Backstepping Control of a Hyperbolic PDE System With Zero Characteristic Speed States

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***Abstract*—While for coupled hyperbolic partial differential equations (PDEs) of first order, there now exist numerous PDE backstepping designs, systems with zero speed, i.e., without convection but involving infinite-dimensional ordinary differential equations (ODEs), which arise in many applications, from environmental engineering to lasers to manufacturing, have received virtually no attention. In this article, we introduce single-input boundary feedback designs for a linear 1-D hyperbolic system with two counterconvecting PDEs and n equations (infinite-dimensional ODEs) with zero characteristic speed. The inclusion of zero-speed states, which we refer to as *atachic*, may result in the nonstabilizability of the plant. We give a verifiable condition for the model to be stabilizable and design a full-state backstepping controller, which exponentially stabilizes the origin in the $\mathcal{L}^2$ sense. In particular, to employ the backstepping method in the presence of atachic states, we use an invertible Volterra transformation only for the PDEs with nonzero speeds, leaving the zero-speed equations unaltered in the target system input-to-state stable with respect to the decoupled and stable counterconvecting nonzero-speed equations. Simulation results are presented to illustrate the effectiveness of the proposed control design.**

***Index Terms*—Boundary control, hyperbolic systems, PDE backstepping, stabilization.**

I. Introduction

In recent decades, boundary stabilization of hyperbolic partial differential equations (PDEs) has been extensively explored due to its relevance in diverse applications, such as oil extraction [1], electrical lines [3], water channels [8], traffic [11], and pipelines [13]. The field has reached a rather advanced ("mature" is the adjective that might come to the creativity-challenged mind) stage, propelled by the effectiveness of the PDE backstepping method for creating control laws and observers. Stabilization for the "general heterodirectional case," consisting on $(n+m)\times(n+m)$ hyperbolic systems with n equations moving in one direction and m controlled ones in the opposite, was solved in [15]. For output-feedback control with disturbances, Deutscher and Gabriel [9] offered a solution. Adaptive observer designs for $(n+1)\times(n+1)$ systems, useful even with unknown parameters, are available in [2]. Control of hyperbolic systems with nonstrict-feedback connections with ordinary differential equations (ODEs) has also been investigated [6]. These advancements, however, are predicated on nonzero characteristic speeds and do not apply otherwise.

Outside of the backstepping literature, only a few works consider zero or vanishing characteristic speeds, as can be seen in [4], [16], or [19]. A methodology employing static output feedback controllers is detailed in [21], albeit under stringent structural stability prerequisites for the system's coefficients.

We refer to hyperbolic systems containing states with zero velocity as *atachic* (meaning "with no velocity" in Greek; recall the term *isotachic* in [15] and the special properties of PDEs of equal characteristic speeds). The disregard in the control literature for such systems does not imply that they are not of practical interest. On the contrary, multiple applications do exist. A first example is a model of heat transfer dynamics in solar thermal plants based on direct steam generation technology [14], where the receiver temperature dynamics is an atachic state. Adiabatic flows, such as the Saint-Venant equations or the isentropic and full Euler equation for gas dynamics [4], which are of practical interest in accounting for the trend to operate combined sewer systems and other channel networks, also admit zero characteristic speed.

Another application that fits into the atachic framework is the intensity dynamics of the laser beam [18]. In several laser applications, the maximization of the energy extracted from the laser pulse is critical for the process's efficiency. For example, in the polysilicon process for manufacturing flat panel displays, one of the main problems is to obtain enough instantaneous laser power to melt as large an area as desired. Photolithography is another example where optical exposures must be accomplished with fewer pulses of higher energy. A few more examples include models with thermoacoustic instabilities—a zero transport velocity in thermoacoustics is a direct consequence of the second law of thermodynamics—double-pass laser amplifiers [18], neurofilament transport in axons [5], and biomass production in photobioreactors [12].

Motivated by these applications, this article aims to extend the infinite-dimensional backstepping methodology to what we denote, extending the $(n+m)\times(n+m)$ notation, as $(1+n+1)\times(1+n+1)$ 1-D hyperbolic systems, which contain one rightward convecting unactuated state, n nonconvecting/zero-speed/atachic unactuated states, and one leftward-convecting state with boundary actuation. Previous results, such as [10] or [15], are inapplicable since they would result in a controller with infinite gain. In addition, it is shown that not all $(1+n+1)\times(1+n+1)$ systems are stabilizable. The

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homogeneous part of the zero-speed equation must be asymptotically stable for the overall system to be stabilizable without other restrictions.

We apply a backstepping transformation only to the PDEs with nonzero speeds, leaving the state of the zero-speed equation unaltered, but making the target zero-speed equation input-to-state stable with respect to the decoupled and stable counterconvecting nonzero-speed target PDEs. Compared with other results in the literature for hyperbolic PDEs containing states with zero characteristic speeds, our approach can be applied to a richer family of hyperbolic systems that can be unstable in the nonzero speed part of the plant. In particular, we provide numerical simulations to show the effectiveness of the method for an open-loop unstable case. Part of these contributions was previously published in preliminary form in the conference paper [7] for a $(1+1+1)\times(1+1+1)$ system.

We do not consider general heterodirectional systems with zero speeds, these are $(n+m+l)\times(n+m+l)$ systems, in order to maintain clarity and provide comprehensive details, as the simplicity of the $(1+n+1)\times(1+n+1)$ case aids in a thorough exposition. While extending our methodology to encompass these larger systems is straightforward, doing so would considerably complicate and lengthen the proofs mainly due to kernel discontinuities that are unavoidable in such designs, muddling the exposition without yielding significant additional insights.

The rest of this article is organized as follows. In Section II, we present the control problem and some properties of the equations that motivate our assumptions on controlling the system. In Section III, we design a stabilizing control law using the backstepping methodology. The results are illustrated using numerical simulations in Section IV. Finally, Section V concludes this article.

Notations

For a given $u:\mathbb{R}_+\times[0,1]\to\mathbb{R}$, we use the notation $u[t]$ to denote the profile at certain $t\geq 0$, i.e., $(u[t])(x)=u(t,x)\forall x\in[0,1]$. $\mathcal{L}^2(0,1)$ denotes the set of equivalence classes of measurable functions $f:[0,1]\to\mathbb{R}$ for which $\|f\|_2=(\int_0^1|f(x)|^2dx)^{1/2}<+\infty$. For an interval $I\subset\mathbb{R}_+$, the space $\mathcal{C}^0(I;\mathcal{L}^2(0,1))$ is the space of continuous mappings $I\ni t\to u[t]\in\mathcal{L}^2(0,1)$. Finally, $\mathcal{H}^1(0,1)$ denotes the Sobolev space of functions in $\mathcal{L}^2(0,1)$ with all its first-order weak derivatives in $\mathcal{L}^2(0,1)$.

II. Problem Statement

For $n\geq 1$, consider the following set of $(1+n+1)$ hyperbolic system:

$$\partial_t u(t,x)=-\lambda_1\partial_x u(t,x)+\sigma_{12}p(t,x)+\Theta_1 v(t,x) \tag{1}$$

$$\partial_t v(t,x)=\Omega_1 u(t,x)+\Omega_2 p(t,x)+\Psi v(t,x) \tag{2}$$

$$\partial_t p(t,x)=\lambda_2\partial_x p(t,x)+\sigma_{21}u(t,x)+\Theta_2 v(t,x) \tag{3}$$

$$u(t,0)=U(t)+qp(t,0) \tag{4}$$

$$p(t,1)=\rho u(t,1) \tag{5}$$

where $t\in[0,\infty)$ is the time, $x\in[0,1]$ is the space, the states are given by u, p, and $v=(v_1,\ \ldots\ ,v_n)$, and the control action is U. The transport speeds satisfy $\lambda_1>0>-\lambda_2$, and ρ and q are nonzero reflection coefficients. The other coefficients of the system are

$$\Theta_1=(\theta_{11}\quad\ldots\quad\theta_{1n}),\qquad \Theta_2=(\theta_{21}\quad\ldots\quad\theta_{2n})$$
$$\Omega_1=(\omega_{11}\quad\ldots\quad\omega_{n1})^T,\qquad \Omega_2=(\omega_{12}\quad\ldots\quad\omega_{n2})^T$$
$$\Psi=\{\psi_{ij}\}_{1\leq i\leq n,1\leq j\leq n}.$$

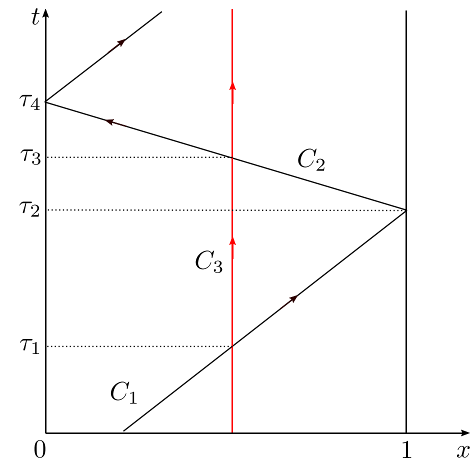

Fig. 1. Characteristic lines of system. The characteristic lines C_1 (with slope λ_1) and C_2 (with slope $-\lambda_2$) correspond to (1) and (3), respectively, whereas C_3 corresponds to (2) with $n=1$. The reflection mechanism is illustrated at the points $x=0$ and $x=1$ at the time instants τ_2 and τ_4, respectively.

Finally, the initial conditions of (1)–(4) are

$$u(0,x)=u_0(x),\ v(0,x)=v_0(x),\ p(0,x)=p_0(x) \tag{6}$$

with $u_0\in\mathcal{L}^2(0,1)$, $v_0\in(\mathcal{L}^2(0,1))^n$, and $p_0\in\mathcal{L}^2(0,1)$.

System (1)–(6) is hyperbolic with two characteristic speeds with opposite signs, associated with (1)–(3), respectively, and n identically zero speeds for (2). The latter means that the characteristics corresponding to (2) are vertical on the $(x,\ t)$ plane, as can be seen in Fig. 1. Note that the vertical characteristic C_3 intersects the characteristics C_1 and C_2 at τ_1 and τ_3, respectively, making the solution change and ultimately destabilizing the system.

As shown in [19], the stabilizability of (1)–(6) can be obtained (without any constraints on the parameters) by imposing boundary controllers and an in-domain controller for the equation with zero characteristic speed. However, if internal controllers are not realizable, then the stabilizability can be obtained only for very strict cases. For that reason, we state the following assumption.

Assumption 2.1: It is assumed that Ψ is a Hurwitz matrix.

Note that this assumption means that (2) would be exponentially stable in the absence of the coupling with (1)–(3). In the next section, we will show, with a simplified version of system (1)–(6), that if Assumption 2.1 is violated, then the system is nonstabilizable. Although this example does not consider counterconvective PDEs, it is to be expected that this result is valid for the general form (1)–(6).

A. Stabilizability of Systems With Zero Characteristic Speeds

The primary goal of this section is to investigate the stabilizability of systems with zero characteristic speeds, with the final objective of justifying Assumption 2.1. We will first establish that stabilizability is not feasible under general conditions with a very simple example based on explicit solutions (see Section II-A1), and then systematically prove, for a more general case, the conditions under which stabilizability is not feasible (see Section II-A2), both pointwise stabilization and $\mathcal{L}^2$ stabilization, thus providing a clear and rigorous mathematical framework to understanding the stabilization limitations of such systems.

1) Particular Case Showing Lack of Stabilizability: Consider first a simplified version of the problem (1)–(3)

$$\partial_t u(t,x)=-\,\partial_x u(t,x) \tag{7}$$

$$\partial_t v(t,x) = \psi v(t,x) + u(t,x) \quad (8)$$

$$u(t,0) = U(t) \quad (9)$$

with $\psi > 0$, and $u(0,x) = 0$ and $v(0,x) = v_0(x)$ for all $x \in [0,1]$.

One way to argue the lack of stabilizability of (7)–(9) is to convert the equations into a form in which a part of the dynamics is autonomous and unstable. To do that, first note that after $t \geq 1$, the solution of (7) is $u(t,x) = U(t-x)$, and thus, the atachic equation (8) can be rewritten to

$$\partial_t v(t,x) = \psi v(t,x) + U(t-x). \quad (10)$$

Now, set any $0 < x_1 \leq 1$, and define $v_1(t) = v(t,0)$ and $v_2(t) = v(t,x_1)$. Then,

$$\dot{v}_1(t) = \psi v_1(t) + U(t) \quad (11)$$

$$\dot{v}_2(t) = \psi v_2(t) + U(t-x_1). \quad (12)$$

For $t \geq x_1$, define $w(t) = v_1(t-x_1) - v_2(t)$. Then, it follows that:

$$\dot{w}(t) = \psi w(t). \quad (13)$$

Note that (13) is autonomous and unstable, and thus, unless $w(x_1) = 0$, we have $|w(t)| \to \infty$ as $t \to \infty$. By definition, the only way to have $w(x_1) = 0$ is if $v_2(x_1) = v_1(0)$. Thus, using the explicit solution of (12), which is $v_2(t) = v_2(0)\mathrm{e}^{\psi t} + \int_0^t U(\tau - x_1)\mathrm{e}^{\psi(t-\tau)}d\tau$, into $v_2(x_1) = v_1(0)$, it follows that $v_2(0)\mathrm{e}^{\psi x_1} + \int_0^{x_1} U(\tau - x_1)\mathrm{e}^{\psi(x_1-\tau)}d\tau = v_1(0)$. Recalling that $v_1(0) = v(0,0) = v_0(0)$, and $v_2(0) = v(0,x_1) = v_0(x_1)$, and calling $x_1 = x$, we get (after a slight change in the variable of integration)

$$v_0(0) = \mathrm{e}^{\psi x} v_0(x) + \int_{-x}^{0} \mathrm{e}^{-\psi\tau} U(\tau) d\tau \quad (14)$$

must be verified by $U(t)$ for $t \in [-1,0]$ so that the system can be stabilized. If that is the case, one has $w(t) = 0$ for all $0 < x_1 \leq 1$, and therefore, it holds that for $t \geq x$, $v(t,x) = v(t-x,0) = v_1(t-x)$, where v_1 satisfies (11), and so designing a stabilizing control law for v_1 also stabilizes for v.

This quick argument shows that one cannot expect stabilizability of (7)–(9) if $\psi > 0$ (it would also hold for $\psi = 0$).

2) Formal Lack of Stabilizability in a More General Case:

We now formalize and generalize the above arguments and prove the lack of stabilizability of the following hyperbolic system:

$$\partial_t u(t,x) = -\lambda \partial_x u(t,x) \quad (15)$$

$$\partial_t v(t,x) = \psi v(t,x) + \omega u(t,x) \quad (16)$$

$$u(t,0) = U(t) \quad (17)$$

where $\lambda > 0$, $\psi \geq 0$, and $\omega \in \mathbb{R}$.

The initial condition of (15)–(17) is

$$u(0,x) = u_0(x), \qquad v(0,x) = v_0(x) \quad (18)$$

with $u_0 \in \mathcal{H}^1(0,1)$ and $v_0 \in \mathcal{H}^1(0,1)$.

The idea consists in proving the existence of a continuous functional $P(u,v)$, with the set $S = \{(u,v) \in \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1) : P(u,v) = 0\}$ is nonempty and $\frac{dP}{dt}(u,v) \geq 0$ for all $(u,v) \in S$ and for all $U(t) \in \mathbb{R}$.

Proposition 2.1: Let $\lambda > 0$, $\psi \geq 0$, and $\omega \in \mathbb{R}$ be given constants. Let $S \subset \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1)$ be the linear subspace

$$S = \{(u,v) \in \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1) : P(u,v) = 0\} \quad (19)$$

where $P : \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1) \to \mathcal{H}^1(0,1)$ is the linear operator defined by

$$(P(u,v))(x) = v(x) - \mathrm{e}^{-\lambda^{-1}\psi x} v(0) + \lambda^{-1}\omega \int_0^x \mathrm{e}^{-\lambda^{-1}\psi(x-s)} u(s) ds \quad (20)$$

for all $(u,v) \in \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1)$ and $x \in [0,1]$.

Then, the following property holds for all $(u_0,v_0) \in \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1) \backslash S$.

(P) For every input $U \in \mathcal{C}^1(\mathbb{R}_+)$ with $U(0) = u_0(0)$, the corresponding unique solution $(u,v) \in (\mathcal{C}^0(\mathbb{R}_+;\mathcal{H}^1(0,1)) \cap \mathcal{C}^1(\mathbb{R}_+;\mathcal{L}^2(0,1)))^2$ of the initial-boundary value problem (15)–(18) does not satisfy $\lim_{t\to\infty}(\|u[t]\|_2) = \lim_{t\to\infty}(\|v[t]\|_2) = 0$.

Proposition 2.2: Let $\lambda > 0$, $\psi \geq 0$, and $\omega \in \mathbb{R}$ be given constants. Then, the linear subspace $S \subset \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1)$ defined by (19) is positively invariant. In other words, for every $(u_0,v_0) \in S$ and for every input $U \in \mathcal{C}^1(\mathbb{R}_+)$, with $U(0) = u_0(0)$, the corresponding unique solution $(u,v) \in (\mathcal{C}^0(\mathbb{R}_+;\mathcal{H}^1(0,1)) \cap \mathcal{C}^1(\mathbb{R}_+;\mathcal{L}^2(0,1)))^2$ of the initial value problem (15)–(18) satisfies $(u[t],v[t]) \in S$ for all $t \geq 0$.

Proposition 2.3: Let $\lambda > 0$, $\psi \geq 0$, and $\omega \in \mathbb{R}$ be given constants. Let $S \subset \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1)$ be the linear subspace defined by (19). Then, the following property holds for all $(u_0,v_0) \in \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1) \backslash S$

(P') For every input $U \in \mathcal{C}^1(\mathbb{R}_+)$ with $U(0) = u_0(0)$, the corresponding unique solution $(u,v) \in (\mathcal{C}^0(\mathbb{R}_+;\mathcal{H}^1(0,1)) \cap \mathcal{C}^1(\mathbb{R}_+;\mathcal{L}^2(0,1)))^2$ of the initial-boundary value problem (15)–(18) does not satisfy $\lim_{t\to\infty} v(t,x) = 0$, for all $x \in [0,1]$.

The proofs of Propositions 2.1–2.3 are given in the Appendix.

Remark 2.1: Property **(P')** is different from property **(P)** since **(P')** deals with pointwise convergence to zero whereas property **(P)** deals with convergence to zero in $\mathcal{L}^2$. It is well known that convergence in $\mathcal{L}^2$ does not imply pointwise convergence and pointwise convergence does not imply convergence in $\mathcal{L}^2$. Moreover, property **(P')** deals with the pointwise convergence of v only, whereas **(P)** deals with convergence in $\mathcal{L}^2$ of (u,v).

Remark 2.2: Considering the transformation

$$w(t,x) = v(t,x) - \mathrm{e}^{-\lambda^{-1}\psi x} v(t,0) + \lambda^{-1}\omega \int_0^x \mathrm{e}^{-\lambda^{-1}\psi(x-s)} u(t,s) ds \quad (21)$$

system (15)–(17) is equivalent to the system

$$\partial_t u(t,x) = -\lambda \partial_x u(t,x) \quad (22)$$

$$\partial_t w(t,x) = \psi w(t,x) \quad (23)$$

$$\dot{y}(t) = \psi y(t) + \omega U(t) \quad (24)$$

$$u(0) = U(t) \quad (25)$$

$$w(t,0) = 0 \quad (26)$$

with state $(u,w,y) \in \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1) \times \mathbb{R}$.

To see this, notice that (21) and the definition $y(t) = v(t,0)$ give us (24). On the other hand, the inverse transformation is given by $v(x) = w(x) + \mathrm{e}^{-\lambda^{-1}\psi x} y - \lambda^{-1}\omega \int_0^x \mathrm{e}^{-\lambda^{-1}\psi(x-s)} u(s) ds$ for $x \in [0,1]$ and transforms (24) to (15)–(17). The system may be defined on the invariant subspace S defined by (19) and corresponds to the case $w = 0$ for (24). Thus, system (15)–(17) with $\omega \neq 0$ on the invariant subspace S may be stabilized by the feedback law

$$U(t) = -\frac{k}{\omega}v(0)$$

where $k > \omega$ is a design constant. When $\omega = 0$, (15)–(18) can be stabilized on the subspace $S' = \{(u, v) \in S : v(0) = 0\}$ with $U(t) = 0$.

III. Control Design

Having justified Assumption 2.1, in this section, we design a backstepping controller so that the null solution of (1)–(6) becomes stable. It will be assumed that the full-state measurements are available for the control law.

Consider the following Volterra transformation:

$$\alpha = u - \int_x^1 K_1(x,\xi)u(t,\xi)d\xi - \int_x^1 K_2(x,\xi)p(t,\xi)d\xi - \int_x^1 G(x,\xi)v(t,\xi)d\xi, \tag{27}$$

$$\beta = p - \int_x^1 Q_1(x,\xi)u(t,\xi)d\xi - \int_x^1 Q_2(x,\xi)p(t,\xi)d\xi - \int_x^1 R(x,\xi)v(t,\xi)d\xi \tag{28}$$

where the kernels K_i and Q_i, for $i \in \{1,2\}$, $G = (G_1 \quad \ldots \quad G_n)$, and $R = (R_1 \quad \ldots \quad R_n)$ satisfy the following PDEs:

$$\lambda_1\partial_x K_1 + \lambda_1\partial_\xi K_1 = -\sigma_{21}K_2 - G\Omega_1 \tag{29}$$

$$\lambda_1\partial_x K_2 - \lambda_2\partial_\xi K_2 = -\sigma_{12}K_1 - G\Omega_2 \tag{30}$$

$$\lambda_1\partial_x G = -K_1\Theta_1 - K_2\Theta_2 - G\Psi \tag{31}$$

$$\lambda_2\partial_x Q_1 - \lambda_1\partial_\xi Q_1 = \sigma_{21}Q_2 + R\Omega_1 \tag{32}$$

$$\lambda_2\partial_x Q_2 + \lambda_2\partial_\xi Q_2 = \sigma_{12}Q_1 + R\Omega_2 \tag{33}$$

$$\lambda_2\partial_x R = Q_1\Theta_1 + Q_2\Theta_2 + R\Psi \tag{34}$$

with boundary conditions

$$K_1(x,1) = \frac{\lambda_2\rho}{\lambda_1}K_2(x,1), \qquad Q_1(x,x) = \frac{\sigma_{21}}{\lambda_1+\lambda_2} \tag{35}$$

$$K_2(x,x) = -\frac{\sigma_{12}}{\lambda_1+\lambda_2}, \qquad Q_2(x,0) = \frac{\lambda_1}{\lambda_2 q}Q_1(x,0) \tag{36}$$

$$G(x,x) = -\frac{\Theta_1}{\lambda_1}, \qquad R(x,x) = \frac{\Theta_2}{\lambda_2} \tag{37}$$

for $k \in \{1,\ldots,n\}$.

These kernels evolve in the triangular domain

$$\mathcal{T} = \{(x,\xi) \in \mathbb{R}^2 : 0 \le x \le \xi \le 1\}.$$

The explicit solution of (31) and (34), together with the boundary condition (37), is given by

$$G(x,\xi) = -\frac{1}{\lambda_1}\Theta_1\Phi_1(x,\xi) - \frac{1}{\lambda_1}\int_\xi^x K_1(\tau,\xi)\Theta_1\Phi_1(x,\tau)d\tau - \frac{1}{\lambda_1}\int_\xi^x K_2(\tau,\xi)\Theta_2\Phi_1(x,\tau)d\tau \tag{38}$$

$$R(x,\xi) = \frac{1}{\lambda_2}\Theta_2\Phi_2(x,\xi) + \frac{1}{\lambda_2}\int_\xi^x Q_1(\tau,\xi)\Theta_1\Phi_2(\tau,x)d\tau + \frac{1}{\lambda_2}\int_\xi^x Q_2(\tau,\xi)\Theta_2\Phi_2(\tau,x)d\tau \tag{39}$$

where $\Phi_k(x,\xi) = \mathrm{e}^{\Psi\frac{(\xi-x)}{\lambda_k}}$, for $k \in \{1,2\}$, is the state-transition matrix.

Plugging (38) and (39) into (29)–(30) and (32)–(33), respectively, one obtains the following system of integro–differential equations:

$$\lambda_1\partial_x K_1 + \lambda_1\partial_\xi K_1 = -\sigma_{21}K_2 + \lambda_1^{-1}\Theta_1\Phi_1(x,\xi)\Omega_1 + \int_\xi^x \left(K_1(\tau,\xi)\Theta_1 + K_2(\tau,\xi)\Theta_2\right)\lambda_1^{-1}\Phi_1(x,\tau)\Omega_1 d\tau \tag{40}$$

$$\lambda_1\partial_x K_2 - \lambda_2\partial_\xi K_2 = -\sigma_{12}K_1 + \lambda_1^{-1}\Theta_1\Phi_1(x,\xi)\Omega_2 + \int_\xi^x \left(K_1(\tau,\xi)\Theta_1 + K_2(\tau,\xi)\Theta_2 d\tau\right)\lambda_2^{-1}\Phi_1(x,\tau)\Omega_2 \tag{41}$$

$$\lambda_2\partial_x Q_1 - \lambda_1\partial_\xi Q_1 = \sigma_{21}Q_2 + \lambda_2^{-1}\Theta_2\Phi_2(x,\xi)\Omega_1 + \int_\xi^x \left(Q_1(\tau,\xi)\Theta_1 + Q_2(\tau,\xi)\Theta_2\right)\lambda_2^{-1}\Phi_2(\tau,x)\Omega_1 d\tau \tag{42}$$

$$\lambda_2\partial_x Q_2 + \lambda_2\partial_\xi Q_2 = \sigma_{12}Q_1 + \lambda_2^{-1}\Theta_2\Phi_2(x,\xi)\Omega_2 + \int_\xi^x \left(Q_1(\tau,\xi)\Theta_1 + Q_2(\tau,\xi)\Theta_2\right)\Phi_2(\tau,x)\lambda_2^{-1}\Omega_2 d\tau. \tag{43}$$

These equations, together with boundary conditions (35) and (36), are well posed as shown in the next lemma.

Lemma 3.1: The PDE system (40)–(43) with boundary conditions (35) and (36) has a unique $\mathcal{C}^1(\mathcal{T})$ solution.

Proof: The well posedness follows the steps of the proof in [20, Appendix], with some slight modifications to account for the integral terms as in [17]. ■

From the previous Lemma and the theory of Volterra integral equations, it follows that the inverse of transformation (27) and (28) always exists and can be defined as:

$$u = \alpha + \int_x^1 L_1(x,\xi)\alpha(t,\xi)d\xi + \int_x^1 L_2(x,\xi)\beta(t,\xi)d\xi + \int_x^1 S(x,\xi)v(t,\xi)d\xi \tag{44}$$

$$p = \beta + \int_x^1 M_1(x,\xi)\alpha(t,\xi)d\xi + \int_x^1 M_2(x,\xi)\beta(t,\xi)d\xi + \int_x^1 E(x,\xi)v(t,\xi)d\xi \tag{45}$$

where L_i and M_i, with $i \in \{1,2\}$, $S = (S_1 \quad \ldots \quad S_n)$, and $E = (E_1 \quad \ldots \quad E_n)$ are the inverse kernels, which verify equations similar to (29)–(37).

Using the above results, we state the following lemma.

Lemma 3.2: Let U be given by the following control law:

$$U(t) = -qp(t,0) + \int_0^1 K_1(0,\xi)u(t,\xi)d\xi + \int_0^1 K_2(0,\xi)p(t,\xi)d\xi + \int_0^1 G(0,\xi)v(t,\xi)d\xi. \tag{46}$$

Then, the transformation (27) and (28) maps (1)–(4) into the following target system:

$$\partial_t \alpha(t,x) = -\lambda_1 \partial_x \alpha(t,x) \tag{47}$$

$$\partial_t \beta(t,x) = \lambda_2 \partial_x \beta(t,x) \tag{48}$$

$$\begin{aligned}\partial_t v(t,x) &= \Omega_1 \alpha(t,x) + \Omega_2 \beta(t,x) + \Psi v(t,x) \\ &\quad + \int_x^1 N_1(x,\xi)\alpha(t,\xi)d\xi + \int_x^1 N_2(x,\xi)\beta(t,\xi)d\xi \\ &\quad + \int_x^1 N_3(x,\xi)v(t,\xi)d\xi\end{aligned} \tag{49}$$

$$\beta(t,1) = \rho\alpha(t,1) \tag{50}$$

$$\alpha(t,0) = 0 \tag{51}$$

with

$$N_j(x,\xi) = \Omega_1 L_j(x,\xi) + \Omega_2 M_j(x,\xi), \qquad \text{for } j \in \{1,2\}$$
$$N_3(x,\xi) = \Omega_1 S(x,\xi) + \Omega_2 E(x,\xi).$$

Proof: Differentiating (27) and (28) with respect to time and space, integrating by parts, substituting the resultant expressions into (1) and (2), and applying (44) and (45), we obtain (47)–(49).

Evaluating (27) at $x = 0$, substituting it into (4) and using (46), we obtain (50). Finally, evaluating (28) for $x = 1$, substituting it into (5) and using (45), we obtain (50). ■

A. Stability of the Target System

The stability properties of the target system (47)–(51) are proved in the following lemma.

Lemma 3.3: The zero equilibrium of system (47)–(51) is exponentially stable in the L_2 sense.

Proof: Consider the following Lyapunov functional:

$$\begin{aligned}V(t) &= \int_0^1 \left(\frac{A}{\lambda_1}\mathrm{e}^{-\mu x}\alpha^2(t,x) + \frac{B}{\lambda_2}\mathrm{e}^{\mu x}\beta^2(t,x)\right)dx \\ &\quad + \frac{1}{2}\int_0^1 v^T(t,x)P(x)v(t,x)dx\end{aligned} \tag{52}$$

where $P(x) = \mathrm{e}^{-\vartheta x} I_{n\times n}$, in which $I_{n\times n}$ stands for the $n \times n$ identity matrix, and A, B, μ, and ϑ are constants to be defined. Differentiating V with respect to time yields

$$\begin{aligned}\dot V(t) &= 2\int_0^1 \Big(- A\mathrm{e}^{-\mu x}\alpha(t,x)\partial_x\alpha(t,x) \\ &\quad + B\mathrm{e}^{\mu x}\beta(t,x)\partial_x\beta(t,x)\Big)dx \\ &\quad + \int_0^1 v^T(t,x)P(x)\Big(\Omega_1\alpha(t,x) \\ &\quad + \Omega_2\beta(t,x) + \Psi v(t,x)\Big)dx \\ &\quad + \int_0^1 v^T(t,x)P(x)\left(\int_x^1 \Big(N_1(x,s)\alpha(t,s) \right. \\ &\quad \left. + N_2(x,s)\beta(t,s) + N_3(x,s)v(t,s)\Big)ds\right)dx.\end{aligned} \tag{53}$$

Integrating by parts the first two terms in the right-hand side of (53) and plugging the boundary conditions (50) and (51) yields

$$\begin{aligned}&\int_0^1 \left(-A\mathrm{e}^{-\mu x}\alpha(t,x)\partial_x\alpha(t,x) + B\mathrm{e}^{\mu x}\beta(t,x)\partial_x\beta(t,x)\right)dx \\ &= -\mu\int_0^1 \Big(A\mathrm{e}^{-\mu x}\alpha^2(t,x) + B\mathrm{e}^{\mu x}\beta^2(t,x)\Big)dx \\ &\quad + (B\rho^2\mathrm{e}^{\mu} - A\mathrm{e}^{-\mu})\alpha^2(t,1) - B\beta^2(t,0).\end{aligned} \tag{54}$$

To make further progress, we will now compute an upper bound for the rest of the terms in the right-hand side of (53).

By using Young's inequality and considering the fact that $\mathrm{e}^{-\vartheta x} \le 1 \le \mathrm{e}^{\mu}\mathrm{e}^{-\mu x}$ and $\mathrm{e}^{-\delta x} \le 1 \le \mathrm{e}^{\mu x}$, we have that

$$\int_0^1 v^T(t,x)P(x)\Big(\Omega_1\alpha(t,x) + \Omega_2\beta(t,x)\Big)dx$$
$$\le \frac{2\mathrm{e}^{\mu}}{\rho(\Psi)}\Omega_1^T\Omega_1\int_0^1 \mathrm{e}^{-\mu x}\alpha^2(t,x)dx \tag{55}$$
$$+ \frac{2}{\rho(\Psi)}\Omega_2^T\Omega_2\int_0^1 \mathrm{e}^{\mu x}\beta^2(t,x)dx$$
$$+ \frac{\rho(\Psi)}{4}\int_0^1 v^T(t,x)P(x)v(t,x)dx \tag{56}$$

where $\rho(\Psi)$ is the spectral radius of Ψ.

Now, define $\overline{N}_i = \|N_i(x,y)\|_\infty$, for $i \in \{1,2,3\}$. Then, using the Cauchy–Schwarz and Young's inequalities, we get

$$\begin{aligned}&\int_0^1\int_x^1 v^T(t,x)P(x)\left(N_1(x,s)\alpha(t,s) + N_2(x,s)\beta(t,s)\right)dsdx \\ &\le \frac{2n\overline{N}_1^2\mathrm{e}^{\mu}}{\rho(\Psi)}\int_0^1 \mathrm{e}^{-\mu x}\alpha^2(t,x)dx + \frac{2n\overline{N}_2^2}{\rho(\Psi)}\int_0^1 \mathrm{e}^{\mu x}\beta^2(t,x)dx \\ &\quad + \frac{\rho(\Psi)}{4}\int_0^1 v^T(t,x)P(x)v(t,x)dx.\end{aligned} \tag{57}$$

Finally

$$\int_1^x |N_3(x,s)v(t,s)|ds$$
$$\le \overline{N}_3\mathrm{e}^{\vartheta/2}\frac{\mathrm{e}^{\vartheta x/2}}{\sqrt{\vartheta}}\sqrt{\int_0^1 v^T(t,s)P(s)v(t,s)ds}$$

where at the last step, Cauchy–Schwarz and Young's inequalities were again applied.

Therefore, we reach

$$\begin{aligned}\dot V \le &-\left(A\sigma - \frac{2(\Omega_1^T\Omega_1 + \overline{N}_1^2)\mathrm{e}^{\sigma}}{\rho(\Psi)}\right)\int_0^1 \mathrm{e}^{-\mu x}\alpha^2(t,x)dx \\ &- \left(B\sigma - \frac{2(\Omega_2^T\Omega_2 + \overline{N}_2^2)}{\rho(\Psi)}\right)\int_0^1 \mathrm{e}^{\mu x}\beta^2(t,x)dx \\ &- \left(\frac{\rho(\Psi)}{2} - \frac{\overline{N}_3}{\sqrt{\vartheta}}\right)\int_0^1 v^T(t,x)P(x)v(t,x)dx \\ &+ (Aq^2 - B)\beta^2(t,0).\end{aligned}$$

Choosing

$$A = \mathrm{e}^{\sigma}, \quad B = Aq^2 + 1, \quad \vartheta = \frac{16\overline{N}_3^2}{\rho(\Psi)^2}$$

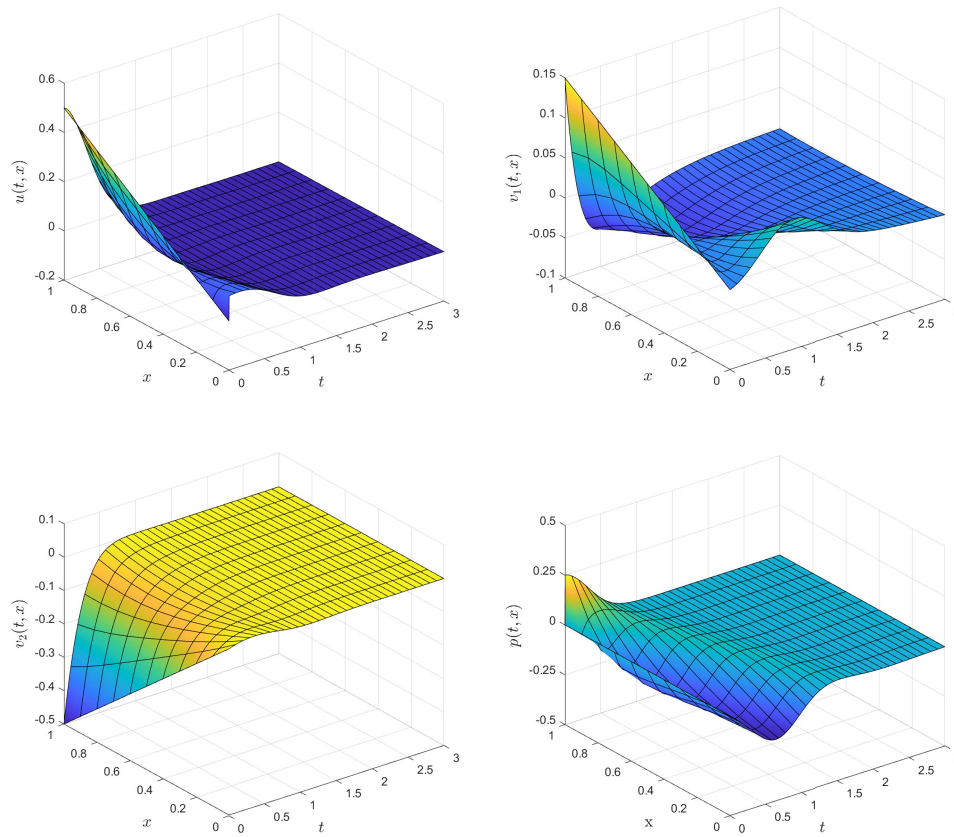

Fig. 2. Distributed states evolution as a function of time and space.

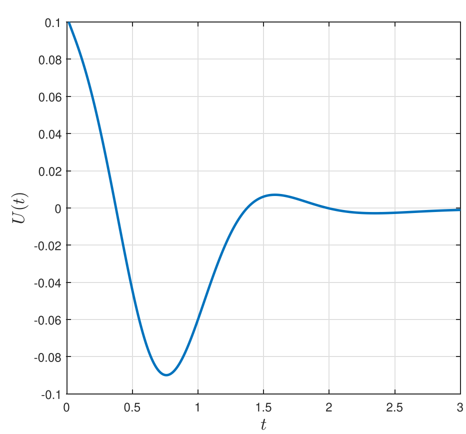

Fig. 3. State feedback control law as a function of time.

$$\mu = \max\left\{\frac{2(\Omega_1^T\Omega_1 + \overline{N}_1^2)}{\rho(\Psi)}, \frac{2(\Omega_2^T\Omega_2 + \overline{N}_2^2)}{\rho(\Psi)}\right\} + 1$$

we get

$$\begin{aligned}\dot{V} \leq & -\int_0^1 \mathrm{e}^{-\mu x}\alpha^2(t,x)dx - \int_0^1 \mathrm{e}^{\mu x}\beta^2(t,x)dx \\ & -\frac{\rho(\Psi)}{4}\int_0^1 v^T(t,x)P(x)v(t,x)(t,x)dx \\ \leq & -KV\end{aligned}$$

for $K = \min\{\frac{2\lambda_1}{A}, \frac{2\lambda_2}{B}, \frac{\rho(\Psi)}{2}\} > 0$, thus proving exponential stability of the equilibrium $\alpha \equiv \beta \equiv v \equiv 0$. ■

IV. Numerical Simulations

In this section, we present numerical simulations of system (1)–(5) with the proposed control law (46) considering $n = 2$. The parameters were chosen as $q = -0.7$, $\rho = 0.5$, $\sigma_{12} = 2.5$, $\sigma_{21} = -3.5$, $\theta_{11} = 0.25$, $\theta_{12} = 0.1$, $\theta_{21} = 0.25$, $\theta_{22} = -0.1$, $\omega_{11} = 0.3$, $\omega_{12} = 0.8$, $\omega_{21} = -0.65$, $\omega_{22} = 0.3$, $\psi_{11} = -1.5$, $\psi_{12} = 2$, $\psi_{21} = -1$, $\psi_{22} = -2$, $\lambda_1 = 1.25$, and $\lambda_2 = 0.9$, which corresponds to an open-loop unstable system. The finite differences method was employed in MATLAB to compute the states of the system and solve the kernel PDEs.

Figs. 2 and 3 show the closed-loop states and the control signal, respectively. As can be seen in Figs. 2 and 3, the system states decay to zero after an initial transient.

V. Conclusion

In this work, we introduced the state feedback stabilization of a class of hyperbolic systems containing an atachic subsystem (zero characteristic velocities), denoted as $(1 + n + 1) \times (1 + n + 1)$ systems. We showed that stabilizability requires that the atachic subsystem be asymptotically stable. Under this condition, we applied the backstepping methodology to guarantee the closed-loop exponential stability in the $\mathcal{L}^2$ sense. Interestingly, we employ the invertible Volterra transformation only for the PDEs with nonzero characteristic speeds, leaving the atachic subsystem unaltered in the target system but making it input-to-state stable (ISS) with respect to the counterconvecting nonzero-speed states. As future work, a Luenberger-type state observer with boundary measurements of states with nonzero characteristic speeds can be designed to obtain the associated output feedback controller. Future discussions should include networks of systems of hyperbolic balance laws coupled with ODEs.

Appendix

A. Proof of Proposition 2.1

Let arbitrary $(u_0, v_0) \in \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1) \backslash S$ be given. For all $x \in [0,1]$, define the functions

$$\varphi(x) \triangleq \psi \int_0^x v_0(s)ds + \omega \int_0^x u_0(s)ds + \lambda v_0(x) \quad (58)$$

$$h(x) \triangleq \varphi(x) - K \quad (59)$$

where

$$K \triangleq \int_0^1 \varphi(x)dx. \quad (60)$$

Definitions (19) and (58)–(60) give the implication

$$h(x) \equiv 0 \Rightarrow (u_0, v_0) \in S. \quad (61)$$

Therefore, since $(u_0, v_0) \notin S$ it follows from (61) that $h \in \mathcal{H}^1(0,1)$ is a nonidentically zero function, i.e., $h \neq 0$. Moreover, definitions (59) and (60) imply that $\int_0^1 h(x)dx = 0$. Since the Cauchy–Schwarz inequality in $\mathcal{L}^2(0,1)$ holds as an equality if and only if the functions are linearly dependent in $\mathcal{L}^2(0,1)$, we obtain from (19) and (58)–(60) that

$$\|\varphi\|_2 = |K| \Leftrightarrow (u_0, v_0) \in S. \quad (62)$$

Consequently, it follows from (62) that:

$$\int_0^1 h(x) \left(\psi \int_0^x v_0 ds + \omega \int_0^x u_0(s)ds + \lambda v_0(x) \right) dx = \|\varphi\|_2^2 - K^2 > 0. \quad (63)$$

We show next that for every input $U \in \mathcal{C}^1(\mathbb{R}_+)$ with $U(0) = u_0(0)$, the unique solution $(u,v) \in (\mathcal{C}^0(\mathbb{R}_+; \mathcal{H}^1(0,1)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{L}^2(0,1)))^2$ of the initial-boundary value problem (15)–(18) does not satisfy $\lim_{t\to\infty}(\|u[t]\|_2) = \lim_{t\to\infty}(\|v[t]\|_2) = 0$. By contradiction, suppose that there exists an input $U \in \mathcal{C}^1(\mathbb{R}_+)$ with $U(0) = u_0(0)$ for which the unique solution $(u,v) \in (\mathcal{C}^0(\mathbb{R}_+; \mathcal{H}^1(0,1)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{L}^2(0,1)))^2$ of the initial-boundary value problem (15)–(18) satisfies $\lim_{t\to\infty}(\|u[t]\|_2) = \lim_{t\to\infty}(\|v[t]\|_2) = 0$. Define the functional

$$R(t) \triangleq \int_0^1 h(x) \left(\psi \int_0^x v(t,s)ds + \omega \int_0^x u(t,s)ds + \lambda v(t,x) \right) dx \quad (64)$$

for $t \geq 0$. Since $(u,v) \in (\mathcal{C}^1(\mathbb{R}_+; \mathcal{L}^2(0,1)))^2$, it follows that $R \in \mathcal{C}^1(\mathbb{R}_+)$. Using (9), (64), and $\int_0^1 h(x)dx = 0$, we get substituting (15)–(17) and integrating by parts, for all $t \geq 0$

$$\dot{R}(t) = \int_0^1 h(x) \left(\psi^2 \int_0^x v(t,s)ds + \psi\omega \times \int_0^x u(t,s)ds + \lambda\psi v(t,x) + \lambda\omega u(t,0) \right) dx = \psi R(t) + \lambda\omega U(t) \int_0^1 h(x)dx = \psi R(t).$$

From the above expression, it follows that:

$$R^2(t) = \mathrm{e}^{2\psi t} R^2(0) \quad (65)$$

for all $t \geq 0$. Notice that definitions (63) and (64) imply that $R(0) \neq 0$ and since $\psi \geq 0$, we get from (65) that

$$R^2(t) \geq R^2(0) > 0 \quad (66)$$

for all $t \geq 0$. Using definition (64) and the Cauchy–Schwarz and Holder inequalities, we obtain the estimate

$$R^2(t) \leq 2(\psi+\lambda)^2 \|h\|_\infty^2 \|v[t]\|_2^2 \quad +2|\omega| \|h\|_\infty^2 \|u[t]\|_2^2 \quad (67)$$

for all $t \geq 0$. However, inequalities (66) and (67) contradict the fact that $\lim_{t\to\infty}(\|u[t]\|_2) = \lim_{t\to\infty}(\|v[t]\|_2) = 0$.

Therefore, for every input $U \in \mathcal{C}^1(\mathbb{R}_+)$ with $U(0) = u_0(0)$, the corresponding unique solution $(u,v) \in (\mathcal{C}^0(\mathbb{R}_+; \mathcal{H}^1(0,1)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{L}^2(0,1)))^2$ of the initial-boundary value problem (15)–(18) does not satisfy $\lim_{t\to\infty}(\|u\|_2) = \lim_{t\to\infty}(\|v\|_2) = 0$. The proof is complete. ■

B. Proof of Proposition 2.2

Consider the transformation (21) for all $t \geq 0$ and $x \in [0,1]$. Note that (15)–(17) imply the following equation:

$$\partial_t w = \psi w. \quad (68)$$

Definitions (19) and (21) give the following equivalence for all $t \geq 0$:

$$(u[t], v[t]) \in S \Leftrightarrow w[t] = 0 \quad (69)$$

while (68) gives the implication $w[0] = 0 \rightarrow w[t] = 0$ for all $t \geq 0$. The proof is complete. ■

C. Proof of Proposition 2.3

Let arbitrary $(u_0, v_0) \in \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1) \backslash S$ be given. We show next that for every input $U \in \mathcal{C}^1(\mathbb{R}_+)$ with $U(0) = u_0(0)$, the unique solution $(u,v) \in (\mathcal{C}^0(\mathbb{R}_+; \mathcal{H}^1(0,1)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{L}^2(0,1)))^2$ of the initial-boundary value problem (15)–(18) does not satisfy $\lim_{t\to\infty} v(t,x) = 0$ for all $x \in [0,1]$.

For the solution of (15)–(18), the following formulas are valid for $t \geq 0$ and $x \in [0,1]$:

$$u(t,x) = \begin{cases} u_0(x - \lambda t), & 0 \leq t \leq \lambda^{-1}x \\ U(t - \lambda^{-1}x), & t > \lambda^{-1}x \end{cases} \quad (70)$$

$$v(t,x) = \mathrm{e}^{\psi t} v_0(x) + \omega \int_0^x \mathrm{e}^{\psi(t-s)} u(s,x)ds. \quad (71)$$

It follows from (15)–(17) that the following equation holds for all $x \in [0,1]$ and $t > \lambda^{-1}x$:

$$\frac{d}{dt}\left(v(t-\lambda^{-1}x, 0) - v(t,x)\right) = \psi\left(v(t-\lambda^{-1}x,0) - v(t,x)\right). \quad (72)$$

Using (72) and continuity of v, we get for all $x \in [0,1]$ and $t \geq \lambda^{-1}x$

$$v(t-\lambda^{-1}x,0) - v(t,x) = \mathrm{e}^{\psi(t-\lambda^{-1}x)}\left(v_0(0) - v(\lambda^{-1}x, x)\right). \quad (73)$$

Since $\psi \geq 0$, we get from (73) for all $x \in [0,1]$ and $t \geq \lambda^{-1}x$

$$\left|v(t-\lambda^{-1}x,0) - v(t,x)\right| \geq \mathrm{e}^{\psi(t-\lambda^{-1}x)}\left|v_0(0) - v(\lambda^{-1}x,x)\right|. \quad (74)$$

Inequality (74) combined with the fact that $\lim_{t\to\infty} v(t,x) = 0$ for all $x \in [0,1]$ implies that $v(\lambda^{-1}x, x) = v_0(0)$ for all $x \in [0,1]$. This equation combined with (71) gives for all $x \in [0,1]$

$$v_0(0) = \mathrm{e}^{\psi\lambda^{-1}x} v_0(x) + \omega \int_0^{\lambda^{-1}x} \mathrm{e}^{\psi(\lambda^{-1}x - s)} u(s,x) ds. \quad (75)$$

Using (70) and (75), we get for all $x \in [0,1]$

$$v_0(0) = v_0(x) + \omega \int_0^{\lambda^{-1}x} \mathrm{e}^{-\psi s} u_0(x - \lambda s) ds. \quad (76)$$

However, (76) in conjunction with definitions (19) and (20) implies that $(u_0, v_0) \in S$; a contradiction. Therefore, for every input $U \in \mathcal{C}^1(\mathbb{R}_+)$ with $U(0) = u_0(0)$, the corresponding unique solution $(u,v) \in (\mathcal{C}^0(\mathbb{R}_+; \mathcal{H}^1(0,1)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{L}^2(0,1)))^2$ of the initial-boundary value problem (15)–(18) does not satisfy $\lim_{t\to\infty} v(t,x) = 0$, for all $x \in [0,1]$. The proof is complete. ■

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