Control of 1-D parabolic PDEs with Volterra nonlinearities, Part I: Design☆

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ABSTRACT

Boundary control of nonlinear parabolic PDEs is an open problem with applications that include fluids, thermal, chemically-reacting, and plasma systems. In this paper we present stabilizing control designs for a broad class of nonlinear parabolic PDEs in 1-D. Our approach is a direct infinite dimensional extension of the finite-dimensional feedback linearization/backstepping approaches and employs spatial Volterra series nonlinear operators both in the transformation to a stable linear PDE and in the feedback law. The control law design consists of solving a recursive sequence of linear hyperbolic PDEs for the gain kernels of the spatial Volterra nonlinear control operator. These PDEs evolve on domains T_n of increasing dimensions $n + 1$ and with a domain shape in the form of a "hyper-pyramid", $0 \leq \xi_n \leq \xi_{n-1} \cdots \leq \xi_1 \leq x \leq 1$. We illustrate our design method with several examples. One of the examples is analytical, while in the remaining two examples the controller is numerically approximated. For all the examples we include simulations, showing blow up in open loop, and stabilization for large initial conditions in closed loop. In a companion paper we give a theoretical study of the properties of the transformation, showing global convergence of the transformation and of the control law nonlinear Volterra operators, and explicitly constructing the inverse of the feedback linearizing Volterra transformation; this, in turn, allows us to prove L^2 and H^1 local exponential stability (with an estimate of the region of attraction where possible) and explicitly construct the exponentially decaying closed loop solutions.

1. Introduction

Boundary control of linear parabolic PDEs is a well established subject with extensive literature. On the other hand, boundary control of *nonlinear* parabolic PDEs is still an open problem as far as general classes of systems are concerned, with many applications of interest including fluids, structures, thermal, chemically-reacting, and plasma systems. Past efforts include the book (Christofides, 2001), which solves problems of nonlinear parabolic PDE control but for *inside-the-domain* actuation, rather than with boundary control, and developments to solve the problem of *motion planning* for boundary controlled nonlinear parabolic PDEs (Meurer, 2005) (using flatness and formal power series) and structural systems (Kugi, Thull, & Kuhnen, 2006) (with a flatness/passivity approach).

When attempting to develop general methods for nonlinear PDEs, it is advisable to take a clue from finite dimensional nonlinear systems. Clearly, one should bet on methods

that have emerged as successful there. This essentially eliminates (direct) optimal control methods, because of the requirement to solve Hamilton–Jacobi–Bellman PDEs, and leaves feedback linearization/backstepping/Lyapunov approaches (Isidori, 1995; Khalil, 2002; Krstic, Kanellakopoulos, & Kokotovic, 1995; Sepulchre, Jankovic, & Kokotovic, 1997) as candidates for extension to PDEs. The backstepping approach for *linear* PDEs has reached the level of maturity where a systematic design procedure (Smyshlyaev & Krstic, 2004) is available for a broad class of parabolic integro-differential equations in 1-D. This systematic procedure has found many applications (Krstic, Smyshlyaev, & Siranosian, 2006; Vazquez & Krstic, 2006), including even extensions to the Navier–Stokes equations (Vazquez & Krstic, 2007) and to adaptive PDE control (Krstic, 2005; Smyshlyaev & Krstic, 2005), and is the starting point for our *nonlinear* developments here.

Our early nonlinear efforts (Aamo & Krstic, 2004; Boskovic & Krstic, 2001, 2002, 2003) were *discretization*-based and were successful in addressing some applications but in general cannot be expected to converge when the discretization step goes to zero, as shown in Balogh and Krstic (2003).

Our approach is a direct infinite dimensional extension of the finite-dimensional feedback linearization/backstepping approaches and employs spatial Volterra series nonlinear operators both in the state transformation to a stable linear PDE and in the

feedback law. The control law design consists of solving a recursive sequence of linear hyperbolic PDEs for the gain kernels of the spatial Volterra nonlinear control operator. These PDEs evolve on domains $\mathcal{T}_n$ of increasing dimensions $n+1$ and with a domain shape in the form of a "hyper-pyramid," $0 \leq \xi_n \leq \xi_{n-1} \cdots \leq \xi_1 \leq x \leq 1$. We illustrate our design method with several examples. One of the examples is analytical, while in the remaining two examples the controller is numerically approximated. For all the examples we include simulations, showing blow up in open loop, and stabilization for large initial conditions in closed loop.

In a companion paper (Vazquez & Krstic, 2008) we study the properties of the transformation, showing global convergence of the transformation and control law nonlinear Volterra operators, and including an explicit construction of the inverse of the feedback linearizing Volterra transformation for both the general case and the analytical example; this, in turn, allows us to prove local L^2 and H^1 exponential stability (with an estimate of the region of attraction where possible) and explicitly construct the exponentially decaying closed loop solutions.

This paper solves the open problem 5.1 in the *Unsolved Problems* volume (Balogh & Krstic, 2003).

2. Class of systems under study

We study the following class of parabolic systems,

$$u_t(t, x) = u_{xx}(t, x) + \lambda(x)u(t, x) + F[u](t, x) + uH[u](t, x), \quad (1)$$

for $x \in (0, 1)$, with the following boundary conditions

$$u_x(t, 0) = qu(t, 0), \qquad u(t, 1) = U(t), \quad (2)$$

where $U(t)$ is the control input and $F[u]$ and $H[u]$ are Volterra series nonlinearities as explained below. In (2), q is a number that can take any value. The particular cases $q = 0$ and $q = \infty$ can be used to model, respectively, Neumann and Dirichlet boundary conditions at $x = 0$. For simplicity we consider a Dirichlet boundary condition at $x = 1$, but different boundary conditions at the controlled end can be accommodated in our design. In the sequel, we will omit time and space dependency of the state when possible.

Define $\xi_0 = x$ and for any $i \leq n$, $\hat{\xi}_i^n = (\xi_i, \ldots, \xi_n)$. Let $\mathcal{T}_n(x, \xi) = \left\{ \hat{\xi}_1^n : 0 \leq \xi_n \leq \cdots \leq \xi_1 \leq x \right\}$ and $\mathcal{T}_n = \mathcal{T}_n(1, \xi)$. Define also

$$\prod^i u = \prod_{j=1}^i u(t, \xi_j), \qquad \prod^{i,k} u = \prod_{\substack{j=1 \\ j \neq k}}^i u(t, \xi_j), \quad (3)$$

$$\int_{\mathcal{T}_n(x, \xi)} f(\hat{\xi}_0^n) d\hat{\xi}_1^n = \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} f(\hat{\xi}_0^n) d\xi_n \ldots d\xi_1. \quad (4)$$

A Volterra series is defined as a functional (i.e., a function that depends on another function), and has the form

$$F[u](t, x) = \sum_{n=1}^\infty F_n[u](t, x), \quad (5)$$

where the notation $F_n[u](t, x)$ emphasizes the fact that each $F_n[u]$ is defined as a functional of $u(t, x)$ and also depends on t and x. The precise definition of each term is, using the notation of (3) and (4),

$$F_n[u](t, x) = \int_{\mathcal{T}_n(x, \xi)} f_n(\hat{\xi}_0^n) \prod^i u \, d\hat{\xi}_1^n, \quad (6)$$

where f_n is called the n-th Volterra (triangular) kernel.

Volterra series (Volterra, 1959) are widely known and studied in the control literature (Boyd, Chua, & Desoer, 1984; Isidori,

1995; Lamnabhi-Lagarrigue, 1996; Sastry, 1999). They are causal functionals (Fliess, 1981) that represent the general solution for nonlinear equations, generalizing the convolution solution for linear systems. An excellent exposition on Volterra series can be found in Rugh (1981).

In the sequel, we will omit time and/or space dependency of the state when possible.

3. Motivating examples

We give two examples of nonlinear plants that fall into the class of systems of Section 2.

3.1. Coupled nonlinear plant

Consider the following nonlinear plant

$$u_t = u_{xx} + \mu v, \quad (7)$$

$$0 = v_{xx} + \omega^2 v + uv + u, \quad (8)$$

where μ and ω are plant parameters, with boundary conditions

$$u(0) = v(0) = 0, \quad (9)$$

$$u(1) = U, \qquad v(1) = V, \quad (10)$$

where $U(t)$ and $V(t)$ are actuation variables.

This kind of plant *structure*, consisting of an evolution equation (Eq. (7), parabolic in this case) coupled with an static equation (Eq. (8), elliptic in this case), arises in some relevant applications, for example fluid mechanics (Vazquez & Krstic, 2007), structural problems (Krstic et al., 2006), or singularly perturbed problems in thermal fluid convection (Vazquez & Krstic, 2006) or chemical reactors (Boskovic & Krstic, 2002).

To obtain a plant equation in the class of (1), we solve for v in terms of u from (8). Define

$$v = \sum_{n=1}^\infty v_n, \qquad V = \sum_{n=1}^\infty V_n, \quad (11)$$

where v_1 verifies

$$0 = v_{1xx} + \omega^2 v_1 + u, \quad (12)$$

and for $n > 1$, v_n verifies

$$0 = v_{nxx} + \omega^2 v_n + uv_{n-1}, \quad (13)$$

with boundary conditions, for each n,

$$v_n(0) = 0, \qquad v_n(1) = V_n. \quad (14)$$

Since V in (10) is one of our two control inputs, we are free to choose V_n in any meaningful way if the series for V in (11) converges and the solution for (12)–(14) also makes the series for v in (11) convergent.

In this case, it is possible to solve (12) and (13) explicitly. Denoting $v_0 = 1$, we get the following recursive solution for $n \geq 1$

$$v_n = -\int_0^x \frac{\sin(\omega(x - \xi))}{\omega} v_{n-1}(\xi)u(\xi)d\xi + \frac{\sin(\omega x)}{\sin \omega}$$
$$\times \left(V_n + \int_0^1 \frac{\sin(\omega(1 - \xi))}{\omega} v_{n-1}u(\xi)d\xi \right). \quad (15)$$

Set the control law V as follows.

$$V_n = -\int_0^1 \frac{\sin(\omega(1 - \xi))}{\omega} v_{n-1}u(\xi)d\xi. \quad (16)$$

The reason to choose this particular control law is to get a spatially strict-feedback (Krstic et al., 1995) solution, i.e., a solution that is causal in space, meaning that $v(x)$ only depends on values of $u(\xi)$

for $0 \leq \xi \leq x$. This is a requirement of the backstepping method and was used in Vazquez and Krstic (2007), Krstic et al. (2006) and Vazquez and Krstic (2006) to control *linear* plants structurally similar to (7) and (8).

With this control law the solution to the Eqs. (12) and (13) is

$$v_n = -\int_0^x \frac{\sin(\omega(x-\xi))}{\omega} v_{n-1}(\xi)u(\xi)\mathrm{d}\xi. \tag{17}$$

We can solve the recursion in (17), getting

$$v_n = \frac{(-1)^n}{\omega^n} \int_{\mathcal{T}_n(x,\xi)} \prod_{j=1}^{n} \left[\sin\left(\omega(\xi_{j-1}-\xi_j)\right) u(\xi_j) \right] \mathrm{d}\hat{\xi}_1^n. \tag{18}$$

Plugging (18) into (16), we obtain a general formula for V_n as follows:

$$V_n = \frac{(-1)^n}{\omega^n} \int_{\mathcal{T}_n} \sin\left(\omega(1-\xi_1)\right) u(\xi_1)$$

$$\times \prod_{j=1}^{n-1} \left[\sin\left(\omega(\xi_j-\xi_{j+1})\right) u(\xi_{j+1}) \right] \mathrm{d}\hat{\xi}_1^n. \tag{19}$$

Assuming that $u(t,x) \in L^2(0,1)$, both series in (11) converge in L^2 since using that $|\sin(\omega)/\omega| \leq 1$, one can bound $\|v_n\|_{L^2}^2$ as follows.

$$\|v_n\|_{L^2}^2 = \int_0^1 v_n^2(x)\mathrm{d}x$$

$$\leq \left| \frac{\sin(\omega)}{\omega} \right|^{2n} \int_0^1 \left(\int_{\mathcal{T}_n(x,\xi)} \prod^i u \mathrm{d}\hat{\xi}_1^n \right)^2 \mathrm{d}x$$

$$\leq \left| \frac{\sin(\omega)}{\omega} \right|^{2n} \frac{1}{n!^2} \int_0^1 \left(\int_0^x u^2(\xi_1)\mathrm{d}\xi_1 \right)^n \mathrm{d}x$$

$$\leq \frac{\|u\|_{L^2}^{2n}}{n!^2}. \tag{20}$$

Hence, using the Cauchy–Schwartz inequality in ℓ_2,

$$\|v\|_{L^2}^2 = \int_0^1 \left(\sum_{n=1}^{\infty} v_n(x) \right)^2 \mathrm{d}x \leq \left(\sum_{n=1}^{\infty} n^2 \|v_n\|_{L^2}^2 \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)$$

$$\leq 2 \sum_{n=1}^{\infty} \frac{\|u\|_{L^2}^{2n}}{(n-1)!^2}, \tag{21}$$

where we used $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6 \leq 2$. Thus, $\|v\|_{L^2}^2 \leq 2\|u\|_{L^2}^2 \mathrm{e}^{\|u\|_{L^2}^2}$. Similarly $|V| \leq 2\|u\|_{L^2}^2 \exp\left(\|u\|_{L^2}^2\right)$.

Plugging the solution for v into (7), we reach

$$u_t = u_{xx} + \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(x,\xi)} f_n(\hat{\xi}_0^n) \prod^i u \, \mathrm{d}\hat{\xi}_1^n, \tag{22}$$

where $f_n = \mu \frac{(-1)^n}{\omega^n} \prod_{j=1}^{n} \sin\left(\omega(\xi_{j-1}-\xi_j)\right)$, an autonomous system in u with boundary conditions

$$u(0) = 0, \qquad u(1) = U, \tag{23}$$

and now the problem is reduced to designing U such that the above system is guaranteed to be stable in L^2.

Eq. (22) is a particular example of (1) with $\lambda = H = 0$, and $q = \infty$.

3.2. Parabolic semilinear equation

Consider the plant

$$v_t = v_{xx} + f(v), \tag{24}$$

where $f(v)$ is a nonlinear function *analytic at the origin*, verifying $f(0) = 0$, with boundary conditions

$$v(0) = 0, \qquad v_x(1) = U, \tag{25}$$

where U is the actuation variable.

To cast (24) into the form of (1) we differentiate (24) in x, getting

$$v_{xt} = v_{xxx} + f'(v)v_x. \tag{26}$$

Call $u = v_x$. Then, $v = \int_0^x u(\xi)\mathrm{d}\xi$ and (26) yields

$$u_t = u_{xx} + uf'\left(\int_0^x u(\xi)\mathrm{d}\xi\right), \tag{27}$$

with boundary conditions

$$u_x(0) = 0, \qquad u(1) = U. \tag{28}$$

The boundary condition at 0 was obtained evaluating (24) at $x = 0$ and using (25) and $f(0) = 0$. Expanding f' in its Taylor series at the origin, and calling

$$\lambda = f'(0), \tag{29}$$

$$h_n = f^{(n+1)}(0), \quad n \geq 1, \tag{30}$$

we can write (27) as

$$u_t = u_{xx} + \lambda u + u \sum_{n=1}^{\infty} \frac{h_n}{n!} \left(\int_0^x u(\xi)\mathrm{d}\xi \right)^n, \tag{31}$$

and since

$$\left(\int_0^x u(\xi)\mathrm{d}\xi \right)^n = n! \int_{\mathcal{T}_n(x,\xi)} \prod^i u \, \mathrm{d}\hat{\xi}_1^n, \tag{32}$$

we get

$$u_t = u_{xx} + \lambda u + u \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(x,\xi)} h_n \prod^i u \, \mathrm{d}\hat{\xi}_1^n, \tag{33}$$

with boundary conditions (28). Eq. (33) falls in the class of (1) with $F = 0, q = 0$, and λ and H given by (29) and (30). Note that stability of u in the L^2 norm implies stability of v in the H^1 norm, as $u(0) = 0$.

Remark 1. For the open-loop plant (24), finite-time blow up instabilities are likely to occur when $f(u)$ is superlinear. This was first studied in a classical paper (Fujita, 1966) for power-like nonlinearities, and has been the subject of systematic study in subsequent years (see the reviews Levine (1990) and Deng and Levine (2000)). More recently the question of *controllability* of these kind of equations has been considered. For superlinear functions which grow faster than $|u|\log^2(1+|u|)$ lack of global controllability is proved in Fernandez-Cara and Zuazua (2000). Therefore, for many nonlinearities $f(v)$ only *local* or *restricted* results can be achieved; for example in Coron and Trelat (2004) boundary control is used to move between sets of steady states for plants with superlinear nonlinearities.

4. Control strategy

The objective is to find a Volterra feedback law $U(t)$, so the controlled system (1) and (2) is stable. To achieve that, a new *target system* PDE is introduced in the form

$$w_t = w_{xx}, \tag{34}$$

with homogeneous boundary conditions

$$w_x(0) = \bar{q}w(0), \qquad w(1) = 0, \tag{35}$$

where $\bar{q} = \max\{0, q\}$. The plant (34) and (35) is an L^2 and H^1 exponentially stable system by standard results from linear PDE theory.

The idea of the method is to transform (1) into (34). For this we use a change of variables based on a Volterra series,

$$w = u - K[u] = u - \sum_{n=1}^{\infty} K_n[u]. \tag{36}$$

Evaluating (36) at $x = 1$ and using (2) and (35), we arrive at the control law

$$U = \sum_{n=1}^{\infty} K_n[u](1). \tag{37}$$

Therefore, the control is computed from the the Volterra kernels that define (36).

Remark 2. Some of the right hand side terms in (1) might be helpful for stability, for instance, a negative reaction term, i.e., $\lambda(x) \leq 0$ for all x, or the Volterra nonlinearity resulting from $-v^3$ in (24). Those terms could be kept in the target system (34), with only minor modifications in the kernel equations that follows.

We assume the series in (36) can be differentiated term by term.[1] Substituing (36) into (34) we get

$$\frac{\partial}{\partial t}\left(u - \sum_{n=1}^{\infty} K_n[u]\right) = \frac{\partial^2}{\partial x^2}\left(u - \sum_{n=1}^{\infty} K_n[u]\right). \tag{38}$$

Using (1) for u_t in (38) the following equation is obtained:

$$\lambda(x)u + \sum_{n=1}^{\infty} F_n[u] + u\sum_{n=1}^{\infty} H_n[u]$$
$$= \sum_{n=1}^{\infty} \left(\frac{\partial}{\partial t}K_n[u] - \frac{\partial^2}{\partial x^2}K_n[u]\right). \tag{39}$$

From (39), we can obtain a set of of partial integro-differential equations (PIDEs) for the kernels k_i that define the control (37). While the details of the derivation are presented in the Appendix, the PIDE verified by the n-th order kernel is given by

$$\partial_{xx}k_n = \sum_{i=1}^{n} \left(\partial_{\xi_i\xi_i}k_n + \lambda(\xi_i)k_n\right) + \sum_{m=1}^{n-1} C_n^m[k_{n-m}, h_m]$$
$$- f_n + I_n[k_n, f_1] + \sum_{m=2}^{n} B_n^m[k_{n-m+1}, f_m]. \tag{40}$$

The functions B_n^m, C_n^m and I_n in (40) have an involved definition that requires additional notation and the introduction of some intermediate functions. Hence for clarity we first finish stating and discussing the kernel equations and then introduce the concepts

towards the precise definition of B_n^m, C_n^m and I_n, which is given respectively in (54), (55) and (56).

The solution to the PIDE (40) needs to satisfy the following boundary conditions. For $n = 1$,

$$k_1(x, x) = \hat{q} - \frac{1}{2}\int_0^x \lambda(s)\mathrm{d}s, \tag{41}$$

$$k_{1\xi_1}(x, 0) = qk_1(x, 0), \tag{42}$$

where $\hat{q} = \min\{0, q\}$, while for $n \geq 2$,

$$k_n(x, x, \hat{\xi}_2^n) = -\frac{1}{2}\int_{\xi_2}^x h_{n-1}(s, \hat{\xi}_2^n)\mathrm{d}s, \tag{43}$$

$$k_{nx}(x, x, \hat{\xi}_2^n) = -\frac{1}{4}\left(3h_{n-1}(\xi_2, \hat{\xi}_2^n) + h_{n-1}(x, \hat{\xi}_2^n)\right)$$
$$+ \frac{1}{2}\int_{\xi_2}^x \phi_n(s, \hat{\xi}_2^n)\mathrm{d}s, \tag{44}$$

$$k_{n\xi_{i-1}}(\hat{\xi}_0 n)|_{\xi_{i-1}=\xi_i} = k_{n\xi_i}(\hat{\xi}_0^n)|_{\xi_{i-1}=\xi_i}, \quad i = 2, \ldots, n, \tag{45}$$

$$k_{n\xi_n}(\hat{\xi}_0^{n-1}, 0) = qk_n(\hat{\xi}_0^{n-1}, 0), \tag{46}$$

which are of mixed kind. In (44), the function ϕ_n is defined as

$$\phi_n = \left[\sum_{i=2}^{n} \partial_{\xi_i\xi_i}k_n + \sum_{i=1}^{n} \lambda(\xi_i)k_n + \sum_{m=1}^{n-1} C_n^m[k_{n-m}, h_m]\right.$$
$$\left. + \sum_{m=2}^{n} B_n^m[k_{n-m+1}, f_m] + I_n[k_n, f_1] - f_n\right]_{x=\xi_1}. \tag{47}$$

Eq. (40) is a hyperbolic PIDE, for each k_n, evolving in the interior of the domain $\mathcal{T}_{n+1}$, which is a "hyper-pyramid" of dimension $n+1$ (in particular, a triangle for $n = 1$, and a pyramid for $n = 2$). Note that, by (32), the volume of $\mathcal{T}_{n+1}$ decreases rapidly as the dimension n increases, as given by the following formula:

$$\mathrm{Vol}\,(\mathcal{T}_{n+1}) = \frac{1}{(n + 1)!}. \tag{48}$$

Remark 3. The domain $\mathcal{T}_{n+1}$ has $n + 2$ "sides" (which belong to n-dimensional hyperplanes) on its boundary. These are

$$\mathcal{R}_0 = \{\hat{\xi}_0^n : 0 < \xi_n < \cdots < \xi_1 < x = 1\}, \tag{49}$$

$$\mathcal{R}_1 = \{\hat{\xi}_0^n : 0 < \xi_n < \cdots < \xi_1 = x < 1\}, \tag{50}$$

$$\mathcal{R}_i = \{\hat{\xi}_0^n : 0 < \xi_n < \cdots < \xi_i = \xi_{i-1} < \cdots < \xi_1 < x < 1\},$$
$$i = 2, \ldots, n \tag{51}$$

$$\mathcal{R}_{n+1} = \{\hat{\xi}_0^n : 0 = \xi_n < \cdots < \xi_1 < x < 1\}. \tag{52}$$

The boundary conditions (43) and (44) are non-homogeneous and given on $\mathcal{R}_1$. Note that the bracket in the definition of ϕ_n in (47), which is needed for (44), is evaluated at $x = \xi_1$ and thus can be computed from (43), without needing to know the kernel k_n a priori (this is explicitly illustrated next with the formula for ϕ_2 in (68)). The boundary condition (45) is given on $\mathcal{R}_i$, for $i = 2, \ldots, n$ and represents the value of the normal derivative of k_n in the boundary $\mathcal{R}_i$, hence it is of Neumann type. The boundary condition (46) is of Robin type and given on $\mathcal{R}_{n+1}$. The value of the function k_n on $\mathcal{R}_0$ is what needs to be found for the control law (37).

Remark 4. Eq. (40) with its boundary conditions can be reinterpreted as a wave equation in spacetime. If one thinks of x as time (time-like variable) and the other variables $\xi_1, \xi_2, \ldots, \xi_n$ as space coordinates (space-like variables), then the domain can be seen as a n-dimensional hyper-pyramid in $\mathbb{R}^n$ that *grows* (linearly in "time" x), with boundaries $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_{n+1}$ that are also

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[1] This assumption requires uniform convergence of the transformation Volterra series which is shown in Vazquez and Krstic (2008, Theorem 2).

growing in time. In particular, it can be seen that the boundaries $\mathcal{R}_2, \ldots, \mathcal{R}_{n+1}$ are time-like (they grow slower than the characteristic speed of wave propagation, lying inside the "causality" cone), but the boundary $\mathcal{R}_1$ is space-like, i.e., it grows faster than the characteristic speed of wave propagation (and lies outside the causality cone). For a wave equation to be well-posed (Courant & Hilbert, 1995), it is necessary that it has exactly one boundary condition on its time-like boundaries and two boundary conditions (initial data) on its space-like boundaries. That is the reason why the boundary $\mathcal{R}_1$ has two boundary conditions. The only exception is k_1, for which $\mathcal{R}_1$ is characteristic (i.e., the boundary condition is of Goursat type) and thus only needs one boundary condition, which is (41).

The term $I_n[k_n, f_1]$ is the homogenous integral term of the PIDE, while $B_n^m[k_{n-m+1}, f_m]$ and $C_n^m[k_{n-m}, h_m]$ are forcing terms, where only terms including previous kernels k_m with $m < n$ appear. This means the set of PIDE's can be solved *recursively* up to any desired order n, beginning with k_1.

We now introduce some additional definitions needed for writing the expressions for $B_n^m[k_{n-m+1}, f_m]$, $C_n^m[k_{n-m}, h_m]$ and $I[k_n, f_1]$ in (40).

Definition 4.1. Given a set $\delta = \{a_1, a_2, \ldots, a_k\}$ of k ordered variables and given m such that $0 \leq m \leq k$, we define $P_m(\delta)$ as the set of all possible ordered k-tuples that can be formed in the following way. The first m elements of the k-tuple are any m members of δ *ordered* by their indices. The last $k - m$ elements of the k-tuple are all the remaining members of δ, also ordered by their indices.

Example 4.1. If $\delta = \{a_1, a_2, a_3, a_4\}$, then:

$P_0(\delta) = \{(a_1, a_2, a_3, a_4)\}$,

$P_1(\delta) = \{(a_1, a_2, a_3, a_4), (a_2, a_1, a_3, a_4), (a_3, a_1, a_2, a_4),$
$\quad (a_4, a_1, a_2, a_3)\}$,

$P_2(\delta) = \{(a_1, a_2, a_3, a_4), (a_1, a_3, a_2, a_4), (a_1, a_4, a_2, a_3),$
$\quad (a_2, a_3, a_1, a_4), (a_2, a_4, a_1, a_3), (a_3, a_4, a_1, a_2)\}$,

$P_3(\delta) = \{(a_1, a_2, a_3, a_4), (a_1, a_2, a_4, a_3), (a_1, a_3, a_4, a_2),$
$\quad (a_2, a_3, a_4, a_1)\}$,

$P_4(\delta) = \{(a_1, a_2, a_3, a_4)\}$.

Remark 5. If δ has k elements, the number of elements of $P_m(\delta)$ is $\binom{k}{m} = \frac{k!}{m!(k-m)!}$.

We finally get to defining B_n^m, C_n^m and I_n. Given a function $g(\hat{\xi}_0^{n+m})$, and $1 \leq j \leq n$, let $D_j^{n,m}[g(\hat{\xi}_0^{n+m})]$ denote

$$D_j^{n,m}[g(\hat{\xi}_0^{n+m})] = \sum_{\hat{\gamma}_1^{n-j+m} \in P_{n-j}(\hat{\xi}_{j+1}^{n+m})} g(\hat{\xi}_0^j, \hat{\gamma}_1^{n-j+m}). \tag{53}$$

Then, the term $B_n^m[k_{n-m+1}, f_m]$ is defined as

$$B_n^m = \sum_{j=1}^{n-m+1} \int_{\xi_j}^{\xi_{j-1}} D_j^{n-m+1,m}$$
$$\times \left[k_{n-m+1}(\hat{\xi}_0^{j-1}, s, \hat{\xi}_j^{n-m}) f_m(s, \hat{\xi}_{n-m+1}^n) \right] \mathrm{d}s, \tag{54}$$

and the term $C_n^m[k_{n-m}, h_m]$ is defined as

$$C_n^m[k_{n-m}, h_m] = \sum_{j=1}^{n-m} D_j^{n-m,m} \left[k_{n-m}(\hat{\xi}_0^{n-m}) h_m(\xi_j, \hat{\xi}_{n-m+1}^n) \right]. \tag{55}$$

The definition of $I_n[k_n, f_1]$ is, using (54),

$$I_n[k_n, f_1] = B_n^1[k_n, f_1]. \tag{56}$$

Remark 6. The number of terms of $B_n^m[k_{n-m+1}, f_m]$ is, using Remark 5,

$$\sum_{j=1}^{n-m+1} \binom{n-j+1}{n-j-m+1}. \tag{57}$$

The number of terms of $I_n[k_n, f_1]$ is

$$\sum_{j=1}^{n} \binom{n-j+1}{n-j} = \sum_{j=1}^{n} (n-j+1) = n(n+1)/2. \tag{58}$$

Hence in the PIDE for k_n, the total number of integrals in I_n and B_n^m is

$$\sum_{m=1}^{n} \sum_{j=1}^{n-m+1} \binom{n-j+1}{n-j-m+1}$$
$$= \sum_{j=1}^{n} \sum_{m=1}^{n-j+1} \binom{n-j+1}{n-j-m+1}$$
$$= \sum_{j=1}^{n} \sum_{m=0}^{n-j} \binom{n-j+1}{m}$$
$$= \sum_{j=1}^{n} (2^{n-j+1} - 1)$$
$$= 2^{n+1} \sum_{j=1}^{n} 2^{-j} - n$$
$$= 2^{n+1}(2(1 - 2^{-n-1}) - 1) - n$$
$$= 2^{n+1} - n - 2. \tag{59}$$

Similarly, the number of terms in C_n^m is $2^n - n - 1$.

We next show, as an illustration of the general case, the PIDE equations that the first two kernels, k_1, k_2, satisfy.

The PIDE equation for k_1 is

$$\partial_{xx} k_1 = \partial_{\xi_1 \xi_1} k_1 + \lambda(\xi_1) k_1 - f_1(x, \xi_1) + \int_{\xi_1}^{x} k_1(x, s) f_1(s, \xi_1) \mathrm{d}s, \tag{60}$$

with boundary conditions

$$k_1(x, x) = \hat{q} - \frac{1}{2} \int_0^x \lambda(s) \mathrm{d}s, \tag{61}$$

$$k_{1\xi_1}(x, 0) = q k_1(x, 0). \tag{62}$$

This equation evolves on the triangle $\mathcal{T}_2 = \{(x, \xi_1) : 0 \leq \xi_1 \leq x \leq 1\}$, which is drawn in Fig. 1(top).

Remark 7. Eq. (60) is an autonomous equation in k_1. It is a particular case of the kernel equation for backstepping control of linear parabolic PDEs. Its well-posedness is already established (Smyshlyaev & Krstic, 2004), where symbolic and numerical methods of solution are proposed.

The PIDE equation verified by k_2 is

$$\partial_{xx} k_2 = \partial_{\xi_1 \xi_1} k_2 + \partial_{\xi_2 \xi_2} k_2 + (\lambda(\xi_1) + \lambda(\xi_2)) k_2$$
$$- f_2 + k_1(x, \xi_1) h_1(\xi_1, \xi_2)$$
$$+ \int_{\xi_1}^{x} k_1(x, s) f_2(s, \xi_1, \xi_2) \mathrm{d}s + \int_{\xi_2}^{\xi_1} k_2(x, \xi_1, s) f_1(s, \xi_2) \mathrm{d}s$$
$$+ \int_{\xi_1}^{x} k_2(x, s, \xi_1) f_1(s, \xi_2) \mathrm{d}s + \int_{\xi_1}^{x} k_2(x, s, \xi_2) f_1(s, \xi_1) \mathrm{d}s, \tag{63}$$
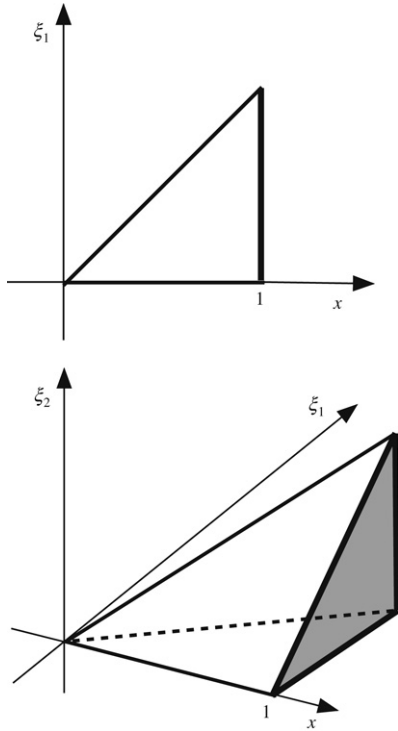
Fig. 1. <u>Top</u>: The domain $\mathcal{T}_2$. Boundary conditions are given at $\xi_1 = 0$ and $x = \xi_1$ (lower and diagonal lines, respectively). The feedback law requires to compute the kernel k_1 at the boundary $x = 1$ (the vertical bold line). <u>Bottom</u>: The domain $\mathcal{T}_3$ shown in perspective. Robin boundary conditions are given at $\xi_2 = 0$ (the ground surface), while at $x = \xi_1$ (normal to the ground and hidden behind the figure due to the perspective) we have both Dirichlet and Neumann boundary conditions (initial-like conditions). A Neumann boundary condition is given at $\xi_1 = \xi_2$ (the surface that lies in front of a viewer looking in the ξ_1 direction). The feedback law requires one to compute the kernel k_2 at the boundary $x = 1$ (the shaded surface).

with boundary conditions

$$k_2(x, x, \xi_2) = -\frac{1}{2} \int_{\xi_2}^{x} h_1(s, \xi_2) \mathrm{d}s, \tag{64}$$

$$k_{2x}(x, x, \xi_2) = -\frac{1}{4} \left(3h_1(\xi_2, \xi_2) + h_1(x, \xi_2) \right) + \frac{1}{2} \int_{\xi_2}^{x} \phi_2(s, \xi_2) \mathrm{d}s \qquad , \tag{65}$$

$$k_{2\xi_2}(x, \xi_1, 0) = qk_2(x, \xi_1, 0), \tag{66}$$

$$k_{2\xi_1}(x, \xi_1, \xi_2)|_{\xi_2 = \xi_1} = k_{2\xi_2}(x, \xi_1, \xi_2)|_{\xi_2 = \xi_1}, \tag{67}$$

where

$$\phi_2 = -\int_{\xi_2}^{x} \frac{h_{1\xi_2\xi_2}(s, \xi_2)}{2} \mathrm{d}s - h_{1\xi_2}(\xi_2, \xi_2) - \frac{h_{1\xi_1}(\xi_2, \xi_2)}{2}$$
$$- \frac{\lambda(x) + \lambda(\xi_2)}{2} \int_{\xi_2}^{x} h_1(s, \xi_2) \mathrm{d}s - f_2(x, x, \xi_2)$$
$$- \int_{\xi_2}^{x} \int_{s}^{x} \frac{h_1(\sigma, s) f_1(s, \xi_2)}{2} \mathrm{d}\sigma \mathrm{d}s - h_1(x, \xi_2) \int_{0}^{x} \frac{\lambda(s)}{2} \mathrm{d}s. \tag{68}$$

This equation evolves on the pyramid $\mathcal{T}_3 = \{(x, \xi_1, \xi_2) : 0 \leq \xi_2 \leq \xi_1 \leq x \leq 1\}$, which is shown in Fig. 1(bottom). Once k_1 is solved from (60), it can be plugged into (63) which becomes an autonomous equation for k_2.

Note the increasing complexity of the kernel PIDEs but also the common recursive structure that underlies all the equations.

5. An example of a stabilizable super-linear system

In Section 6 we discuss a numerical approach that would be used for solving for the controller gain kernels in a general case. However, in this section we consider a particularly "simple" example which is tractable analytically because it is formulated in an "inverse" manner–we choose a simple Volterra nonlinear controller and then derive a plant for which this controller is stabilizing. To be precise, for $\lambda = 0$, $H = 0$, and $q = \infty$ (Dirichlet boundary conditions for the plant), instead of solving for the k-kernels with the f-kernels as given, we solve for the f-kernels with the k-kernels as given. This is not possible in general, however, in the case where $f_1 = 0$, i.e., the "purely nonlinear" case where the plant doesn't have a linear term in its Volterra series, it is possible to find the f-kernels when the k-kernels are given, i.e., it is possible to find the plant that is stabilized by a pre-assigned controller. This is easy to see by examining the Eqs. (60)–(67). First, when $f_1 = 0$, then $k_1 = 0$. Second, for any k_2 that satisfies the boundary conditions (64)–(67), the kernel f_2 is obtained by direct evaluation of the derivatives of k_2 from (63). And so on for $f_3, f_4, \ldots$.

So, starting with a controller as simple as possible–yet nonlinear–in this section we illustrate how it is possible to solve (40)–(46) to find the (nonlinear) plant which is stabilized by the preassigned controller

The simplest possible (nonlinear) controller we can think of comes from a single *second order* control kernel, $k_2 = \sigma_1\sigma_2(x - \sigma_1)(x - \sigma_2)$, whose particular form is chosen to satisfy (64)–(67). All other control kernels are set to zero, i.e., $k_1 = k_3 = \cdots = k_n = \cdots = 0$. Then the control input, $U(t) = K[u](t, 1)$, is:

$$U = K[u](1)$$
$$= \int_0^1 \int_0^{\xi_1} \xi_1\xi_2(x - \xi_1)(x - \xi_2)u(\xi_1)u(\xi_2)\mathrm{d}\xi_2\mathrm{d}\xi_1, \tag{69}$$

which can be written shorter thanks to the symmetry of the kernel:

$$U = \frac{1}{2} \left(\int_0^1 \xi(x - \xi)u(\xi)\mathrm{d}\xi \right)^2. \tag{70}$$

The plant kernels derived from (40) are $f_1 = 0$,

$$f_2 = 2\xi_2\xi_1 + 2\xi_2x - 2\xi_2^2 + 2\xi_1x - 2\xi_1^2, \tag{71}$$

$$f_n = B_n^{n-1}[k_2, f_{n-1}], \quad n \geq 3 \tag{72}$$

where we can write (72) using definition (54) as

$$f_n = \int_{\xi_1}^{x} \sum_{\hat{\gamma}_1^n \in P_1(\hat{\xi}_1^n)} k_2(x, s, \gamma_1)f_{n-1}(s, \gamma_2, \ldots, \gamma_n)\mathrm{d}s$$
$$+ \int_{\xi_2}^{\xi_1} k_2(x, \xi_1, s)f_{n-1}(s, \xi_2, \ldots, \xi_n)\mathrm{d}s. \tag{73}$$

Using this definition and employing a symbolic calculation program, it is possible to get all the kernels up to a desired order. Higher order kernels get smaller and smaller, and their influence becomes negligible. This is stated in the following lemma, that guarantees convergence of the Volterra series of the plant defined by (71) and (72).

Proposition 5.1. *The kernels $f_2, \ldots, f_n, \ldots$ defined by (71) and (72) verify the following bound.*

$$|f_n(x, \xi_1, \ldots, \xi_n)| \leq 3x^{5n-8}. \tag{74}$$

Hence, the Volterra series defined by $\sum_{i=2}^{\infty} F_i[u](t, x)$ converges for $u \in L^2(0, 1)$.

Proof. For every n one has that $\xi_n \le \xi_{n-1} \le \cdots \le \xi_1 \le x$. Note that

$$|k_2| = |\xi_1\xi_2(x-\xi_1)(x-\xi_2)| \le \frac{x^4}{16}. \tag{75}$$

For $n = 2$,

$$\begin{aligned}|f_2| &= \left|2\xi_2\xi_1 + 2\xi_2 x - 2\xi_2^2 + 2\xi_1 x - 2\xi_1^2\right| \\ &= |2\xi_2\xi_1 + 2\xi_2(x-\xi_2) + 2\xi_1(x-\xi_1)| \\ &\le x^2(2 + 1/2 + 1/2) = 3x^2.\end{aligned} \tag{76}$$

Assume now the claim of the theorem is true for $n - 1$. Then, for n,

$$\begin{aligned}|f_n| &= \left| \int_{\xi_2}^{\xi_1} k_2(x,\xi_1,s)f_{n-1}(s,\xi_2,\dots,\xi_n)\mathrm{d}s \right. \\ &\quad \left. + \int_{\xi_1}^{x} \sum_{\hat{\gamma}_1^n \in P_1(\hat{\xi}_1^n)} k_2(x,s,\gamma_1)f_{n-1}(s,\gamma_2,\dots,\gamma_n)\mathrm{d}s \right| \\ &\le \frac{x^4}{16}\left(\int_{\xi_2}^{\xi_1} 3s^{5n-13}\mathrm{d}s + \int_{\xi_1}^{x} \sum_{\hat{\gamma}_1^n \in P_1(\hat{\xi}_1^n)} 3s^{5n-13}\mathrm{d}s \right) \\ &= \frac{x^4}{16}\left(3\frac{n+1}{5n-12}x^{5n-12} \right) \\ &= 3x^{5n-8}\frac{n+1}{16(5n-12)} \le 3x^{5n-8},\end{aligned} \tag{77}$$

since for $n \ge 3$, $\frac{n+1}{16(5n-12)} \le 1$. This gives us (74).

Since $x \in (0,1)$, we have that $|f_n| \le 3$. Hence if $u \in L^2(0,1)$,

$$\begin{aligned}&\int_0^1 \left(\sum_{n=2}^{\infty} F_n[u] \right)^2 \mathrm{d}x \\ &= \int_0^1 \left(\sum_{n=2}^{\infty} \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} f_n(x,\xi_1,\dots,\xi_n) \right. \\ &\quad \left. \times \left(\prod_{j=1}^{n} u(\xi_j) \right) \mathrm{d}\xi_n \dots \mathrm{d}\xi_1 \right)^2 \mathrm{d}x \\ &\le 9\int_0^1 \left(\sum_{n=2}^{\infty} \frac{\left(\int_0^x u(\xi)\mathrm{d}\xi \right)^n}{n!} \right)^2 \\ &\le 18\|u\|_{L^2}^2 \exp\left(\|u\|_{L^2}^2 - 1 \right),\end{aligned} \tag{78}$$

where we have followed similar steps as in (20). This completes the proof. □

For the purpose of illustrating the effect of the functional operators K and F, we plot the effect of both of them on an example function, $u(t,x) = 100\sin(2\pi x)$, in Fig. 2. The order of magnitude of K is much less than the order of magnitude of F, so we plot $20K$ for the sake of clarity.

6. Numerical simulations

Before we consider some challenging numerical demonstrations of solving the gain kernel PIDEs, we present numerical simulations of the nonlinear plant introduced in Section 5. Starting with a large enough initial condition (of the order of 200), the uncontrolled system diverges to infinity in finite time, as seen in Fig. 3. With the controller (69), this behavior is suppressed and the system is stabilized, as shown in Fig. 3.
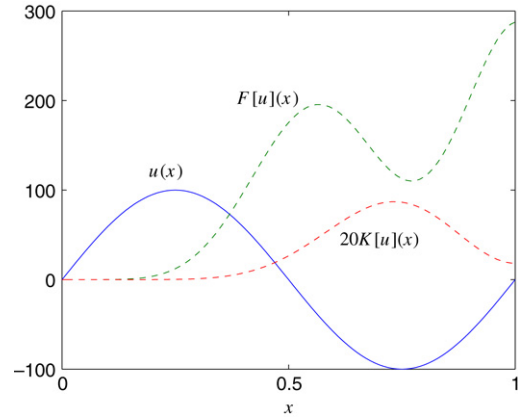

Fig. 2. Effect of K and F on $u(x) = 100\sin(2\pi x)$.

Next we discuss numerical techniques for computing the Volterra kernels k_n. The first-order kernel k_1 is computed with a finite differences scheme from Smyshlyaev and Krstic (2004). Using a similar finite difference scheme, we are able to compute the second-order kernel k_2 for the examples of Sections 3.1 and 3.2. We then use the k_1 and k_2 kernels to approximate[2] control law (37) to do closed-loop simulations of the systems. For computing k_2, we have to use the extra boundary condition (65) and use a smaller discretization step for x than for the ξ variables, which is essential for numerical stability (Lines, Slawinski, & Bording, 1999).

6.1. Coupled nonlinear plant

Consider the example plant given in Section 3.1. Its Volterra nonlinearity is explicitly written in Eq. (22). We set the numerical values for the parameters of the plant as $\mu = 50, \omega = 2.5$. A simple linear stability analysis shows that the equilibrium at the origin is unstable for these values.

To find a control law to stabilize the system, we apply the design method outlined in Section 4, and numerically solve for the kernels. In Fig. 4 we show the numerical value of the first two kernels, k_1 and k_2, at $x = 1$, which is the value appearing in the control formula (37). We find that using just the linear kernel k_1 in the feedback law (37),[3] stabilizes the system for a wide range of initial conditions. However, for initial conditions of large enough size (with a peak of the order of 1000), the linear controller fails to stabilize the system, as shown in Fig. 5. In Fig. 6 we show how the same initial condition is stabilized when the second-order kernel is used in (37), i.e., truncating the control law to second order is enough for stabilization for that size of initial conditions.

6.2. Quadratic nonlinearity

Consider the plant

$$u_t = u_{xx} + u^2, \tag{79}$$

with boundary conditions

$$u(0) = 0, \quad u_x(1) = U. \tag{80}$$

This plant is in the class of the example of Section 3.2, with $f(u) = u^2$. Then, in (33), $\lambda = 0$, $h_1 = 2$, and for $n > 1$, $h_n = 0$. In this case, $k_1 = 0$ as the plant does not have linear terms. In Fig. 7 we show the numerical value of the second order

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[2] Since the Volterra series for the control law is convergent (Vazquez & Krstic, 2008, Theorem 2), truncation yields a good approximation if "sufficiently many" terms are used. In the examples, Volterra series are rapidly convergent and two terms suffice.

[3] This is equivalent to applying the result of Smyshlyaev and Krstic (2004) to the *linearized* system.
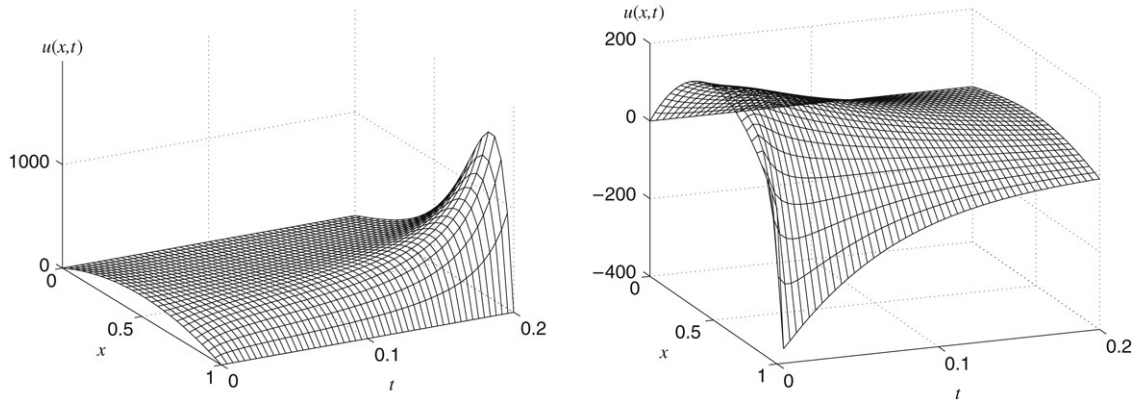
Fig. 3. Uncontrolled (left) and controlled system (right) for the example of Section 5. The solution of the uncontrolled system blows up in finite time. The trajectory of the control input (right) is $u(t, 1)$. The size of the control effort (-400) is reasonable given the size of the initial condition (with a peak about 200).
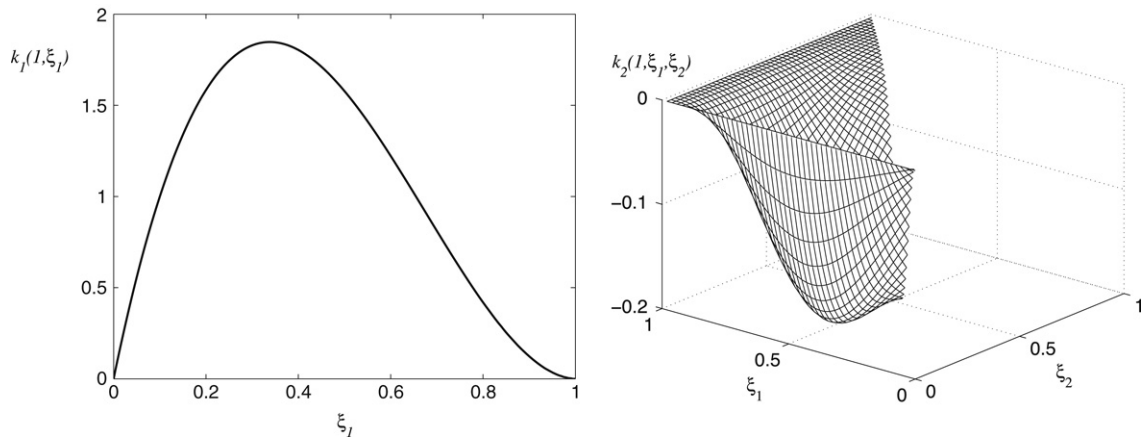

Fig. 4. Control kernels $k_1(1, \xi_1)$ (left) and $k_2(1, \xi_1, \xi_2)$ (right) for the example of Section 3.1, with $\mu = 50$, $\omega = 2.5$. Note that the kernel $k_2(1, \xi_1, \xi_2)$ is only defined for $\xi_2 \leq \xi_1$.
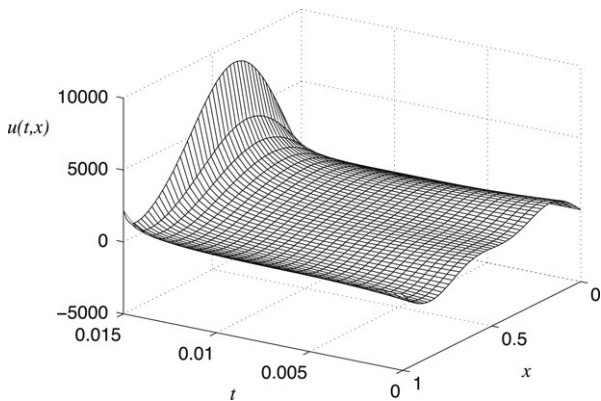

Fig. 5. Closed-loop simulation for $u(t, x)$ using only the first (linear) order kernel k_1, in the example of Section 3.1.

control kernel k_2. We tested numerically the control law (37) using only k_2. We found that, for initial conditions of size large enough (with a peak value approximately more than 4), the open-loop system blows up (in finite time), as shown in Fig. 8(left). In Fig. 8(right), we show how the second-order controller is able to prevent the blow-up and stabilize the system for the same initial conditions. However, the same controller fails to stabilize u for larger initial conditions (with peaks over 8). This restricted local result is not only due to truncation of (37), but to the fact that (79) is *not* globally stabilizable (see Remark 1). Thus increasing the order of the controller may enlarge the basin of attraction of the

equilibrium at the origin for the closed-loop system, but only up to a certain limit.

7. Conclusions

We have presented a new approach for stabilization of a class of parabolic 1-D nonlinear partial differential equations based on feedback linearization methods for finite-dimensional systems. For nonlinear ODEs, feedback linearization recursively absorbs all the plant nonlinearities into a feedback transformation. The resulting transformation often involves nonlinearities of much higher growth than the plant nonlinearities. For example, systems with n states and only quadratic nonlinearities lead to feedback linearizing controllers of polynomial power up to $n + 1$. Intuitively, one would worry that an infinite-step feedback linearization procedure may result in polynomial powers that go to infinity. We handle this problem introducing a framework, based on Volterra series, which allows one to design feedback linearizing boundary controllers with a well defined limit. The convergence of our state transformation (36) and feedback (37) is proved in a companion paper (Vazquez & Krstic, 2008) by deriving norm estimates of the solutions k_n of the kernel equations (40)–(47).

The class of stabilizable systems is given by (1) and (2), which are 1-D parabolic equations with Volterra nonlinearities. We provide examples of unstable nonlinear plants commonly found in applications that can be written in the form (1) and (2) or converted into this form by an invertible transformation. For such systems, we show closed-loop stabilization in simulations for large initial conditions, where the controller is approximated by truncating
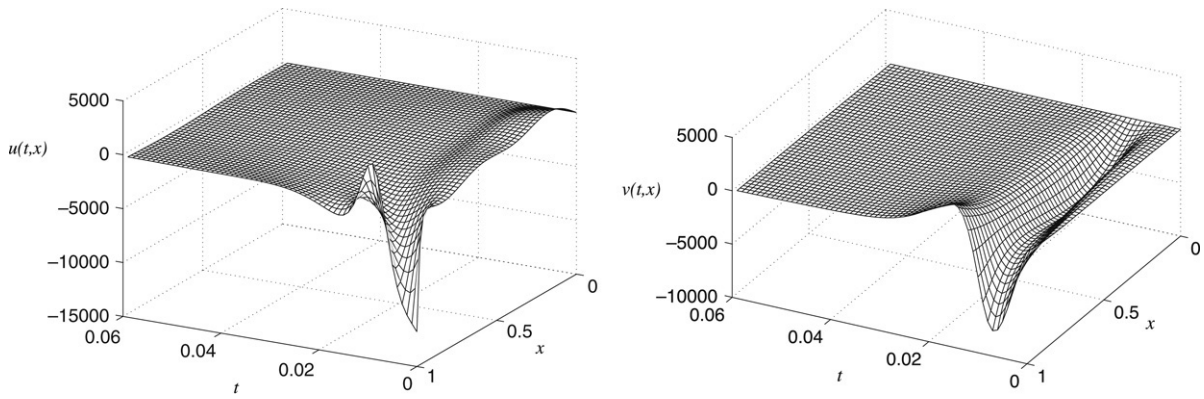
Fig. 6. Closed-loop simulation for $u(t, x)$ (left) and $v(t, x)$ (right) in the example of Section 3.1. The control law is approximated to second order using control kernels k_1 and k_2.
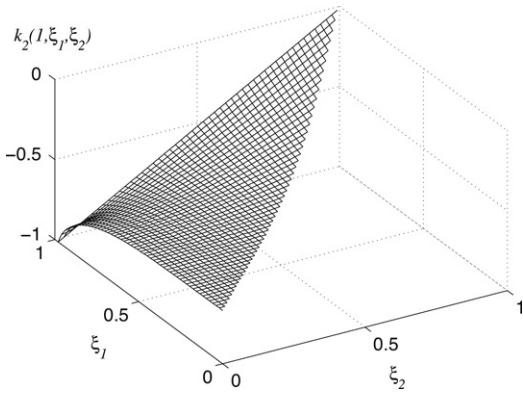

Fig. 7. Second-order control kernel $k_2(1, \xi_1, \xi_2)$ for the example of Section 3.2 with $f(u) = u^2$ (quadratic nonlinearity). Note that the kernel $k_2(1, \xi_1, \xi_2)$ is only defined for $\xi_2 \leq \xi_1$.

the series to only the first and second order Volterra kernels. The kernels are numerically pre-computed from the k_1 and k_2 equation, (60)–(62) and (63)–(68) respectively.

In the companion paper (Vazquez & Krstic, 2008) we also study closed-loop system properties, deriving local L^2 and H^1 exponential stability using the inverse of the transformation. Since the inverse of operators of the form "identity minus Volterra series" is, in general, only locally defined, and since the class of systems considered includes systems that are not globally null controllable (Remark 1), the form of stability achieved is not global but local, with an estimate of the region of attraction in cases where the inverse backstepping transformation can be quantified.
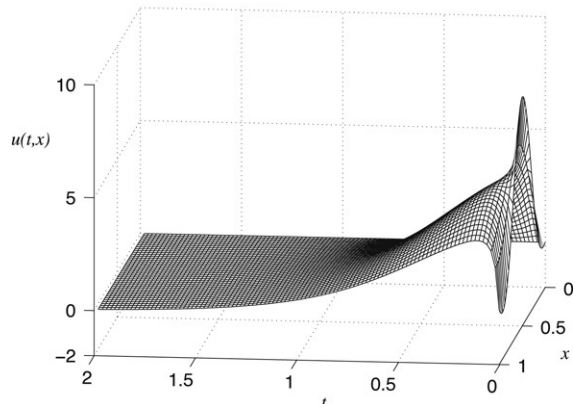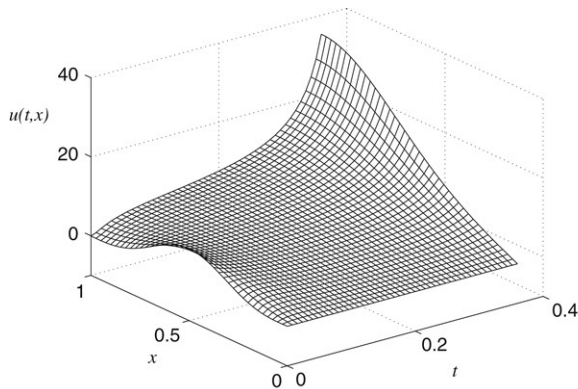
The development of feedback linearization/backstepping designs for nonlinear PDEs still has a long way to go. Coordinate-free tests of linearizability are needed, as well as methods for finding the "flat output" when the input is not at a boundary but of "point-actuator" type. We point out that in Section 3.2 we found a flat output $v_x(0, t)$ and a pre-transformation of the type $u = v_x$ that casts the system into the form with Volterra series nonlinearities. Systematic procedures for achieving this for broader classes of PDE systems would be welcome.

Appendix

Here we show the derivation of the general kernel PIDE equation for any order n.

We first state a technical result.

Lemma A.1. *The following two identities hold.*

$$\int_{\mathcal{T}_n(x,\xi)} f_n(\hat{\xi}_0^n) \mathrm{d}\hat{\xi}_1^n = \int_{\mathcal{T}_{n-1}(x,\xi)} \int_{\xi_m}^{\xi_{m-1}} f_n(\hat{\xi}_0^{m-1}, s, \hat{\xi}_m^{n-1}) \mathrm{d}s \mathrm{d}\hat{\xi}_1^{n-1},$$

(A.1)

$$\int_{\mathcal{T}_n(x,\xi)} f_n(\hat{\xi}_0^n) \int_{\mathcal{T}_m(\xi_j,\sigma)} g_m(\xi_j, \hat{\sigma}_1^m) \mathrm{d}\hat{\sigma}_1^m \mathrm{d}\hat{\xi}_1^n$$

$$= \int_{\mathcal{T}_{n+m}(x,\xi)} D_j^{n,m}[f_n(\hat{\xi}_0^n) g_m(\xi_j, \hat{\xi}_{n+1}^{n+m})] \mathrm{d}\hat{\xi}_1^{n+m},$$

(A.2)

where $D_j^{n,m}$ is defined as in (53).

Fig. 8. Uncontrolled (left) and controlled system (right) for the example of Section 3.2 with $f(u) = u^2$ (quadratic nonlinearity). The control law is truncated to second order. The solution of the uncontrolled system blows up in finite time, while the controlled system converges to the origin.

Proof. Identity (A.1) is derived directly from Fubini's theorem. For (A.2), write

$$\int_{\mathcal{T}_n(x,\xi)} f_n(\hat{\xi}_0^n) \left(\int_{\mathcal{T}_m(\xi_j,\sigma)} g_m(\xi_j,\hat{\sigma}_1^m) d\hat{\sigma}_1^m \right) d\hat{\xi}_1^n$$

$$= \int_{\Omega_j^m(\hat{\xi}_1^{n+m})} f_n(\hat{\xi}_0^n) g_m(\xi_j,\hat{\xi}_{n+1}^{n+m}) d\hat{\xi}_1^{n+m}, \tag{A.3}$$

where

$$\Omega_j^m(\hat{\xi}_1^{n+m}) = \{x \geq \xi_1 \geq \cdots \geq \xi_n \geq 0; \xi_j$$

$$\geq \xi_{n+1} \geq \cdots \geq \xi_{n+m} \geq 0\}. \tag{A.4}$$

For any m and $1 \leq j \leq n$, it holds that

$$\Omega_j^m(\hat{\xi}_1^{n+m}) = \bigcup_{\hat{\gamma}_1^{n-j+m} \in \mathcal{P}_{n-j}(\hat{\xi}_{j+1}^{n+m})} \{\xi_1 \geq \xi_2 \geq \cdots \geq \xi_j$$

$$\geq \gamma_1 \geq \cdots \geq \gamma_{n+m-j} \geq 0\}. \tag{A.5}$$

To prove (A.5), we first note that if $j = n$ or $m = 0$, $\Omega_j^m(\hat{\xi}_1^{n+m}) = \mathcal{T}_n(x,\xi)$, while if $j < n$ and $m \geq 1$, since

$$\{\xi_j \geq \xi_{j+1} \geq 0\} = \{\xi_j \geq \xi_{j+1} \geq \xi_{n+1}\}$$

$$\bigcup_{l=2}^{n-1} \{\xi_{n+l} \geq \xi_{j+1} \geq \xi_{n+l+1}\}$$

$$\cup \{\xi_{n+m} \geq \xi_{j+1} \geq 0\}, \tag{A.6}$$

we get

$$\Omega_j^m(\hat{\xi}_1^{n+m}) = \Omega_{j+1}^m(\hat{\xi}_1^{n+m}) \cup \Omega_{j+1}^0(\hat{\xi}_1^j,\hat{\xi}_{n+1}^{n+m},\hat{\xi}_{j+1}^n)$$

$$\bigcup_{l=2}^{m-1} \Omega_{j+1}^{m-l}(\hat{\xi}_1^j,\hat{\xi}_{n+1}^{n+l-1},\hat{\xi}_{j+1}^n,\hat{\xi}_{n+l}^{n+m}). \tag{A.7}$$

Similarly, for $j = n$ or $m = 0$, the symbol $\mathcal{P}_{n-j}(\hat{\xi}_{j+1}^{n+m}) = \{\hat{\xi}_{j+1}^{n+m}\}$, and if $j < n$ and $m \geq 1$, verifies that

$$\mathcal{P}_{n-j}(\hat{\xi}_{j+1}^{n+m}) = \{\xi_{j+1}, P_{n-j-1}(\hat{\xi}_{j+2}^{n+m})\}$$

$$\cup \left(\bigcup_{l=2}^{m-1} \{\hat{\xi}_{n+1}^{n+l-1}, \xi_{j+1}, P_{n-j-1}(\hat{\xi}_{j+2}^n,\hat{\xi}_{n+l}^{n+m})\} \right)$$

$$\cup \{\hat{\xi}_{n+1}^{n+m} \hat{\xi}_{j+1}^n\}. \tag{A.8}$$

Note (A.8) and (A.7) is essentially the same identity (the former expressed as a combinatorial identity and the later given as a geometric identity). This fact allows one to prove (A.5) by double induction on j and m.

With (A.5) established, we have that

$$\int_{\mathcal{T}_n(x,\xi)} f_n(\hat{\xi}_0^n) \left(\int_{\mathcal{T}_m(\xi_j,\sigma)} g_m(\xi_j,\hat{\sigma}_1^m) d\hat{\sigma}_1^m \right) d\hat{\xi}_1^n$$

$$= \sum_{\hat{\gamma}_1^{n-j+1} \in \mathcal{P}_{n-j}(\hat{\xi}_{j+1}^{n+m})} \int_{\mathcal{T}_{n+m}(x,\xi)} f_n(\hat{\xi}_0^j,\hat{\gamma}_1^{n-j}) g_m(\xi_j,\hat{\gamma}_{n-j+1}^{n-j+m}) d\hat{\xi}_1^{n+m}$$

$$= \int_{\mathcal{T}_{n+m}(x,\xi)} D_j^{n,m}[f_n(\hat{\xi}_0^n) g_m(\xi_j,\hat{\xi}_{n+1}^{n+m})] d\hat{\xi}_1^{n+m}, \tag{A.9}$$

where we have used (53). Then, (A.2) follows. □

We next derive the general kernel equation for $n \geq 2$ (the case $n = 1$ is covered in Smyshlyaev and Krstic (2004)). The idea is to, starting from Eq. (39),

$$0 = \lambda(x)u + \sum_{n=1}^{\infty} \left(u(x) \int_{\mathcal{T}_n(x,\xi)} h_n(\hat{\xi}_0^n) \prod^n u d\hat{\xi}_1^n \right.$$

$$+ \int_{\mathcal{T}_n(x,\xi)} f_n(\hat{\xi}_0^n) \prod^n u d\hat{\xi}_1^n$$

$$\left. + \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right) \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) \prod^n u d\hat{\xi}_1^n \right), \tag{A.10}$$

evaluate the derivatives in (A.10) and apply integration by parts to reach a formula that contains the least possible number of derivatives in u.

We begin computing the second spatial derivative in (A.10), which is

$$\frac{\partial^2}{\partial x^2} \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) \prod^n u d\hat{\xi}_1^n$$

$$= \int_{\mathcal{T}_n(x,\xi)} \partial_{xx} k_n(\hat{\xi}_0^n) \prod^n u d\hat{\xi}_1^n + u_x(x)$$

$$\times \int_{\mathcal{T}_{n-1}(x,\xi)} k_n(x,\hat{\xi}_0^{n-1}) \prod^{n-1} u d\hat{\xi}_1^{n-1}$$

$$+ u(x) \int_{\mathcal{T}_{n-1}(x,\xi)} \left(2k_{nx}(x,\hat{\xi}_0^{n-1}) \right.$$

$$\left. + k_{n\xi_1}(x,\hat{\xi}_0^{n-1}) \right) \prod^{n-1} u d\hat{\xi}_1^{n-1}$$

$$+ u(x)^2 \int_{\mathcal{T}_{n-2}(x,\xi)} k_n(x,x,\hat{\xi}_0^{n-2}) \prod^{n-2} u d\hat{\xi}_1^{n-2}. \tag{A.11}$$

Next we compute the time derivative in (A.10), which yields

$$\frac{\partial}{\partial t} \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) \prod^n u d\hat{\xi}_1^n$$

$$= \sum_{j=1}^n \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) u_{xx}(\xi_j) \prod^{n,j} u d\hat{\xi}_1^n$$

$$+ \sum_{j=1}^n \int_{\mathcal{T}_n(x,\xi)} \lambda(\xi_j) k_n(\hat{\xi}_0^n) \prod^n u d\hat{\xi}_1^n$$

$$+ \sum_{j=1}^n \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) H[u](\xi_j) \prod^n u d\hat{\xi}_1^n$$

$$+ \sum_{j=1}^n \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) F[u](\xi_j) \prod^{n,j} u d\hat{\xi}_1^n. \tag{A.12}$$

We need to simplify (A.12). We show how it can be done for each line in the equation. Using (A.1) for the second line in (A.12),

$$\sum_{j=1}^n \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) u_{xx}(\xi_j) \prod^{n,j} u d\hat{\xi}_1^n$$

$$= \sum_{j=1}^{n-1} \int_{\mathcal{T}_{n-1}(x,\xi)} \int_{\xi_j}^{\xi_{j-1}} k_n(\hat{\xi}_0^{j-1},s,\hat{\xi}_j^{n-1}) u_{xx}(s) \prod^{n-1} u ds d\hat{\xi}_1^{n-1}$$

$$+ \int_{\mathcal{T}_{n-1}(x,\xi)} \int_0^{\xi_{n-1}} k_n(\hat{\xi}_0^{n-1},s) u_{xx}(s) \prod^{n-1} u ds d\hat{\xi}_1^{n-1}, \tag{A.13}$$

and integrating by parts,

$$\sum_{j=1}^n \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) u_{xx}(\xi_j) \prod^{n,j} u d\hat{\xi}_1^n$$

$$= \sum_{j=1}^{n} \int_{\mathcal{T}_n(x,\xi)} \partial_{\xi_j \xi_j} k_n(\hat{\xi}_0^n) \prod u \, \mathrm{d}\hat{\xi}_1^n$$

$$+ \sum_{j=1}^{n} \int_{\mathcal{T}_{n-1}(x,\xi)} k_n(\hat{\xi}_0^{j-1}, \xi_{j-1}, \hat{\xi}_j^{n-1}) u_x(\xi_{j-1})$$

$$\times \prod u \, \mathrm{d}\hat{\xi}_1^{n-1} - \sum_{j=1}^{n-1} \int_{\mathcal{T}_{n-1}(x,\xi)} k_n(\hat{\xi}_0^{j-1}, \xi_j, \hat{\xi}_j^{n-1})$$

$$\times u_x(\xi_j) \prod u \, \mathrm{d}\hat{\xi}_1^{n-1} - \sum_{j=1}^{n} \int_{\mathcal{T}_{n-1}(x,\xi)} \prod u$$

$$\times k_{n\xi_j}(\hat{\xi}_0^{j-1}, \xi_{j-1}, \hat{\xi}_j^{n-1}) u(\xi_{j-1}) \mathrm{d}\hat{\xi}_1^{n-1}$$

$$+ \sum_{j=1}^{n-1} \int_{\mathcal{T}_{n-1}(x,\xi)} k_{n\xi_j}(\hat{\xi}_0^{j-1}, \xi_j, \hat{\xi}_j^{n-1}) u(\xi_j)$$

$$\times \prod u \, \mathrm{d}\hat{\xi}_1^{n-1} - u_x(0) \int_{\mathcal{T}_{n-1}(x,\xi)} k_n(\hat{\xi}_0^{n-1}, 0)$$

$$\times \prod u \, \mathrm{d}\hat{\xi}_1^{n-1} + u(0) \int_{\mathcal{T}_{n-1}(x,\xi)} k_{n\xi_n}(\hat{\xi}_0^{n-1}, 0) \prod u \, \mathrm{d}\hat{\xi}_1^{n-1}. \quad \text{(A.14)}$$

Since $(\hat{\xi}_0^{j-1}, \xi_j, \hat{\xi}_j^{n-1}) = (\hat{\xi}_0^j, \hat{\xi}_j^{n-1})$ and $(\hat{\xi}_0^{j-1}, \xi_{j-1}, \hat{\xi}_j^{n-1}) = (\hat{\xi}_0^{j-1}, \hat{\xi}_{j-1}^{n-1})$, the fourth to sixth lines in (A.14) summed over j simplify as

$$\sum_{j=1}^{n} \int_{\mathcal{T}_{n-1}(x,\xi)} k_n(\hat{\xi}_0^{j-1}, \xi_{j-1}, \hat{\xi}_j^{n-1}) u_x(\xi_{j-1}) \prod u \, \mathrm{d}\hat{\xi}_1^{n-1}$$

$$- \sum_{j=1}^{n-1} \int_{\mathcal{T}_{n-1}(x,\xi)} k_n(\hat{\xi}_0^{j-1}, \xi_j, \hat{\xi}_j^{n-1}) u_x(\xi_j) \prod u \, \mathrm{d}\hat{\xi}_1^{n-1}$$

$$= u_x(x) \int_{\mathcal{T}_{n-1}(x,\xi)} k_n(x, x, \hat{\xi}_1^{n-1}) \prod u \, \mathrm{d}\hat{\xi}_1^{n-1}. \quad \text{(A.15)}$$

Hence (A.14) can be simplified as

$$\sum_{j=1}^{n} \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) u_{xx}(\xi_j) \prod^{n,j} u \, \mathrm{d}\hat{\xi}_1^n$$

$$= u(0) \int_{\mathcal{T}_{n-1}(x,\xi)} \left(k_{n\xi_n}(\hat{\xi}_0^{n-1}, 0) - q k_n(\hat{\xi}_0^{n-1}, 0) \right)$$

$$\times \prod u \, \mathrm{d}\hat{\xi}_1^{n-1} + \sum_{j=1}^{n} \int_{\mathcal{T}_n(x,\xi)} \partial_{\xi_j \xi_j} k_n(\hat{\xi}_0^n) \prod u \, \mathrm{d}\hat{\xi}_1^n$$

$$+ u_x(x) \int_{\mathcal{T}_{n-1}(x,\xi)} k_n(x, x, \hat{\xi}_1^{n-1}) \prod u \, \mathrm{d}\hat{\xi}_1^{n-1}$$

$$- u(x) \int_{\mathcal{T}_{n-1}(x,\xi)} k_{n\xi_1}(x, x, \hat{\xi}_1^{n-1}) \prod u \, \mathrm{d}\hat{\xi}_1^{n-1}$$

$$+ \sum_{j=1}^{n-1} \int_{\mathcal{T}_{n-1}(x,\xi)} \left(k_{n\xi_j}(\hat{\xi}_0^{j-1}, \xi_j, \hat{\xi}_j^{n-1}) \right.$$

$$\left. - k_{n\xi_{j+1}}(\hat{\xi}_0^{j-1}, \xi_j, \hat{\xi}_j^{n-1}) \right) u(\xi_j) \prod u \, \mathrm{d}\hat{\xi}_1^{n-1}. \quad \text{(A.16)}$$

In the second line of (A.16) we have used the Robin boundary condition for u at $x = 0$.

The fourth line in (A.12) can be written as

$$\sum_{j=1}^{n} \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) H[u](\xi_j) \prod u \, \mathrm{d}\hat{\xi}_1^n$$

$$= \sum_{j=1}^{n} \sum_{m=1}^{\infty} \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) \left(\int_{\mathcal{T}_i(\xi_j, \sigma)} h_m(\xi_j, \hat{\sigma}_1^m) \prod^{m} u \, \mathrm{d}\hat{\sigma}_1^m \right)$$

$$\times \prod u \, \mathrm{d}\hat{\xi}_1^n$$

$$= \sum_{j=1}^{n} \sum_{m=1}^{\infty} \int_{\mathcal{T}_{n+m-1}(x,\xi)} D_j^{n,m}[k_n(\hat{\xi}_0^n) h_m(\xi_j, \hat{\xi}_{n+1}^{m+n})]$$

$$\times \prod u \, \mathrm{d}\hat{\xi}_1^{n+m}, \quad \text{(A.17)}$$

where we have applied (A.2). Then, since the time derivative computed in (A.12) is summed from $n = 1$ to infinity in (A.10), we consider the infinite sum for the result in (A.17), which yields

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n} \sum_{m=1}^{\infty} \int_{\mathcal{T}_{n+m}(x,\xi)} D_j^{n,m}[k_n(\hat{\xi}_0^n) h_m(\xi_j, \hat{\xi}_{n+1}^{m+n})] \prod^{n+m} u \, \mathrm{d}\hat{\xi}_1^{n+m}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{j=1}^{n-m+1} \int_{\mathcal{T}_n(x,\xi)} D_j^{n-m,m}[k_{n-m}(\hat{\xi}_0^{n-m}) h_m(\xi_j, \hat{\xi}_{n-m+1}^n)]$$

$$\times \prod^{n} u \, \mathrm{d}\hat{\xi}_1^n$$

$$= \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(x,\xi)} \sum_{m=1}^{n-1} C_n^m[k_{n-m}, h_m] \prod^{n} u \, \mathrm{d}\hat{\xi}_1^n, \quad \text{(A.18)}$$

where we have used the definition of C_n^m given in (55). Similarly, the fifth line in (A.12) can be written as

$$\sum_{j=1}^{n} \int_{\mathcal{T}_n(x,\xi)} k_n(\hat{\xi}_0^n) F[u](\xi_j) \prod^{n,j} u \, \mathrm{d}\hat{\xi}_1^n$$

$$= \sum_{j=1}^{n} \sum_{m=1}^{\infty} \int_{\mathcal{T}_{n+m}(x,\xi)} D_j^{n,m}[k_n(\hat{\xi}_0^n) f_m(\xi_j, \hat{\xi}_{n+1}^{m+n})] \prod^{n+m,j} u \, \mathrm{d}\hat{\xi}_1^{n+m}$$

$$= \sum_{j=1}^{n} \sum_{m=1}^{\infty} \int_{\mathcal{T}_{n+m-1}(x,\xi)} \int_{\xi_j}^{\xi_{j-1}} D_j^{n,m}[k_n(\hat{\xi}_{j-1}^n, s, \hat{\xi}_j^{n-1})$$

$$\times f_m(s, \hat{\xi}_n^{m+n-1})] \mathrm{d}s \prod^{n+m-1} u \, \mathrm{d}\hat{\xi}_1^{n+m-1}, \quad \text{(A.19)}$$

where we have applied (A.1) and (A.2). Then, again computing the infinite sum as in (A.18),

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n} \sum_{m=1}^{\infty} \int_{\mathcal{T}_{n+m-1}(x,\xi)} \int_{\xi_j}^{\xi_{j-1}} D_j^{n,m}[f_m(s, \hat{\xi}_n^{m+n-1})$$

$$\times k_n(\hat{\xi}_{j-1}^n, s, \hat{\xi}_j^{n-1})] \mathrm{d}s \prod^{n+m-1} u \, \mathrm{d}\hat{\xi}_1^{n+m-1}$$

$$= \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(x,\xi)} \left(I[k_n, f_1] + \sum_{m=2}^{n} B_n^m[k_{n-m+1}, f_m] \right) \prod^{n} u \, \mathrm{d}\hat{\xi}_1^n, \quad \text{(A.20)}$$

where the definitions (56) and (54) of respectively I and B_n^m have been used.

Collecting all the terms (A.11), (A.16), (A.18) and (A.20) into (A.10), we get

$$0 = \sum_{n=1}^{\infty} \int_{\mathcal{T}_n(x,\xi)} \left[\partial_{xx} k_n - \sum_{j=1}^{n} \partial_{\xi_j \xi_j} k_n - \sum_{j=1}^{n} \lambda(\xi_j) k_n \right.$$

$$+ f_n - \sum_{m=2}^{n} B_n^i[k_{n-m+1}, f_m] - \sum_{m=1}^{n-1} C_n^m[k_{n-m}, h_m]$$

$$\left. - I[k_n, f_1] \right] \prod^{n} u \, \mathrm{d}\hat{\xi}_1^n + u(0) \times \sum_{n=1}^{\infty} \int_{\mathcal{T}_{n-1}(x,\xi)} \prod^{n-1} u$$

$$\times \left(k_{n\xi_n}(\hat{\xi}_0^{n-1}, 0) - q k_n(\hat{\xi}_0^{n-1}, 0) \right) d\hat{\xi}_1^{n-1}$$

$$+ \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} \int_{\mathcal{T}_{n-1}(x,\xi)} \left(k_{n\xi_j}(\hat{\xi}_0^{j-1}, \xi_j, \hat{\xi}_j^{n-1}) \right.$$

$$\left. - k_{n\xi_{j+1}}(\hat{\xi}_0^{j-1}, \xi_j, \hat{\xi}_j^{n-1}) \right) u(\xi_j) \prod^{n-1} u d\hat{\xi}_1^{n-1}$$

$$+ u^2(x) \sum_{n=2}^{\infty} \int_{\mathcal{T}_{n-2}(x,\xi)} k_n(x,x,\hat{\xi}_1^{n-2}) \prod^{n-2} u d\hat{\xi}_1^{n-2}$$

$$+ u(x) \left[\lambda(x) + \sum_{n=1}^{\infty} \int_{\mathcal{T}_{n-1}(x,\xi)} \left(h_{n-1} \right. \right.$$

$$\left. \left. + 2 \frac{d}{dx} k_n(x,x,\hat{\xi}_1^{n-1}) \right) \prod^{n-1} u d\hat{\xi}_1^{n-1} \right], \tag{A.21}$$

where we define the total derivative $\frac{d}{dx} k_n(x,x,\hat{\xi}_1^{n-1}) = k_{nx}(x,x,\hat{\xi}_1^{n-1}) + k_{n\xi_1}(x,x,\hat{\xi}_1^{n-1})$. Since (A.21) has to be verified for arbitrary u, we get that the terms inside the integrals must be zero. Hence, we get from the first three lines that

$$\partial_{xx} k_n = \sum_{i=1}^{n} \partial_{\xi_i \xi_i} k_n + \sum_{j=1}^{n} \lambda(\xi_j) k_n + \sum_{m=1}^{n-1} C_n^m[k_{n-m}, h_m]$$

$$+ \sum_{m=2}^{n} B_n^m[k_{n-m+1}, f_m] - f_n + I_n[k_n, f_1], \tag{A.22}$$

and from the rest of (A.21), we obtain

$$k_{n\xi_{i-1}}(\hat{\xi}_0^n)|_{\xi_{i-1}=\xi_i} = k_{n\xi_i}(\hat{\xi}_0^n)|_{\xi_{i-1}=\xi_i}, \quad i = 2, \ldots, n, \tag{A.23}$$

$$k_{n\xi_n}(\hat{\xi}_0^{n-1}, 0) = q k_n(\hat{\xi}_0^{n-1}, 0), \tag{A.24}$$

$$\frac{d}{dx} k_n(x, x, \hat{\xi}_2^n) = -\frac{1}{2} h_{n-1}(x, \hat{\xi}_2^n), \tag{A.25}$$

$$k_n(x, x, \hat{\xi}_3^n) = 0. \tag{A.26}$$

Eqs. (A.22)–(A.26) are the general kernel equations, but we still need to derive boundary conditions (43) and (44)

Integrating (A.25) and using (A.26) to determine the constant of integration, we get (43):

$$k_n(x, x, \hat{\xi}_2^n) = -\frac{1}{2} \int_{\xi_2}^{x} h_{n-1}(s, \hat{\xi}_2^n) ds. \tag{A.27}$$

Boundary condition (44) is built into (A.22). Defining ϕ_n as in (47), when $\xi_1 = x$ (A.22) reduces to

$$\partial_{xx} k_n|_{x=\xi_1} = \partial_{\xi_1 \xi_1} k_n(x, x, \hat{\xi}_2^n) + \phi_n(x, \hat{\xi}_2^n). \tag{A.28}$$

Taking derivative with respect to x in (A.25),

$$\left(\partial_{xx} k_n + \partial_{\xi_1 \xi_1} k_n + 2 \partial_{x\xi_1} k_n \right)_{x=\xi_1} = -\frac{1}{2} \partial_x h_{n-1}(x, \hat{\xi}_2^n), \tag{A.29}$$

which substituted in (A.28) gives

$$2 \left(\partial_{xx} k_n + \partial_{x\xi_1} k_n \right)_{x=\xi_1} = -\frac{1}{2} \partial_x h_{n-1}(x, \hat{\xi}_2^n) + \phi_n(x, \hat{\xi}_2^n), \tag{A.30}$$

hence

$$\frac{d}{dx} k_{nx}(x, x, \hat{\xi}_2^n) = -\frac{1}{4} \partial_x h_{n-1}(x, \hat{\xi}_2^n) + \frac{1}{2} \phi_n(x, \hat{\xi}_2^n). \tag{A.31}$$

From (A.23) at $i = 2$, $\xi_1 = x$, we get that

$$k_{n\xi_1}(x, x, x, \hat{\xi}_3^n) = k_{n\xi_2}(x, x, x, \hat{\xi}_3^n) \tag{A.32}$$

and since (A.26) implies

$$\left(k_{nx} + k_{n\xi_1} + k_{n\xi_2} \right)(x, x, x, \hat{\xi}_3^n) = 0, \tag{A.33}$$

we get, from (A.25), that

$$k_{nx}(x, x, x, \hat{\xi}_3^n) = -h_{n-1}(x, \hat{\xi}_2^n). \tag{A.34}$$

Integrating (A.31) and using (A.34) to find the constant of integration, we get

$$k_{nx}(x, x, \hat{\xi}_2^n) = -\frac{1}{4} h_{n-1}(x, \hat{\xi}_2^n) - \frac{3}{4} h_{n-1}(\xi_2, \hat{\xi}_2^n)$$

$$+ \frac{1}{2} \int_{\xi_2}^{x} \phi_n(s, \hat{\xi}_2^n) ds, \tag{A.35}$$

which was the remaining boundary condition (44).

References

Aamo, O. M., & Krstic, M. (2004). Global stabilization of a nonlinear Ginzburg-Landau model of vortex shedding. *European Journal of Control*, 10, 105–116.

Balogh, A., & Krstic, M. (2003). Infinite dimensional backstepping for nonlinear parabolic pdes. In V. Blondel, & A. Megretski (Eds.), *Sixty open problems in the mathematics of systems and control*. Princeton, NJ: Princeton University Press.

Boskovic, D., & Krstic, M. (2001). Nonlinear stabilization of a thermal convection loop by state feedback. *Automatica*, 37, 2033–2040.

Boskovic, D., & Krstic, M. (2002). Backstepping control of chemical tubular reactors. *Computers and Chemical Engineering*, 26, 1077–1085.

Boskovic, D., & Krstic, M. (2003). Stabilization of a solid propellant rocket instability by state feedback. *International Journal of Robust and Nonlinear Control*, 13, 483–495.

Boyd, S., Chua, L. O., & Desoer, C. A. (1984). Analytical foundations of volterra series. *Journal of Mathematical Control and Information*, 1, 243–282.

Christofides, P. D. (2001). *Nonlinear and robust control of partial differential equation systems: Methods and applications to transport-reaction processes*. Boston: Birkhauser.

Coron, J. M., & Trelat, E. (2004). Global steady-state controllability of 1-D semilinear heat equation. *SIAM Journal on Control and Optimization*, 43(2), 549–569.

Courant, R., & Hilbert, D. (1995). *Methods of mathematical physics: Vol. 2*. New York: Interscience Publishers.

Deng, K., & Levine, H. A. (2000). The role of critical exponents in blow-up theorems: The sequel. *Journal of Mathematical Analysis and Applications*, 243, 85–126.

Fernandez-Cara, E., & Zuazua, E. (2000). Null and approximate controllability for weakly blowing up semilinear heat equations. *Annales de l'IHP. Analyse non Linéaire*, 17, 583–616.

Fliess, M. (1981). Fonctionnelles causales non lineaires et indeterminees non commutatives. *Bulletin de la Société Mathématique de France*, 109, 3–40.

Fujita, H. (1966). On the blowing up of solutions of the cauchy problem for $u_t = \triangle u + u^{1+\alpha}$. *Journal of the Faculty of Sciences, University of Tokyo, Section IA Mathematics*, 13, 105–113.

Isidori, A. (1995). *Nonlinear control systems*. Berlin: Springer-Verlag.

Khalil, H. K. (2002). *Nonlinear systems* (3rd ed.). Englewood Cliffs, NJ: Prentice-Hall.

Krstic, M., Kanellakopoulos, I., & Kokotovic, P. V. (1995). *Nonlinear and adaptive control design*. Wiley.

Krstic, M. (2005). Lyapunov adaptive stabilization of parabolic PDEs: Part I—A benchmark for boundary control. In *Proceedings of the 2005 CDC*.

Krstic, M., Smyshlyaev, A., & Siranosian, A. (2006). Backstepping boundary controllers and observers for the slender Timoshenko beam: Part I—Design. In *Proceedings of the 2006 American control conference*.

Kugi, A., Thull, D., & Kuhnen, K. (2006). An infinite-dimensional control concept for piezoelectric structures with complex hysteresis. *Structural Control and Health Monitoring*, 13, 1099–1119. 2006.

Lamnabhi-Lagarrigue, F. (1996). Volterra and Fliess series expansions for nonlinear systems. In W. S. Levine (Ed.), *The control handbook* (pp. 879–888). Boca Raton, FL: CRC Press.

Levine, H. A. (1990). The role of critical exponents in blow-up theorems. *SIAM Review*, 32, 262–288.

Lines, L. R., Slawinski, R., & Bording, R. P. (1999). A recipe for stability of finite-difference wave-equation computations. *Geophysics*, 64, 967–969.

Meurer, T. (2005). Feedforward and feedback tracking control of diffusion-convection-reaction systems using summability methods. *Doctoral dissertation*. Stuttgart University.

Rugh, W. J. (1981). Nonlinear sytem theory: The Volterra/Wiener approach. Web version http://www.ece.jhu.edu/rugh/volterra/book.pdf.

Sastry, S. (1999). *Nonlinear systems: Analysis, stability and control*. New York: Springer-Verlag.

Sepulchre, R., Jankovic, M., & Kokotovic, P. V. (1997). *Constructive nonlinear control*. New York: Springer-Verlag.

Smyshlyaev, A., & Krstic, M. (2004). Closed form boundary state feedbacks for a class of partial integro-differential equations. *IEEE Transactions on Automatic Control*, 49, 2185–2202.

Smyshlyaev, A., & Krstic, M. (2005). Passive identifiers for boundary adaptive control of 3D reaction-advection-diffusion PDEs. In *Proceedings of the 2005 CDC*.

Vazquez, R., & Krstic, M. (2006). Explicit integral operator feedback for local stabilization of nonlinear thermal convection loop PDEs. *Systems and Control Letters*, 55, 624–632.

Vazquez, R., & Krstic, M. (2007). A closed-form feedback controller for stabilization of the linearized 2-D Navier–Stokes Poisseuille system. *IEEE Transactions on Automatic Control, 52*, 2298–2312.

Vazquez, R., & Krstic, M. (2008). Control of 1-D Parabolic PDEs with Volterra Nonlinearities—Part II: Analysis. *Automatica.*

Volterra, V. (1959). *Theory of functionals and of integral and integro-differential equations.* New York: Dover.

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