 339-343, 1859. (In German)

ANNALEN

DER

PHYSIK

UND

CHEMIE.

HERAUSGEGEBEN ZU BERLIN

VON

J. C. POGGENDORFF.

HUNDERT UND SIEBENTER BAND.

DER GANZEN FOLGE HUNDERT UND DREI UND ACHTZIGSTER.

NEBST VIER KUPFERTAFELN.

LEIPZIG, 1859.

VERLAG VON JOHANN AMBROSIUS BARTH.

XV. *Notiz über eine neue Art, die in einer an beiden Enden offenen Röhre enthaltene Luft in Schwingungen zu versetzen; von P. L. Rijke.*

1. Meine ersten Versuche wurden mit einer Glasröhre von $0^{m},8$ Länge gemacht. Ihr Durchmesser betrug in dem oberen Theil 37^{mm} und in dem unteren 30^{mm}. Im Innern, $0^{m},2$ von diesem letzteren Ende ab, hatte ich eine Scheibe von Metallgeflecht, etwa 50^{mm} im Durchmesser haltend, angebracht. Ihre Ränder waren umgebogen, so daſs sie durch den Druck, den diese gegen die Röhrenwandung ausübten, in jeder beliebigen Höhe gehalten werden konnte. Das Metallgeflecht war von $0^{mm},2$ dicken Eisendraht und hielt auf ein Quadrat-Centimeter ungefähr 81 Maschen.

Nachdem der Apparat so vorgerichtet worden, hatte man nur das Metallgeflecht mittelst einer Alkohol- oder Wasserstoff-Lampe in Rothgluth zu versetzen, um einige Augenblicke nach dem Auslöschen oder Fortnehmen der Lampe einen Ton zu vernehmen. Der Ton war beinah der Grundton der Röhre. Er hatte viele Stärke (*éclat*), hielt aber nur einige Sekunden an.

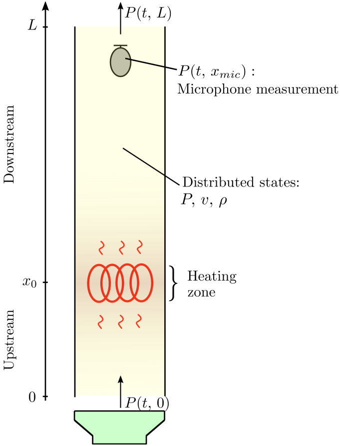
2. Wenn man, statt einer einzigen Scheibe, deren mehre in der Röhre anbringt, so hält der Ton, den man bekommt, länger an.

3. Der Ton hört augenblicklich auf, so wie man die obere Mündung der Röhre verschliefst. Daraus folgt, daſs das Daseyn eines aufsteigenden Luftstromes eine der Bedingungen des Phänomens ausmacht. Auch darf man die Zahl der Scheiben nicht übermäſsig vergröſsern, weil die Verlangsamung des Luftstroms nicht gewisse Gränzen überschreiten darf.

4. Der Versuch gelingt auch, wenn man die Scheibe mittelst einer Kohlenoxyd-Flamme erhitzt. Ich bereitete dieses Gas, indem ich Nordhäuser Schwefelsäure auf Oxal-

The Rijke Tube Experiment

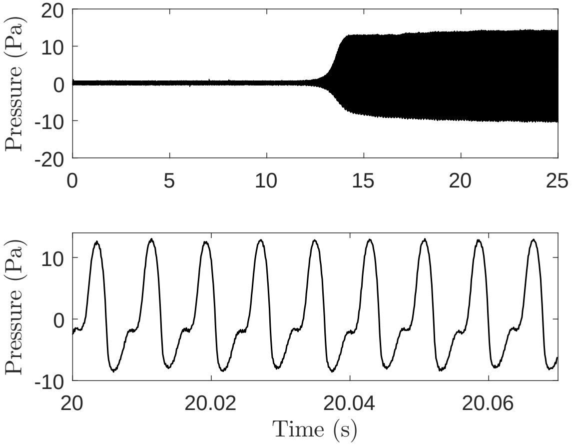
- A vertical tube opened in both ends.
- A heat source is inserted in the lower half of the tube.
- Under the right conditions, the tube begins to hum loudly (thermoacoustic instability).
- A microphone at the top of the tube can be used for measurement of acoustic pressure.
- A speaker at the bottom is used as actuator to stabilize the system.



Click for video

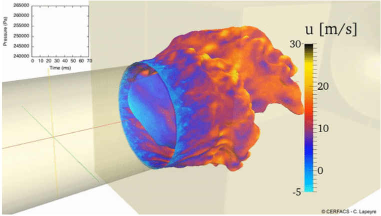
The Rijke Tube Experiment

Microphone signal at the onset of instability showing growth, and then saturation of the limit cycle. A zoomed-in picture shows the periodic, but nonsymetric, limit-cycle behavior.



Motivation

- Thermoacoustic instabilities are often encountered in steam and gas turbines, industrial burners, and jet and ramjet engines.



- These instabilities are undesirable and notorious difficult to model and study.

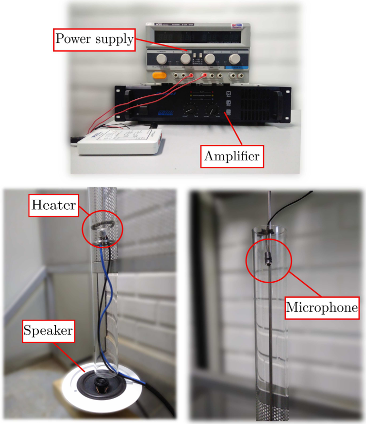
- The absence of combustion process in the Rijke tube makes the modeling and analysis more tractable.

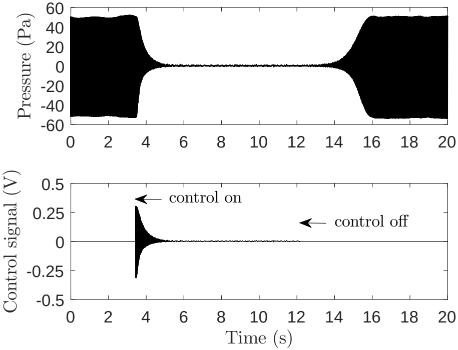
- The Rijke tube experiment provides an accessible platform to explore and study stabilization and state estimation of thermoacoustic oscillations.

Previous works

We have developed an output feedback control law to stabilize the system in

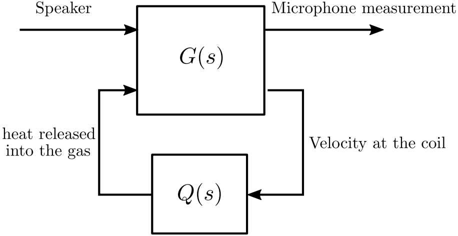
de Andrade, G. A., Vazquez, R., and Pagano, D. J. (2017). Boundary control of a Rijke tube using irrational transfer functions with experimental validation. In Proceedings of the 20th IFAC world congress (pp. 4528–4533).





Previous works

The closed-loop stability analysis was made using an irrational transfer function of the system and frequency domain tools for infinite dimensional systems.



The result guarantees **input-output stability**, but nothing is said about exponential stability (although the experiments show exponential convergence).

A clear and formal way to design a stabilizing control law guaranteeing exponential stability is by the backstepping method.

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  - Simplification assumptions and linearization
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  - Backstepping transformation
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Thermoacoustic dynamics

Starting from the conservation of mass, momentum, and energy, we arrive at

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \text{mass conservation}$$

$$\partial_t(\rho v) + \partial_x\left(\rho v^2 + P\right) = 0, \qquad \text{momentum balance}$$

$$\partial_t\left(\rho U + \frac{\rho v^2}{2}\right) + \partial_x\left(v\left(\rho U + \frac{\rho v^2}{2}\right) + Pv\right) = q, \qquad \text{energy balance}$$

with boundary conditions

$$P(t,0) = P_0 + u(t), \qquad \text{open end with speaker}$$

$$P(t,L) = P_0 + f(v(t,\, L)) \qquad \text{open end}$$

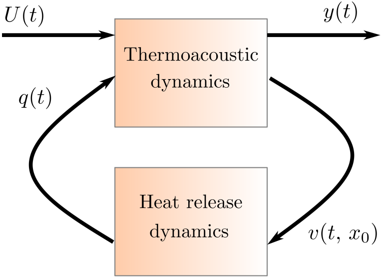
Heat release dynamics

We assume that the heat input is concentrated at a single point x_0:

$$q(x,t) = \frac{1}{A}\delta(x - x_0)Q(t).$$

King's Law describes the dependence of heat transfer on gas velocity:

$$\begin{aligned} \tau_{hr}\dot{Q}(t) &= -Q(t) + Q_K(t), \\ Q_K(t) &= l_w(T_w - T)(\kappa + \kappa_v\sqrt{|v(t,x_0)|}). \end{aligned}$$



Linearization of thermoacoustic dynamics

Assume constant steady-state solution, $(\rho,\, v,\, P) = (\overline{\rho},\, \overline{v},\, \overline{P}), \forall t \in [0,\, +\infty), \forall x \in [0,\, L]$. Then, we can obtain the following linearized model:

$$
\begin{aligned}
&\partial_t \tilde{\rho} + \overline{v}\partial_x \tilde{\rho} + \overline{\rho}\partial_x \tilde{v} = 0,\\
&\partial_t \tilde{v} + \overline{v}\partial_x \tilde{v} + \frac{1}{\overline{\rho}}\partial_x \tilde{P} = 0,\\
&\partial_t \tilde{P} + \gamma \overline{P}\partial_x \tilde{v} + \overline{v}\partial_x \tilde{P} = \frac{\overline{\gamma}}{A}\delta(x - x_0)\tilde{Q},
\end{aligned}
$$

Taking into account that $\overline{v}$ is very small if compared to the speed of sound, it is easy to see that the contribution of $\overline{v}$ to the gas dynamics is negligible. Therefore, one can make $\overline{v} = 0$. In addition, the density ρ is decoupled from the velocity and pressure (which are the states that matter for control), and the remaining coupled part of the dynamics is a 2×2 hyperbolic system!

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Linearization of heat release dynamics and boundary Conditions

Linearizing King's Law yields

$$\tilde{Q}_K(t) = f(\overline{v})\frac{\overline{T}}{\overline{\rho}}\tilde{\rho} + f'(\overline{v})(T_w - \overline{T})\tilde{v} - f(\overline{v})\frac{\overline{T}}{\overline{P}}\tilde{P}.$$

Comparing the size of the gains of each state in the above equation it is possible to conclude that the velocity fluctuations are the main driver of the heat dynamics, hence it is reasonable to drop out the density and pressure influence of the above equation

$$\tilde{Q}_K(t) \approx f'(\overline{v})(T_w - \overline{T})\tilde{v}(t, x_0).$$

Thus,

$$\tau_{hr}\dot{\tilde{Q}}(t) = -\tilde{Q}(t) + \tilde{Q}_K(t).$$

Linearization of the boundary conditions yields

$$\tilde{P}(t, 0) = U(t),$$
$$\tilde{P}(t, 0) = Z_l\tilde{v}(t, L).$$

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Model in Terms of Characteristic Coordinates

Since the system is hyperbolic, there exists an invertible linear mapping

$$\mathbf{T} : \mathscr{L}^{\infty}(0,\, L) \times \mathscr{L}^{\infty}(0,\, L) \to \mathscr{L}^{\infty}(0,\, L) \times \mathscr{L}^{\infty}(0,\, L)$$

such that the system can be diagonalized

$$\begin{pmatrix} \tilde{v} \\ \tilde{P} \end{pmatrix} = \mathbf{T} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{\gamma \overline{P}\overline{\rho}}} & -\frac{1}{2\sqrt{\gamma \overline{P}\overline{\rho}}} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

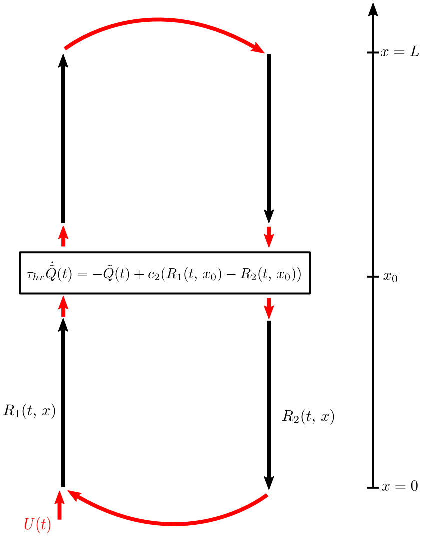
Then, the linearized system is rewritten to

$$\begin{aligned}
&\partial_t R_1 + \lambda \partial_x R_1 = c_1 \delta(x - x_0) \tilde{Q}(t), \\
&\partial_t R_2 - \lambda \partial_x R_2 = c_1 \delta(x - x_0) \tilde{Q}(t), \\
&R_1(t, 0) = -R_2(t,\, 0) + 2U(t), \\
&R_2(t, L) = \alpha R_1(t, L). \\
&\tau_{hr} \dot{\tilde{Q}}(t) = -\tilde{Q}(t) + c_2(R_1(t,\, x_0) - R_2(t,\, x_0)).
\end{aligned}$$

Schematic view of the jumping point at the solution of the PDE system

Integrating along the characteristic lines, the next relations are satisfied:

$$R_1(t,\, x_0^+) = R_1(t,\, x_0^-) + c_1\tilde{Q}(t),$$
$$R_2(t,\, x_0^-) = R_2(t,\, x_0^+) + c_1\tilde{Q}(t).$$



PDE-ODE-PDE system!

Representation in characteristic coordinates

Now, we introduce the following state variables

$$\begin{aligned}
R_{11}(t,\, x) &\triangleq R_1(t,\, x), && \text{if } x \in [0,\, x_0] \\
R_{12}(t,\, x) &\triangleq R_2(t,\, x), && \text{if } x \in [0,\, x_0] \\
R_{21}(t,\, x) &\triangleq R_1(t,\, x), && \text{if } x \in [x_0,\, L] \\
R_{22}(t,\, x) &\triangleq R_2(t,\, x), && \text{if } x \in [x_0,\, L]
\end{aligned}$$

and the rescaled spatial variable, so that everything evolves on the same domain:

$$z = \begin{cases} \frac{x}{x_0} & \text{if } x \in [0,\, x_0] \\ \frac{L-x}{L-x_0} & \text{if } x \in [x_0,\, L] \end{cases}$$

Representation in characteristic coordinates

Then, the system linearized system is equivalent to

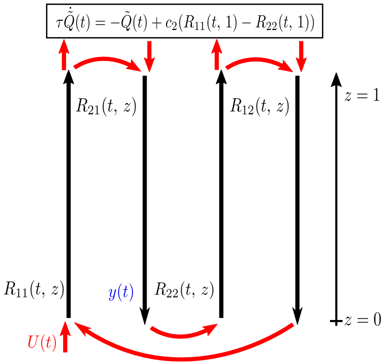
$$
\begin{aligned}
\partial_t R_{11}(t,\, z) + \lambda_1 \partial_z R_{11}(t,\, z) &= 0,\\
\partial_t R_{12}(t,\, z) - \lambda_1 \partial_z R_{12}(t,\, z) &= 0,\\
\partial_t R_{21}(t,\, z) - \lambda_2 \partial_z R_{21}(t,\, z) &= 0,\\
\partial_t R_{22}(t,\, z) + \lambda_2 \partial_z R_{22}(t,\, z) &= 0,
\end{aligned}
$$

where λ_i depends on the value of λ and the geometry (position of x_0), with boundary conditions

$$
\begin{aligned}
R_{11}(t,0) &= -R_{12}(t,\, 0) + 2U(t),\\
R_{12}(t,1) &= R_{22}(t,\, 1) + c_1 \tilde{Q}(t),\\
R_{21}(t,1) &= R_{11}(t,\, 1) + c_1 \tilde{Q}(t),\\
R_{22}(t,0) &= \alpha R_{21}(t,\, 0),\\
\tau_{hr} \dot{\tilde{Q}}(t) &= -\tilde{Q}(t) + c_2 (R_{11}(t,\, 1) - R_{22}(t,\, 1)).
\end{aligned}
$$

Representation in characteristic coordinates

- The boundary conditions represent two effects: reflection of the acoustic waves; and the feedback coupling between R_{21} and R_{22}, and between R_{11} and R_{12}.
- Under the right conditions the system becomes unstable due to this feedback between the states.
- Our objective is to design an output feedback control law that **exponentially stabilize** the zero equilibrium of the system.



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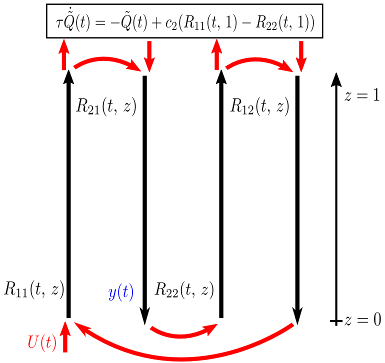

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Backstepping for PDEs: a brief introduction

Roughly speaking, backstepping is a constructive method that achiees Lyapunov stabilization by **transforming** the system into a stable “target system”, which is often achieved by collectively shifting all the eigenvalues in a favorable direction in the complex plane, rather than by assigning individual eigenvalues.

Backstepping is not “one-size-fits-all”. Requires structure-specific effort by designer.

Reward: elegant controller/observer, clear closed-loop behavior.

Backstepping for PDEs: a brief introduction

Basic steps in the backstepping methodology:

1. Identify the undesirable terms in the PDE.

2. Choose a target system in which the undesirable terms are to be eliminated by state transformation and feedback.

3. Find the state transformation.

4. Obtain the boundary feedback/observer gains from the transformation. The transformation alone cannot eliminate the undesirable terms, but the transformation brings them to the boundary, so control can cancel them.

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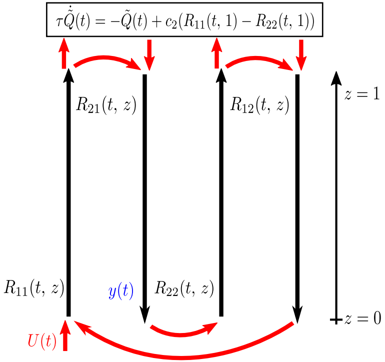
Backstepping-based controller design: original system

- The instabilities of the system raise from the couplings at the boundary.
- The system is “underactuated” (not really, but from the perspective of previous backstepping designs).



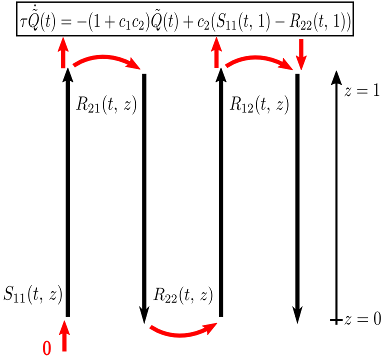
Backstepping-based controller design: target system

We want to map the Rijke tube model into the following target system

$$\begin{aligned}
\partial_t S_{11}(t,\,z) + \lambda_1 \partial_z S_{11}(t,\,z) &= 0,\\
\partial_t R_{12}(t,\,z) - \lambda_1 \partial_z R_{12}(t,\,z) &= 0,\\
\partial_t R_{21}(t,\,z) - \lambda_2 \partial_z R_{21}(t,\,z) &= 0,\\
\partial_t R_{22}(t,\,z) + \lambda_2 \partial_z R_{22}(t,\,z) &= 0,
\end{aligned}$$

with the following boundary conditions

$$\begin{aligned}
&S_{11}(t,0) = 0,\\
&R_{12}(t,1) = R_{22}(t,1) + c_1 \tilde{Q}(t),\\
&R_{21}(t,1) = S_{11}(t,1),\\
&R_{22}(t,0) = \alpha R_{21}(t,0),\\
&\tau_{hr} \dot{\tilde{Q}}(t) = -(1 + c_1 c_2)\tilde{Q}(t) + c_2(S_{11}(t,1) - R_{22}(t,1)).
\end{aligned}$$



Backstepping-based controller design: transformation

To do that, we consider the following backstepping transformation

$$\begin{aligned} S_{11}(t,\, z) \quad &= \quad R_{11}(t,\, z) - \varphi(z)\tilde{Q}(t) - \int_z^1 R_{11}(t,\, \xi)K(z,\, \xi)d\xi \\ &\quad - \int_0^1 R_{22}(t, \xi)G(z,\, \xi)d\xi - \int_0^1 R_{21}(t,\, \xi)H(z,\, \xi)d\xi. \end{aligned}$$

- Domain of the K kernel:

$$\mathscr{T}_0 = \{(z,\, \xi) \in \mathbb{R}^2 | 0 \leq z \leq \xi \leq 1\},$$

- Domain of G and H kernels:

$$\mathscr{T}_1 = \{(z,\, \xi) \in \mathbb{R}^2 | 0 \leq \xi \leq 1, 0 \leq z \leq 1\},$$

- φ is a finite dimensional kernel defined on the interval $z \in [0,\, 1]$.

Backstepping-based controller design: kernel equations

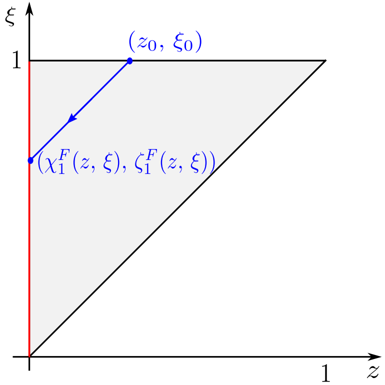
Differentiating the transformation with respect to space and time, integrating by parts, and plugging the target system equation, we obtain that the original system is mapped into the target system if and only if the kernels satisfy the following equations:

$$
\begin{aligned}
&\partial_\xi K(z,\,\xi) + \partial_z K(z,\,\xi) = 0,\\
&\partial_\xi G(z,\,\xi) + \frac{\lambda_1}{\lambda_2}\partial_z G(z,\,\xi) = 0,\\
&\partial_\xi H(z,\,\xi) - \frac{\lambda_1}{\lambda_2}\partial_z H(z,\,\xi) = 0,\\
&\lambda_1\varphi'(z) - \frac{1}{\tau_{hr}}\varphi(z) + \lambda_2 c_1 H(z,\,1) = 0,
\end{aligned}
$$

with

$$
\begin{aligned}
&\lambda_1 K(z,\,1) - \frac{c_2}{\tau}\varphi(z) - \lambda_2 H(z,\,1) = 0, && G(1,\xi) = 0,\\
&\lambda_2 G(z,\,1) + \frac{c_2}{\tau}\varphi(z) = 0, && H(1,\xi) = 0,\\
&\alpha G(z,\,0) - H(z,\,0) = 0, && \varphi(1) = -c_1.
\end{aligned}
$$

Backstepping-based controller design: kernel equations



The boundary value problem for the K kernel:

$$\partial_\xi K(z,\,\xi) + \partial_z K(z,\,\xi) = 0,$$

$$\lambda_1 K(z,\,1) - \frac{c_2}{\tau_{hr}}\varphi(z) - \lambda_2 H(z,\,1) = 0.$$

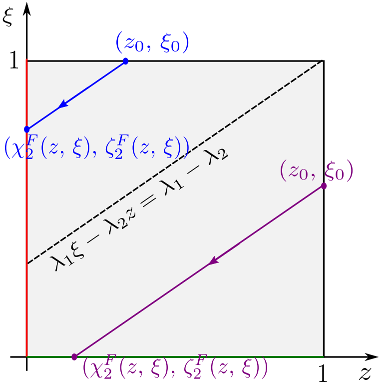
Solution:

$$K(z,\,\xi) = \frac{c_2}{\lambda_1 \tau_{hr}}\varphi(z-\xi+1) + \frac{\lambda_2}{\lambda_1}H(z-\xi+1, 1).$$

Backstepping-based controller design: kernel equations

The boundary value problem for the G kernel:

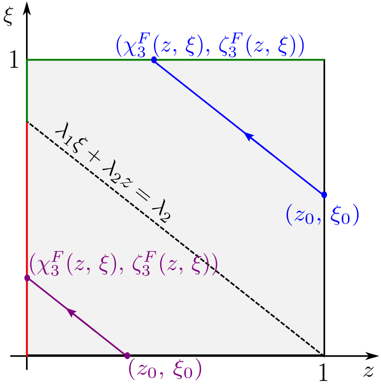
$$\partial_\xi G(z,\,\xi) + \frac{\lambda_1}{\lambda_2}\partial_z G(z,\,\xi) = 0$$
$$\lambda_2 G(z,\,1) + \frac{c_2}{\tau_{hr}}\varphi(z) = 0,$$
$$G(1,\xi) = 0.$$



Solution:

$$G(z,\,\xi) = \begin{cases} 0 & \xi - 1 \leq \frac{\lambda_2}{\lambda_1}(z-1), \\ -\frac{c_2}{\lambda_2\tau_{hr}}\varphi\left(z - \frac{\lambda_1}{\lambda_2}(\xi-1)\right), & \text{otherwise.} \end{cases}$$

Backstepping-based controller design: kernel equations



The boundary value problem for the H kernel:

$$\partial_\xi H(z,\,\xi)-\frac{\lambda_1}{\lambda_2}\partial_z H(z,\,\xi)=0,$$
$$\alpha G(z,\,0)-H(z,\,0)=0,$$
$$H(1,\xi)=0.$$

Solution:

$$H(z,\,\xi)=\begin{cases}0 & \xi+1\geq\frac{\lambda_2}{\lambda_1}(1-z),\\ -\frac{\alpha c_2}{\lambda_2\tau_{hr}}\varphi\left(z+\frac{\lambda_1}{\lambda_2}(\xi+1)\right), & \text{otherwise.}\end{cases}$$

Backstepping-based controller design: kernel equations

The boundary value problem for the φ kernel:

$$\lambda_1\varphi'(z) - \frac{1}{\tau_{hr}}\varphi(z) + \lambda_2 c_1 H(z,\,1) = 0,$$
$$\varphi(1) = -c_1.$$

φ may be piecewise-differentiable depending on the values of $H(z,\,1)$.

Case I ($\lambda_1 \geq \lambda_2$): In this case $H(z,\,1) = 0, \forall z \in [0,\,1]$. Then,

$$\left.\begin{aligned} &\lambda_1\dot{\varphi}(z) - \tfrac{1}{\tau_{hr}}\varphi(z) = 0 \\ &\varphi(1) = -c_1 \end{aligned}\right\} \Rightarrow \varphi(z) = -c_1 \mathrm{e}^{\frac{z-1}{\lambda_1 \tau}}$$

Backstepping-based controller design: kernel equations

Case II ($\lambda_1 < \lambda_2$): In this case, the φ kernel equations can be solved backwards.

Observing the behavior of $H(z,1)$ backwards, one can note that it is zero for all $z \in [1-2\frac{\lambda_1}{\lambda_2}, 1]$. Therefore, the solution φ for $z \in [1-2\frac{\lambda_1}{\lambda_2}, 1]$ satisfies

$$\left.\begin{array}{l} \lambda_1\dot{\varphi}(z) - \frac{1}{\tau_{hr}}\varphi(z) = 0 \\ \varphi(1) = -c_1 \end{array}\right\} \Rightarrow \varphi(z) = -c_1 \mathrm{e}^{\frac{z-1}{\lambda_1\tau}}$$

Once the solution φ is known on $[1-2\frac{\lambda_1}{\lambda_2}, 1]$, we can repeat the same procedure, starting with the solution on $[1-2\frac{\lambda_1}{\lambda_2}, 1]$, to find the solution φ for $z \in [1-4\frac{\lambda_1}{\lambda_2}, 1-2\frac{\lambda_1}{\lambda_2}]$ by computing the solution of the following boundary value problem:

$$\lambda_1\dot{\varphi}(z) - \frac{1}{\tau}\varphi(z) + \frac{\alpha c_1^2 c_2}{\tau}\mathrm{e}^{\frac{\lambda_2(z-1)+2\lambda_1}{\lambda_2\lambda_1\tau}} = 0,$$

$$\varphi\left(1-2\frac{\lambda_1}{\lambda_2}\right) = -c_1\mathrm{e}^{-\frac{2}{\lambda_2\tau}}.$$

We can repeat the same procedure, starting with the solution on $[1-4\frac{\lambda_1}{\lambda_2}, 1-2\frac{\lambda_1}{\lambda_2}]$, to find the solution φ for $z \in [1-8\frac{\lambda_1}{\lambda_2}, 1-4\frac{\lambda_1}{\lambda_2}]$, and so on.

Backstepping-based controller design: kernel equations

Case II ($\lambda_1 < \lambda_2$): In this case, the φ kernel equations can be solved backwards.

Observing the behavior of $H(z,\,1)$ backwards, one can note that it is zero for all $z \in [1-2\frac{\lambda_1}{\lambda_2},\,1]$. Therefore, the solution φ for $z \in [1-2\frac{\lambda_1}{\lambda_2},\,1]$ satisfies

$$\left.\begin{array}{l} \lambda_1\dot{\varphi}(z) - \frac{1}{\tau_{hr}}\varphi(z) = 0 \\ \varphi(1) = -c_1 \end{array}\right\} \Rightarrow \varphi(z) = -c_1 \mathrm{e}^{\frac{z-1}{\lambda_1\tau}}$$

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$$\lambda_1\dot{\varphi}(z) - \frac{1}{\tau}\varphi(z) + \frac{\alpha c_1^2 c_2}{\tau}\mathrm{e}^{\frac{\lambda_2(z-1)+2\lambda_1}{\lambda_2\lambda_1\tau}} = 0,$$
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Backstepping-based controller design: kernel equations

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Observing the behavior of $H(z,1)$ backwards, one can note that it is zero for all $z \in [1-2\frac{\lambda_1}{\lambda_2}, 1]$. Therefore, the solution φ for $z \in [1-2\frac{\lambda_1}{\lambda_2}, 1]$ satisfies

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$$\begin{aligned} &\lambda_1 \dot{\varphi}(z) - \frac{1}{\tau}\varphi(z) + \frac{\alpha c_1^2 c_2}{\tau} \mathrm{e}^{\frac{\lambda_2(z-1)+2\lambda_1}{\lambda_2 \lambda_1 \tau}} = 0, \\ &\varphi\left(1 - 2\frac{\lambda_1}{\lambda_2}\right) = -c_1 \mathrm{e}^{-\frac{2}{\lambda_2 \tau}}. \end{aligned}$$

We can repeat the same procedure, starting with the solution on $[1-4\frac{\lambda_1}{\lambda_2}, 1-2\frac{\lambda_1}{\lambda_2}]$, to find the solution φ for $z \in [1-8\frac{\lambda_1}{\lambda_2}, 1-4\frac{\lambda_1}{\lambda_2}]$, and so on.

Backstepping-based controller design: kernel equations

Applying this iterative procedure n times, where $n \in \mathbb{N}$ is the largest integer such that $\frac{L-x_0}{x_0} > \frac{1}{2^n}$, yields a unique, globally defined, solution φ when $\lambda_1 < \lambda_2$.

The closer the heat element is to the uncontrolled boundary, the larger the number of pieces of the solution φ.

From a practical point of view, the case $\lambda_1 > \lambda_2$ occurs when the heat release is located in the lower half of the tube. Similarly, $\lambda_1 < \lambda_2$ if the heat release is located in the upper half of the tube.

Backstepping-based controller design: invertibility of the transformation

To ensure that the closed-loop system and the target system have the same stability property, the backstepping transformation has to be invertible.

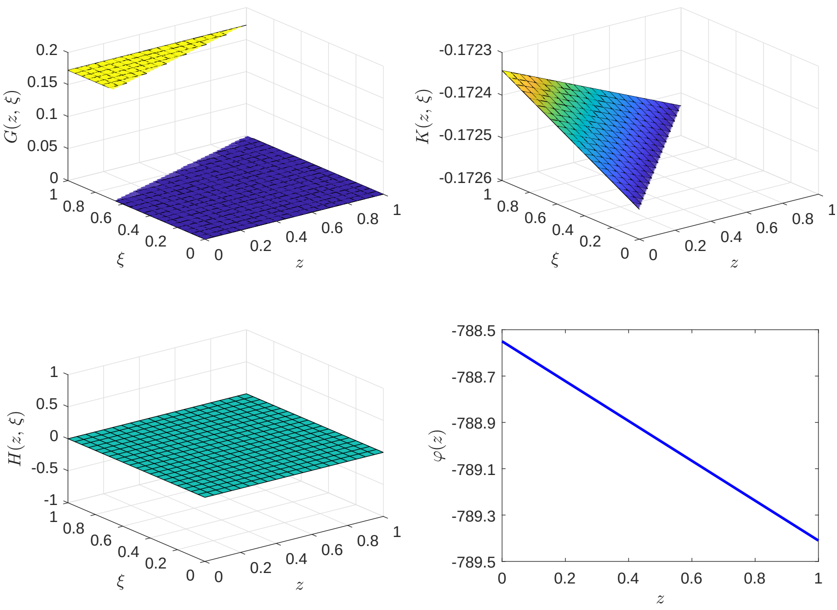
Rewrite the transformation as

$$R_{11}(t, z) = \int_z^1 R_{11}(t, \xi) K(z, \xi) d\xi + \psi(t, z).$$

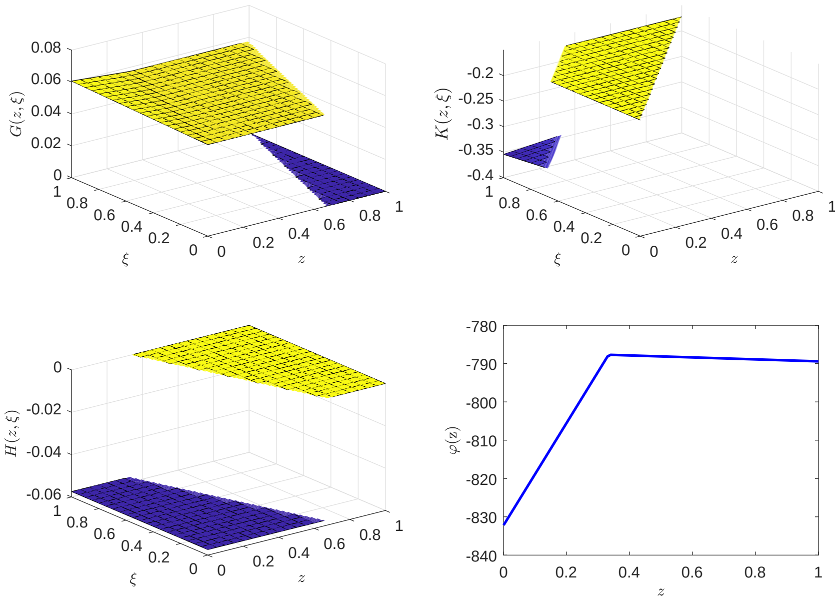
This equation can be seen as a Volterra integral equation of the second kind.

Since $K(z, \xi)$ is bounded, the equation has a unique solution, allowing us to write an inverse transformation, thus proving the invertibility of the transformation.

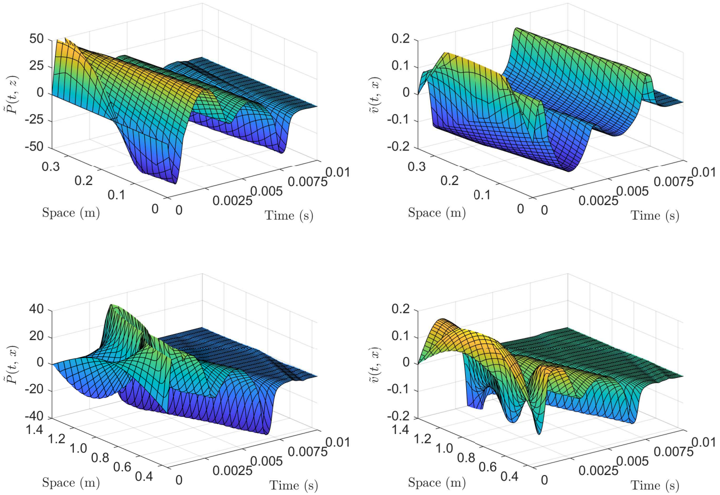
Backstepping-based controller design: kernel visuals (case I)



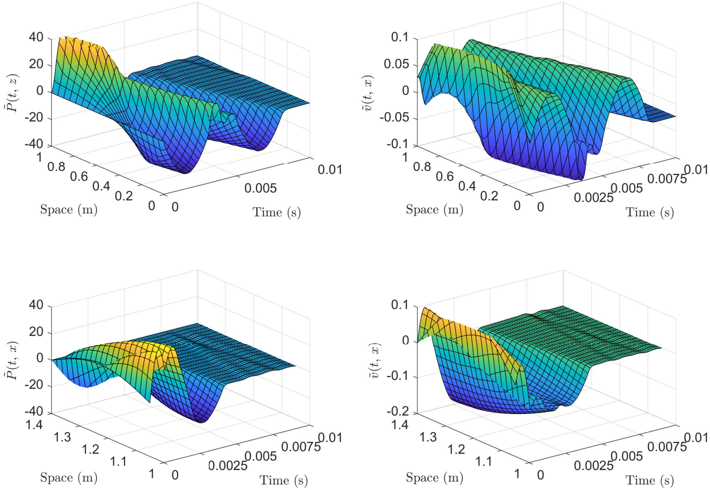
Backstepping-based controller design: kernel visuals (case II)



Backstepping-based controller design: simulation results (case I)



Backstepping-based controller design: simulation results (case II)



Remarks

The backstepping control law requires full-state measurement

$$U(t) = \frac{1}{2}\left(R_{12}(t,0) + \varphi(0)\tilde{Q}(t) + \int_0^1 R_{11}(t,\,\xi)K(0,\,\xi)d\xi \right.$$
$$\left. + \int_0^1 R_{22}(t,\,\xi)G(0,\,\xi)d\xi + \int_0^1 R_{21}(t,\,\xi)H(0,\,\xi)d\xi\right).$$

Therefore, the control law must be applied together with a **state-observer** in order to produce experiments.

Outline

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  - Target system
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  - Well-posedness of the kernel equations and invertibility of the transformation
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Backstepping-based observer design

- Measure: $R_{21}(t,\,0)$.
- We design the observe as a copy of the plant plus output injection terms:

$$\begin{aligned}
\partial_t \hat{R}_{11}(t,\,z) + \lambda_1 \partial_z \hat{R}_{11}(t,\,z) &= -p_{11}(z)\tilde{Y}(t),\\
\partial_t \hat{R}_{12}(t,\,z) - \lambda_1 \partial_z \hat{R}_{12}(t,\,z) &= -p_{12}(z)\tilde{Y}(t),\\
\partial_t \hat{R}_{21}(t,\,z) - \lambda_2 \partial_z \hat{R}_{21}(t,\,z) &= -p_{21}(z)\tilde{Y}(t),\\
\partial_t \hat{R}_{22}(t,\,z) + \lambda_2 \partial_z \hat{R}_{22}(t,\,z) &= -p_{22}(z)\tilde{Y}(t),\\
\tau_{hr}\dot{\hat{Q}}(t) = -\hat{Q}(t) + c_2(\hat{R}_{11}(t,\,1) - \hat{R}_{22}(t,\,1)) - p_Q\tilde{Y}(t),
\end{aligned}$$

  with $\tilde{Y}(t) = R_{21}(t,\,0) - \hat{R}_{21}(t,\,0)$.
- The boundary conditions are given by

$$\begin{aligned}
\hat{R}_{11}(t,0) &= -\hat{R}_{12}(t,\,0) + 2U(t),\\
\hat{R}_{12}(t,1) &= \hat{R}_{22}(t,\,1) + c_1\hat{Q}(t),\\
\hat{R}_{21}(t,1) &= \hat{R}_{11}(t,\,1) + c_1\hat{Q}(t),\\
\hat{R}_{22}(t,0) &= \alpha R_{21}(t,\,0) + p_0\tilde{Y}(t),
\end{aligned}$$

- $p_{11}, p_{12}, p_{21}, p_{22}, p_0$. and p_Q are gains to be found.

Backstepping-based observer design: observed error dynamics

Define the error estimation $\tilde{R}_{ij} = R_{ij} - \hat{R}_{ij}$, $i,\, j = 1,\, 2$, whose dynamics is given by

$$\begin{aligned}
&\partial_t \tilde{R}_{11}(t,\, z) + \lambda_1 \partial_z \tilde{R}_{11}(t,\, z) = p_{11}(z)\tilde{Y}(t),\\
&\partial_t \tilde{R}_{12}(t,\, z) - \lambda_1 \partial_z \tilde{R}_{12}(t,\, z) = p_{12}(z)\tilde{Y}(t),\\
&\partial_t \tilde{R}_{21}(t,\, z) - \lambda_2 \partial_z \tilde{R}_{21}(t,\, z) = p_{21}(z)\tilde{Y}(t),\\
&\partial_t \tilde{R}_{22}(t,\, z) + \lambda_2 \partial_z \tilde{R}_{22}(t,\, z) = p_{22}(z)\tilde{Y}(t),\\
&\tau_{hr}\dot{\tilde{Q}}(t) = -\tilde{Q}(t) + c_2(\tilde{R}_{11}(t,\, 1) - \tilde{R}_{22}(t,\, 1)) + p_Q\tilde{Y}(t),
\end{aligned}$$

and boundary conditions

$$\begin{aligned}
\tilde{R}_{11}(t,0) &= -\tilde{R}_{12}(t,\, 0),\\
\tilde{R}_{12}(t,1) &= \tilde{R}_{22}(t,\, 1) + c_1\tilde{Q}(t),\\
\tilde{R}_{21}(t,1) &= \tilde{R}_{11}(t,\, 1) + c_1\tilde{Q}(t),\\
\tilde{R}_{22}(t,0) &= \alpha\tilde{R}_{21}(t,\, 0) - p_0\tilde{Y}.
\end{aligned}$$

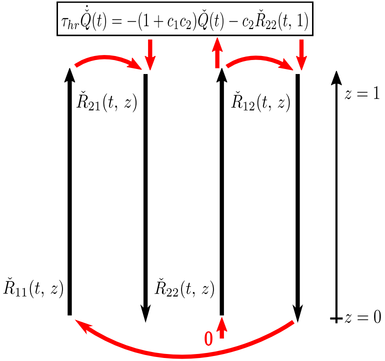
Backstepping-based observer design: target system

To design the observer output injection gains, we map the error estimation dynamics to the following appropriate target system:

$$\partial_t \check{R}_{11}(t,\, z) + \lambda_1 \partial_z \check{R}_{11}(t,\, z) = 0,$$
$$\partial_t \check{R}_{12}(t,\, z) - \lambda_1 \partial_z \check{R}_{12}(t,\, z) = 0,$$
$$\partial_t \check{R}_{21}(t,\, z) - \lambda_2 \partial_z \check{R}_{21}(t,\, z) = 0,$$
$$\partial_t \check{R}_{22}(t,\, z) + \lambda_2 \partial_z \check{R}_{22}(t,\, z) = 0,$$
$$\tau_{hr} \dot{\check{Q}}(t) = -(1 + c_1 c_2)\check{Q}(t) - c_2 \check{R}_{22}(t,\, 1),$$

with boundary conditions (setting $p_0 = \alpha$):

$$\check{R}_{11}(t, 0) = -\check{R}_{12}(t,\, 0),$$
$$\check{R}_{12}(t, 1) = \check{R}_{22}(t,\, 1) + c_1 \check{Q}(t),$$
$$\check{R}_{21}(t, 1) = \check{R}_{11}(t,\, 1) + c_1 \check{Q}(t),$$
$$\check{R}_{22}(t, 0) = 0.$$



Backstepping-based observer design: transformation

We consider the following backstepping transformation (dual to control transformation!):

$$\tilde{R}_{11}(t,\, z) = \check{R}_{11}(t,\, z) - \int_0^1 P_{11}(z,\xi)\check{R}_{21}(t,\xi)d\xi,$$
$$\tilde{R}_{12}(t,\, z) = \check{R}_{12}(t,\, z) - \int_0^1 P_{12}(z,\xi)\check{R}_{21}(t,\xi)d\xi,$$
$$\tilde{R}_{21}(t,\, z) = \check{R}_{21}(t,\, z) - \int_0^z P_{21}(z,\xi)\check{R}_{21}(t,\xi)d\xi,$$
$$\tilde{Q}(t) = \check{Q}(t) - \int_0^1 P_Q(\xi)\check{R}_{21}(t,\xi)d\xi,$$

- Domain of the P_{21} kernel:

$$\mathscr{T}_0 = \{(z,\, \xi) \in \mathbb{R}^2 | 0 \leq z \leq \xi \leq 1\},$$

- Domain of P_{11} and P_{12} kernels:

$$\mathscr{T}_1 = \{(z,\, \xi) \in \mathbb{R}^2 | 0 \leq \xi \leq 1, 0 \leq z \leq 1\},$$

- P_Q is a finite dimensional kernel defined on the interval $\xi \in [0,\, 1]$.

Backstepping-based observer design: kernel equations

Differentiating the transformation with respect to space and time, plugging the target system equation and integrating by parts, we obtain that the error system dynamics is mapped into the target system if and only if the kernels satisfy the following equations:

$$
\begin{aligned}
&\lambda_2 \partial_\xi P_{11}(z,\xi) - \lambda_1 \partial_z P_{11}(z,\xi) = 0,\\
&\lambda_2 \partial_\xi P_{12}(z,\xi) + \lambda_1 \partial_z P_{12}(z,\xi) = 0,\\
&\partial_\xi P_{21}(z,\xi) + \partial_z P_{21}(z,\xi) = 0,\\
&\tau_{hr}\lambda_2 P_Q'(\xi) = P_Q(\xi) - c_2 P_{11}(1,\xi),
\end{aligned}
$$

and

$$
\begin{aligned}
P_{21}(1,\xi) &= P_{11}(1,\xi) + c_1 P_Q(\xi), & P_{11}(z,1) &= 0,\\
P_Q(1) &= -\frac{c_2}{\tau_{hr}\lambda_2}, & P_{12}(z,1) &= 0,\\
P_{11}(0,\xi) &= -P_{12}(0,\xi), & P_{12}(1,\xi) &= c_1 P_Q(\xi).
\end{aligned}
$$

The existence and uniqueness of the solution of the kernel equations can be proved in a similar way to the backstepping control design.

Backstepping-based observer design: kernel equations

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&\lambda_2 \partial_\xi P_{11}(z,\xi) - \lambda_1 \partial_z P_{11}(z,\xi) = 0, \\
&\lambda_2 \partial_\xi P_{12}(z,\xi) + \lambda_1 \partial_z P_{12}(z,\xi) = 0, \\
&\partial_\xi P_{21}(z,\xi) + \partial_z P_{21}(z,\xi) = 0, \\
&\tau_{hr} \lambda_2 P_Q'(\xi) = P_Q(\xi) - c_2 P_{11}(1,\xi),
\end{aligned}
$$

and

$$
\begin{aligned}
P_{21}(1,\xi) &= P_{11}(1,\xi) + c_1 P_Q(\xi), & P_{11}(z,1) &= 0, \\
P_Q(1) &= -\frac{c_2}{\tau_{hr}\lambda_2}, & P_{12}(z,1) &= 0, \\
P_{11}(0,\xi) &= -P_{12}(0,\xi), & P_{12}(1,\xi) &= c_1 P_Q(\xi).
\end{aligned}
$$

The existence and uniqueness of the solution of the kernel equations can be proved in a similar way to the backstepping control design.

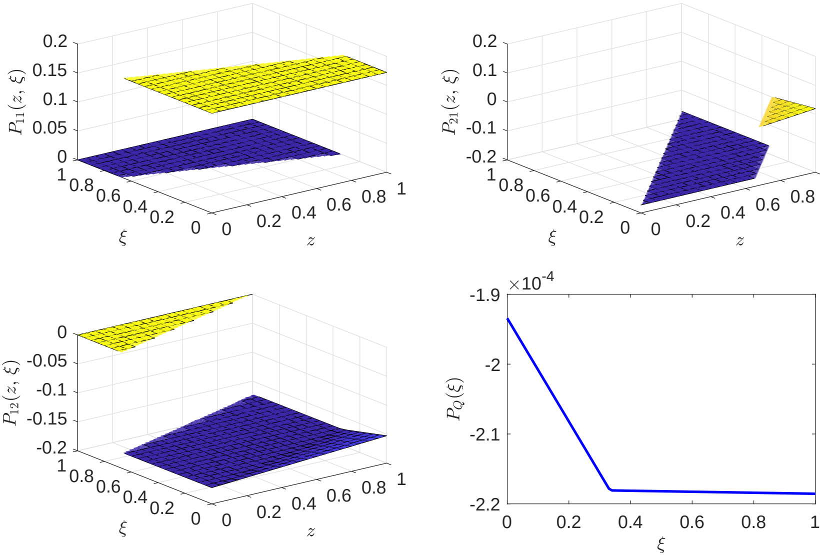
Backstepping-based observer design: observer gains

The closer the heat element is to the unmeasured boundary, the larger the number of pieces of the solution φ and thus, the higher the complexity of the observer.

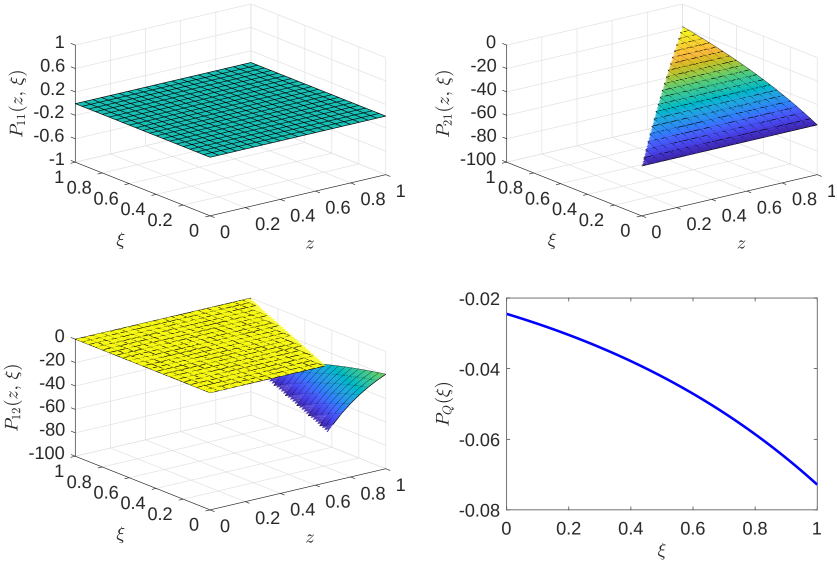
Besides, the observer gains are given by

$$
\begin{aligned}
p_{11}(z) &= \lambda_2 P_{11}(z,0), \\
p_{12}(z) &= \lambda_2 P_{12}(z,0), \\
p_{21}(z) &= \lambda_2 P_{21}(z,0), \\
p_Q &= \tau_{hr}\lambda_2 P_Q(0).
\end{aligned}
$$

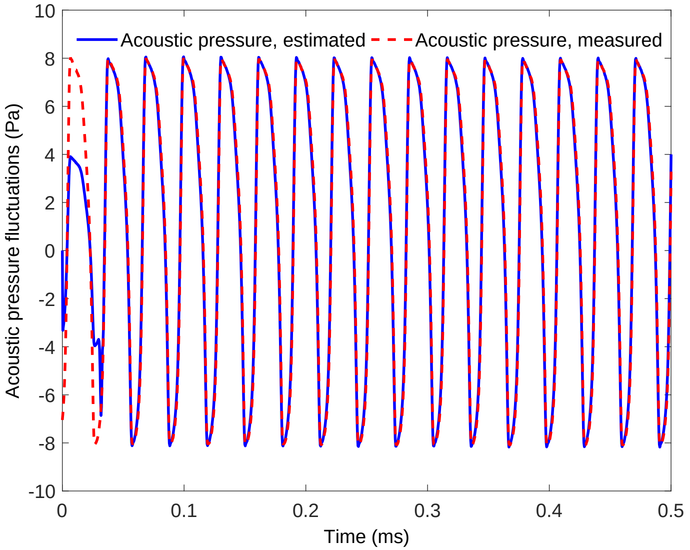
Backstepping-based observer design: kernel visuals (case I)



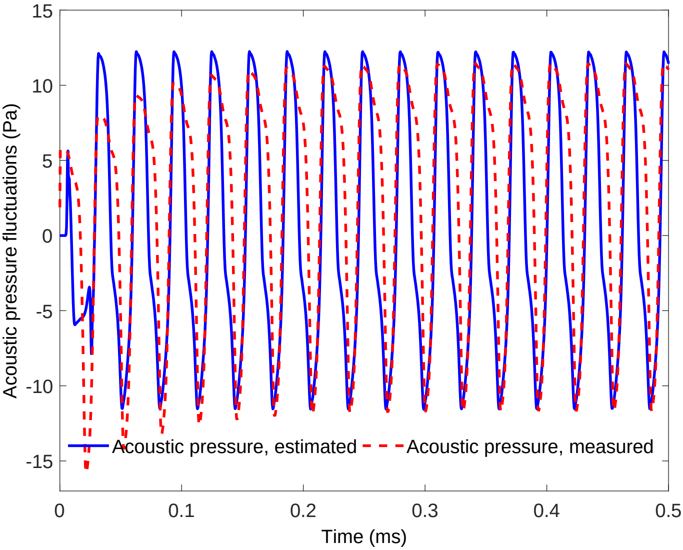
Backstepping-based observer design: kernel visuals (case II)



Experimental results ($x = L$)



Experimental results ($x = L/2$)



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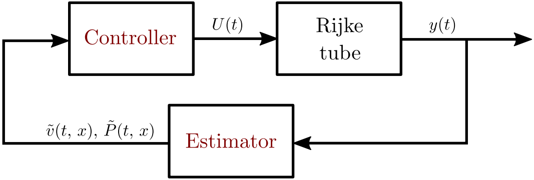
Final remarks

- The Rijke tube experiment can be described by a 4×4 model of hyperbolic PDEs coupled with an ODE (PDE-ODE-PDE model). The couplings are the source of instability.
- A full-state feedback law and a boundary observer for the Rijke tube have been designed using a non-standard backstepping-based design, which works by removing the couplings.
- Explicit exact gains were found, allowing us to find explicit closed loop solutions. The complexity of the gains depend on the position of the heat release element relative to actuation and measurement.
- Experiments for the observer reveal that it works well close to where measurements are taken. Not so well in the middle of the domain, only capturing the oscillations but not fine details.

Next steps

An output feedback control law can be designed using the reconstruction of the estimated states profile through the exponential convergent observer with the acoustic pressure measurement.



Since our design is based on the linear system, the separation principle holds; i.e., the combination of a separately designed state feedback controller and observer results in a stabilizing output-feedback controller.

To be seen: will this work for the real system?

Extension to more complex PDE-ODE-PDE systems: "Delay robust control design of under-actuated PDE-ODE-PDE systems", by Ulf Jakob F. Aarsnes, Rafael Vazquez, Florent Di Meglio, Miroslav Krstic, submitted to 2019 ACC

Some references

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Thanks for your attention