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Boundary Control of Coupled Reaction-Advection-Diffusion Systems with Spatially-Varying Coefficients

Rafael Vazquez and Miroslav Krstic

***Abstract*—Recently, the problem of boundary stabilization for unstable linear constant-coefficient coupled reaction-diffusion systems was solved by means of the backstepping method. The extension of this result to systems with advection terms and spatially-varying coefficients is challenging due to complex boundary conditions that appear in the equations verified by the control kernels. In this paper we address this issue by showing that these equations are essentially equivalent to those verified by the control kernels for first-order hyperbolic coupled systems, which were recently found to be well-posed. The result therefore applies in this case, allowing us to prove H^1 stability for the closed-loop system. It also unveils a previously unknown connection between backstepping kernels for coupled parabolic and hyperbolic problems.**

***Index Terms*—Boundary control; backstepping; parabolic equations; advection-reaction-diffusion systems; distributed parameter systems.**

I. Introduction

In a recent work [2], the problem of boundary stabilization for general linear *constant-coefficient* coupled reaction-diffusion systems was resolved by means of the backstepping method [14]. However, the extension of this result to systems with *advection* terms and *spatially-varying* coefficients—as usually found in applications—is far from trivial. The main difficulty arises when trying to solve the partial differential equations verified by the control kernels (usually known as the "backstepping kernel equations"). For n states in a system of coupled parabolic equations, one needs to find n^2 control kernel verifying n^2 fully coupled *second-order* hyperbolic equations in a triangular domain, with complicated boundary conditions. In the constant-coefficient case, it is possible to simplify the boundary conditions by assuming a certain kernel structure, and then the equations can be readily solved [2]. However, this procedure does not extend to the spatially-varying case and/or advection terms. In this work, we show that the kernel equations can be written (using some non-trivially-defined intermediate kernels) as a coupled system of $2n^2$ *first-order* hyperbolic equations. Interestingly, these kernel equations are very similar to those found when applying backstepping to find boundary controllers for first-order hyperbolic coupled systems [11]. A result recently obtained for this problem showed that the resulting kernel equations were well-posed and had piecewise differentiable solutions [12]. Applying this result in our case allows us to find a backstepping controller, and to prove H^1 exponential stability for the origin of the closed-loop system with arbitrary convergence rate. Our result unveils an interesting and non-trivial connection between backstepping controllers for coupled parabolic and hyperbolic systems.

The problem presented in this paper could be addressed by other methods, including the semigroup approach (see for instance [17]), eigenvalue assignment [3], flatness [18] or LQR [19]. The main advantage of backstepping is that, once the well-posedness of the kernel equations has been established, analytical and numerical results are simple to obtain, often including well-posedness of the closed loop system in high-order Sobolev spaces or even explicit exact controllers [26]. Backstepping has proved itself to be an ubiquitous method for PDE control, with many other applications including, among others, flow control [23], [28], nonlinear PDEs [24], disturbance rejection [1], [9], hyperbolic 1-D systems [6], [7], [16], adaptive control [22], wave equations [21], Korteweg-de Vries equations [5], and delays [15]. Other recent results related to boundary control of parabolic systems include [20], where backstepping is applied to find multi-agent deployments in 3-D space, output-feedback boundary control for ball-shaped domains in any dimension [27], and design of output feedback laws for convection problems on annular domains (see [25]).

The structure of the paper is as follows. In Section II we introduce the problem and state our main result. We explain our design method (backstepping) and show the stability of the closed-loop system in Section III. Next, we prove that there is a solution to the backstepping kernel equations in Section IV. We conclude the paper with some remarks in Section V.

II. Coupled reaction-advection-diffusion systems

Consider the following general linear spatially-varying reaction-advection-diffusion system

$$u_t = \partial_x \left(\Sigma(x) u_x\right) + \Phi(x) u_x + \Lambda(x) u, \tag{1}$$

for $x \in [0,1]$, $t > 0$, with $u \in \mathbb{R}^n$ defined as

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \tag{2}$$

R. Vazquez is with the Department of Aerospace Engineering, University of Seville, Seville, 41092, Spain (e-mail: rvazquez1@us.es)

M. Krstic is with the Department of Mechanical Aerospace Engineering, University of California, San Diego, CA 92093-0411, USA (e-mail: krstic@ucsd.edu)

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and the various coefficients appearing in (1) defined as

$$\Sigma(x) = \begin{bmatrix} \epsilon_1(x) & 0 & \dots & 0 \\ 0 & \epsilon_2(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \epsilon_N(x) \end{bmatrix}, \tag{3}$$

$$\Lambda(x) = \begin{bmatrix} \lambda_{11}(x) & \lambda_{12}(x) & \dots & \lambda_{1n}(x) \\ \lambda_{21}(x) & \lambda_{22}(x) & \dots & \lambda_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1}(x) & \lambda_{n2}(x) & \dots & \lambda_{nn}(x) \end{bmatrix}, \tag{4}$$

$$\Phi(x) = \begin{bmatrix} \phi_{11}(x) & \phi_{12}(x) & \dots & \phi_{1n}(x) \\ \phi_{21}(x) & \phi_{22}(x) & \dots & \phi_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1}(x) & \phi_{n2}(x) & \dots & \phi_{nn}(x) \end{bmatrix}. \tag{5}$$

and with boundary conditions

$$u(0,t) = 0, \tag{6}$$

$$u(1,t) = U(t), \tag{7}$$

where $U(t)$ is the actuation, defined as

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_n(t) \end{bmatrix}. \tag{8}$$

The only assumption on (1),(6)–(7) is that these coefficients are sufficiently regular; in particular, it is required that the entries of $\Sigma(x)$ are three times differentiable, those of $\Phi(x)$ twice differentiable and those of $\Lambda(x)$ differentiable. In addition, we assume that the states are ordered so that $\bar{\epsilon} \geq \epsilon_1(x) > \epsilon_2(x) > \ldots > \epsilon_n(x) \geq \underline{\epsilon} > 0$. The diffusion coefficients could also be equal at some (or all) values of x but to avoid technical complications we confine ourselves to the case of strict inequality.

Since (1), (6)–(7) is potentially unstable depending on the values of the coefficients, the problem we consider is the design of a (full-state) feedback control law for $U(t)$ that makes the system stable for any possible value of the coefficients.

We will make use of the $L^2([0,1])$ and $H^1([0,1])$ spaces, defined, respectively, as the space of square-integrable vector functions in the $[0,1]$ interval and the space of vector functions whose derivative (with respect to x, defined in the weak sense [8]) is square-integrable in the $[0,1]$ interval. For simplicity we will simply write L^2 and H^1. If $f \in L^2$ or $f \in H^1$ its norm will be written as $\|f\|_{L^2}$ or $\|f\|_{H^1}$, respectively, and computed with the following expressions

$$\|f\|_{L^2}^2 = \int_0^1 |f(x)|^2 dx,\ \|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \int_0^1 \left|\frac{\partial f(x)}{\partial x}\right|^2 dx, \tag{9}$$

where $|\cdot|$ denotes the regular Euclidean norm. In addition we will use L^2 spaces with respect to time, which are analogously defined. Rather than using a more complex notation, we will denote the L^2 norm with respect to time equally as $\|\cdot\|_{L^2}$, and since it will only be used for functions only depending on time it should be clear from the context what L^2 norm we are referring to.

Define C as a diagonal matrix of constant positive coefficients, i.e.,

$$C = \begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_n \end{bmatrix}, \tag{10}$$

with $c_1, c_2, \ldots, c_n > 0$, whose values can be chosen but should be sufficiently large (see Section III-C).

Next, we state our main result that solves the stabilization problem in H^1.

Theorem 1. *Consider system (1), (6)–(7) with initial condition $u_0 \in H^1$, $u_0(0) = 0$, and feedback control law*

$$U(t) = \int_0^1 K(1,\xi)u(\xi,t)d\xi + b(t), \tag{11}$$

where the kernel matrix $K(x,\xi)$ is a solution from the following hyperbolic matrix *system of PDEs*

$$\begin{aligned} &\partial_x\left(\Sigma(x)K_x\right) - \partial_\xi\left(K_\xi\Sigma(\xi)\right) + \Phi(x)K_x + K_\xi\Phi(\xi) \\ &= K(x,\xi)\Lambda(\xi) + CK(x,\xi) + K(x,\xi)\Phi'(\xi), \end{aligned} \tag{12}$$

in the domain $\mathcal{T} = \{(x,\xi) : 0 \leq \xi \leq x \leq 1\}$, with boundary conditions

$$\Phi(x)K(x,x) - K(x,x)\Phi(x) + \Lambda(x) + C + K_\xi(x,x)\Sigma(x)$$

$$+\Sigma(x)K_x(x,x) + \frac{d}{dx}\left(\Sigma(x)K(x,x)\right) = 0, \tag{13}$$

$$\Sigma(x)K(x,x) - K(x,x)\Sigma(x) = 0, \tag{14}$$

$$K_{ij}(x,0) = 0, \quad i \leq j \tag{15}$$

and $b(t)$ is defined as

$$b(t) = \left(u_0(1,t) - \int_0^1 K(1,\xi)u_0(\xi)d\xi\right)\mathrm{e}^{-\alpha_1 t}, \tag{16}$$

for any chosen $\alpha_1 > 0$. Assuming that, under these conditions, there is a unique $u(\cdot,t) \in H^1$ solution to (1), (6)–(7), there exists a number c^* depending only on the coefficients $\Sigma(x)$ and $\Phi(x)$, so that if the values of the coefficients of C verify $c_i \geq c^* + \delta$, for all $i = 1,\ldots,n$, and for some $\delta > 0$, then the origin $u \equiv 0$ is exponentially stable in the H^1 norm, i.e.,

$$\|u(\cdot,t)\|_{H^1} \leq C_1\mathrm{e}^{-C_2 t}\|u_0\|_{H^1}, \tag{17}$$

with $C_1, C_2 > 0$, where $C_2 = \min\{\alpha_1, 2\delta\}$.

Note that, in Theorem 1, well-posedness of the closed-loop system is assumed. The extent of this assumption is clarified in Section III-C. In Theorem 1, the main question is if the kernel equations (12)–(15) do indeed have a solution, as implicitly assumed in the theorem's statement. The next result answers this question.

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Theorem 2. *The kernel equations (12)–(15) possess a piecewise differentiable solution in the domain $\mathcal{T}$. In addition, the transformation defined by*

$$g(x) = f(x) - \int_0^x K(x,\xi) f(\xi) d\xi, \tag{18}$$

is an invertible transformation. Both the transformation and its inverse map H^1 functions into H^1 functions, verifying

$$\|g\|_{H^1} \leq K_1 \|f\|_{H^1},\ \|f\|_{H^1} \leq K_2 \|g\|_{H^1}.$$

In the next sections we prove Theorem 1 and 2, respectively in sections III and IV.

III. Control law design and closed-loop stability (proof of Theorem 1)

We stabilize (1), (6)–(7) by applying the backstepping method. Next we explain the method and show that the origin of the resulting closed-loop system is stable in the H^1 norm.

A. Backstepping transformation and target system

The main idea of backstepping is to use a transformation mapping (1), (6)–(7) into a stable *target* system, which has to be adequately chosen. We select the following system

$$w_t = \partial_x \left(\Sigma(x) w_x\right) + \Phi(x) w_x - Cw - G(x) w_x(0,t), \tag{19}$$

where the target state w is defined as

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}. \tag{20}$$

with boundary conditions

$$\begin{aligned} w(0,t) &= 0, & (21)\\ w(1,t) &= b(t), & (22) \end{aligned}$$

whose stability properties will be studied in Section III-C. The matrix $G(x)$ appearing in (19) is a lower triangular matrix with zero diagonal, i.e.,

$$G = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ g_{21}(x) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{(n-1)1}(x) & g_{(n-1)2}(x) & \dots & 0 & 0 \\ g_{n1}(x) & g_{n2}(x) & \dots & g_{n(n-1)}(x) & 0 \end{bmatrix}. \tag{23}$$

The values of the non-zero entries of $G(x)$ are not arbitrary and will be set later in the design.

The main idea behind choosing the cascade structure of $G(x)$ is based on previous works, namely [12] and [11], where as similar target system is used. In Section IV we will see how this selection of target system results in well-posed kernel equations.

The backstepping transformation that maps u into w is defined as

$$w(x,t) = u(x,t) - \int_0^x K(x,\xi) u(\xi,t) d\xi, \tag{24}$$

where the kernel matrix $K(x,\xi)$ is given by

$$K(x,\xi) = \begin{bmatrix} K^{11}(x,\xi) & K^{12}(x,\xi) & \dots & K^{1n}(x,\xi) \\ K^{21}(x,\xi) & K^{22}(x,\xi) & \dots & K^{2n}(x,\xi) \\ \vdots & \vdots & \ddots & \vdots \\ K^{n1}(x,\xi) & K^{n2}(x,\xi) & \dots & K^{nn}(x,\xi) \end{bmatrix}. \tag{25}$$

Next we explain how to find conditions for $K(x,\xi)$ so that in fact (19) holds.

B. Finding the kernel equations

First we establish (12)–(15). To find the equations that the kernel matrix $K(x,\xi)$ must verify, we take time and space derivatives in (24)

$$w_t = u_t - \int_0^x K u_t(\xi,t) d\xi, \tag{26}$$

$$w_x = u_x - \int_0^x K_x u(\xi,t) d\xi - K(x,x) u, \tag{27}$$

$$\begin{aligned} \partial_x \left(\Sigma(x) w_x\right) = {}& \partial_x \left(\Sigma(x) u_x\right) - \int_0^x \partial_x \left(\Sigma(x) K_x\right) u(\xi,t) d\xi \\ & - \partial_x \left(\Sigma(x) K(x,x) u(x,t)\right) \\ & - \Sigma(x) K_x(x,x) u(x,t). \end{aligned} \tag{28}$$

and substituting (1) and (19) inside (26) we find

$$\begin{aligned} & \partial_x \left(\Sigma(x) w_x\right) + \Phi(x) w_x - Cw - G(x) w_x(0,t) \\ = {}& \partial_x \left(\Sigma(x) u_x\right) + \Phi(x) u_x + \Lambda(x) u \\ & - \int_0^x K(x,\xi) \left[\partial_\xi \left(\Sigma(\xi) u_\xi(\xi,t)\right) + \Phi(\xi) u_\xi(\xi,t)\right. \\ & \left. + \Lambda(\xi) u(\xi,t)\right] d\xi. \end{aligned} \tag{29}$$

Using now (27) and (28) we find

$$\begin{aligned} & - \int_0^x \partial_x \left(\Sigma(x) K_x\right) u(\xi,t) d\xi - \Sigma(x) K_x(x,x) u(x,t) \\ & - \partial_x \left(\Sigma(x) K(x,x) u(x,t)\right) - \int_0^x \Phi(x) K_x u(\xi,t) d\xi \\ & - \Phi(x) K(x,x) u - C u(x,t) + \int_0^x C K u(\xi,t) d\xi \\ & - G(x) u_x(0,t) \\ = {}& \Lambda(x) u - \int_0^x K \left[\partial_\xi \left(\Sigma(\xi) u_\xi(\xi,t)\right) + \Phi(\xi) u_\xi(\xi,t)\right. \\ & \left. + \Lambda(\xi) u(\xi,t)\right] d\xi, \end{aligned} \tag{30}$$

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and integrating by parts twice in the right-hand side integral of (30) we find

$$\begin{aligned}&-\int_0^x \partial_x\left(\Sigma(x)K_x\right)u(\xi,t)d\xi-\Sigma(x)K_x(x,x)u(x,t)\\&-\partial_x\left(\Sigma(x)K(x,x)u(x,t)\right)-\int_0^x \Phi(x)K_x u(\xi,t)d\xi\\&-\Phi(x)K(x,x)u-Cu(x,t)+\int_0^x CKu(\xi,t)d\xi\\&-G(x)u_x(0,t)\\=\;&\Lambda(x)u-K(x,x)\Sigma(x)u_x(x,t)+K(x,0)\Sigma(0)u_x(0,t)\\&+K_\xi(x,x)\Sigma(x)u(x,t)-\int_0^x \partial_\xi\left(K_\xi\Sigma(\xi)\right)u(\xi,t)d\xi\\&+\int_0^x K_\xi\Phi(\xi)u(\xi,t)d\xi+\int_0^x K\Phi'(\xi)u(\xi,t)d\xi\\&-K(x,x)\Phi(x)u(x,t)-\int_0^x K\Lambda(\xi)u(\xi,t)d\xi, \qquad (31)\end{aligned}$$

where the boundary condition of u at $x=0$ has been used. We separately collect the terms in (31) affecting $u(x,t)$, $u_x(x,t)$, $u_x(0,t)$ and in the integrals, reaching four equations that need to be independently verified if (31) is to hold for any value of u. These equations are as follows. First we find a hyperbolic matrix PDE

$$\begin{aligned}&\partial_x\left(\Sigma(x)K_x\right)-\partial_\xi\left(K_\xi\Sigma(\xi)\right)+\Phi(x)K_x+K_\xi\Phi(\xi)\\=\;&K\Lambda(\xi)+CK-K\Phi'(\xi), \qquad (32)\end{aligned}$$

where we have omitted the dependence of $K(x,\xi)$. Next, we find three additional conditions

$$\begin{aligned}G(x)&=-K(x,0)\Sigma(0), &(33)\\K(x,x)\Sigma(x)&=\Sigma(x)K(x,x), &(34)\\C+\Lambda(x)&=-\Sigma(x)K_x(x,x)-\partial_x\left(\Sigma(x)K(x,x)\right)\\&\quad-\Phi(x)K(x,x)-K_\xi(x,x)\Sigma(x)\\&\quad+K(x,x)\Phi(x). &(35)\end{aligned}$$

Finally, using the structure of G given in (23) in the boundary condition (34), we find that, on the one hand,

$$K_{ij}(x,0)=0, \qquad \forall j\geq i,$$

which is the boundary condition explicitly named in (15), and on the other hand,

$$g_{ij}(x)=-K_{ij}(x,0)\epsilon_j(0), \qquad \forall j<i,$$

which is the *definition* of the non-zero coefficients of $G(x)$.

C. Target system stability

First, note that from (24), the initial conditions for the target variable w are

$$w_0(x)=u_0(x)-\int_0^x K(x,\xi)u_0(\xi)d\xi, \qquad (36)$$

and therefore $w_0(0)=u_0(0)=0$, and from the definition (16), $w_0(1)=b(0)$. First we consider the following Assumption, which will be shown in Section III-D to be equivalent to the assumption of well-posedness in Theorem 1.

Assumption 1. *The system (19) with boundary conditions (21–22) and initial conditions $w_0\in H^1$, verifying $w_0(0)=0$ and $w_0(1)=b(0)$ and with $b,\dot b\in L^2((0,\infty])$ has an unique solution $w(\cdot,t)\in H^1$.*

A proof of Assumption 1 is beyond the scope of this Note, however next we give some supporting arguments. Consider first redefining the state as $\breve w(x,t)=w(x,t)-x^2b(t)$, noting that $\breve w_0(x)=w_0(x)-x^2b(0)$. Thus, $\breve w_0(0)=\breve w_0(1)=0$ and $\breve w_0\in H_0^1([0,1])$. The equation verifed by $\breve w$ is

$$\breve w_t=\partial_x\left(\Sigma(x)\breve w_x\right)+\Phi(x)\breve w_x-C\breve w-G(x)\breve w_x(0,t)+f(x,t), \qquad (37)$$

where

$$f(x,t)=\left(2x\Sigma'(x)+2\Sigma(x)+2x\Phi(x)-Cx^2\right)b(t)-\dot b(t), \qquad (38)$$

with boundary conditions

$$\breve w(0,t)=\breve w(1,t)=0. \qquad (39)$$

Thus, $f(x,t)\in L^2(0,\infty;L^2([0,1]))$. Now, if $G(x)=0$ in (37), this is a standard parabolic problem, see e.g. [8, p.382], and it follows that $\breve w\in L^2(0,\infty;H^2([0,1]))$ and $\breve w_t\in L^2(0,\infty;L^2([0,1]))$, and the same results follows for $w(t,x)$.

However, the trace term $\breve w_x(0,t)$ in (37) is not classically considered. From the theory of traces [4, p.315–316], if $\breve w\in L^2(0,\infty;H^2([0,1]))$ then $\breve w_x(0,t)\in L^2(0,\infty;L^2([0,1]))$. Thus, reasoning in an informal way, as a right-hand side term in (37), $G(x)\breve w_x(0,t)$ would have the same degree of regularity as $f(x,t)$ and the result should still hold. To give a rigorous proof, one should define $L=\partial_x\left(\Sigma(x)\breve w_x\right)+\Phi(x)\breve w_x-C\breve w-G(x)\breve w_x(0,t)$ and show that despite the trace terms this is still a parabolic operator.

It is interesting to consider the case $\Phi=0$, which decouples the system allowing for a simpler demonstration. In that case, (37) becomes a cascade system for the trace terms, due to the structure of $G(x)$ as seen in (23) and the other coefficients (which are diagonal). Thus, in the equation for $\breve w_1(t,x)$ becomes

$$\breve w_{1t}=\partial_x\left(\epsilon_1(x)\breve w_{1x}\right)-c_1\breve w_1+f_1(x,t), \qquad (40)$$

and the result of [8, p.382] can be applied, obtaining $\breve w_1\in L^2(0,\infty;H^2([0,1]))$. Next, the equation for $\breve w_2(t,x)$ becomes

$$\breve w_{2t}=\partial_x\left(\epsilon_2(x)\breve w_{2x}\right)-c_2\breve w_2+g_{21}(x)w_{1x}(0,t)+f_2(x,t), \qquad (41)$$

and since $\breve w_{1x}(0,t)\in L^2(0,\infty;L^2([0,1]))$ the result of [8, p.382] can be applied again, obtaining $\breve w_2\in L^2(0,\infty;H^2([0,1]))$. By induction, the proof follows for all the components of $\breve w(x,t)$, and thus of $w(x,t)$.

Based on Assumption 1, the following result holds for the target system.

Proposition 1. *Consider the system (19) with boundary conditions (21–22) under the conditions of Assumption 1. Then, there exists a number c^* depending only on the coefficients of the system so that if the coefficients of C verify $c_i\geq c^*+\delta$, for*

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all $i = 1, \ldots, n$, and for some $\delta > 0$, then the origin $w \equiv 0$ is exponentially stable in the H^1 norm, i.e.,

$$\|w(\cdot,t)\|_{H^1} \leq D_1 \mathrm{e}^{-2\delta t}\|w_0\|_{H^1} + D_3\left(\|b\|_{L^2} + \|\dot{b}\|_{L^2}\right). \tag{42}$$

Proof. Now, to show the stability result, consider the following Lyapunov functionals

$$\begin{aligned} V_1(t) &= \frac{1}{2}\int_0^1 w^T Q w dx, &(43)\\ V_2(t) &= \frac{1}{2}\int_0^1 w_x^T Q w_x dx,, &(44)\\ V_3(t) &= \frac{1}{2}\int_0^1 w_{xx}^T Q w_{xx} dx, &(45)\end{aligned}$$

where the space and time dependence of w has been omitted for simplicity. The matrix Q is a square diagonal matrix, with diagonal elements denoted as $q_1, \ldots, q_n$ and chosen positive, so that $\underline{q} \leq q_i \leq \bar{q}$, so that $Q > 0$. It is obvious that $V_1 + V_2$, is equivalent to the H^1 norm of u, i.e., $K_3(V_1+V_2) \leq \|u(x,\cdot)\|^2_{H^1} \leq K_4(V_1+V_2)$ for $K_3, K_4 > 0$.

Taking derivatives we obtain, for $\dot{V}_1$,

$$\begin{aligned} \dot{V}_1 &= \int_0^1 w^T Q\left(\partial_x\left(\Sigma(x)w_x\right)\right)dx \\ &\quad + \int_0^1 w^T Q\left(\Phi(x)w_x - Cw - G(x)w_x(0,t)\right)dx \\ &= -\int_0^1 w_x^T Q\Sigma(x)w_x dx + b^T(t)Q\Sigma(1)w_x(1,t) \\ &\quad + \int_0^1 w^T Q\Phi(x)w_x dx - \int_0^1 w^T QCw dx \\ &\quad - \left(\int_0^1 w^T QG(x)dx\right)w_x(0,t) \end{aligned} \tag{46}$$

Now, assuming that, for all $x \in [0,1]$, the coefficients verify the following bounds: $\|\Phi(x)\| \leq p$, $\underline{c} \leq c_i \leq \bar{c}$, $|g_{ij}(x)| \leq g$, where $\|\cdot\|$ is the matrix operator 2-norm. Then

$$\begin{aligned} \dot{V}_1 &\leq -2\underline{\epsilon}V_2 + \bar{\epsilon}\bar{q}\left|b(t)^T w_x(1,t)\right| + 2p(V_1+V_2) \\ &\quad -(2\underline{c}-1)V_1 + \frac{g^2}{2}|\sqrt{Q}Lw_x(0,t)|^2, \end{aligned} \tag{47}$$

where L is a lower triangular matrix with zero diagonal and unity coefficients, i.e.,

$$L = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 1 & 0 \end{bmatrix}. \tag{48}$$

Similarly, taking derivative in V_2 we obtain

$$\begin{aligned} \dot{V}_2 &= \int_0^1 w_x^T Q w_{xt} dx \\ &= -\int_0^1 w_{xx}^T Q w_t dx + w_x^T Q w_t\Big|_0^1 \\ &= -\int_0^1 w_{xx}^T Q\left(\partial_x\left(\Sigma(x)w_x\right)\right)dx + w_x(1,t)^T Q\dot{b}(t) \\ &\quad -\int_0^1 w_{xx}^T Q\left(\Phi(x)w_x - Cw - G(x)w_x(0,t)\right)dx \\ &= -\int_0^1 w_{xx}^T Q\Sigma(x)w_{xx}dx - \int_0^1 w_{xx}^T Q\Sigma'(x)w_x dx \\ &\quad + w_x(1,t)^T Q\dot{b}(t) + \int_0^1 w_{xx}^T Q\Phi(x)w_x dx \\ &\quad - \int_0^1 w_x^T QCw_x dx + w_x(1,t)^T QCb(t) \\ &\quad - \left(\int_0^1 w_{xx}^T QG(x)dx\right)w_x(0,t). \end{aligned} \tag{49}$$

Therefore, defining $\epsilon_i'(x) \leq \bar{\epsilon}'$ and using the previously defined bounds,

$$\begin{aligned} \dot{V}_2 &\leq -\left[\underline{\epsilon} - \left(\frac{\alpha_2\bar{\epsilon}'}{2} + \frac{p\alpha_3}{2} + \frac{g\alpha_4}{2}\right)\right]V_3 - 2\underline{c}V_2 \\ &\quad + \left(\frac{\bar{\epsilon}'}{\alpha_2} + \frac{p}{\alpha_3}\right)V_2 + \bar{q}\left|\left(\dot{b}(t) + Cb(t)\right)^T w_x(1,t)\right| \\ &\quad + \frac{g}{2\alpha_4}|\sqrt{Q}Lw_x(0,t)|^2), \end{aligned} \tag{50}$$

for $\alpha_2, \alpha_3, \alpha_4 > 0$. Now we have the following inequality

$$|w_x(1,t)|^2 \leq \frac{2}{\underline{q}}(V_2+V_3) \tag{51}$$

which is proven by considering that

$$w_x(1,t) = \int_0^1 \left[(x-1)w_x(x,t)\right]_x dx, \tag{52}$$

therefore

$$|w_x(1,t)| \leq \int_0^1 \left[|w_x(x,t)| + |w_{xx}(x,t)|\right]dx, \tag{53}$$

which squared, gives the inequality. In a similar fashion, we can prove that

$$\left|\sqrt{Q}Lw_x(0,t)\right| \leq \int_0^1 \left[|\sqrt{Q}Lw_x| + |\sqrt{Q}Lw_{xx}|\right]dx, \tag{54}$$

thus

$$\left|\sqrt{Q}Lw_x(0,t)\right|^2 \leq \int_0^1 \frac{w_x^T L^T QLw_x + w_{xx}^T L^T QLw_{xx}}{2}dx, \tag{55}$$

Considering now $V = V_1 + V_2$, we obtain

$$\begin{aligned}\dot{V} &\leq -V_1\left[2\underline{c}-(2p+1)\right] \\ &\quad -V_2\left[2\underline{\epsilon}+2\underline{c}-\left(2p+\left(\frac{\bar{\epsilon}'}{\alpha_2}+\frac{p}{\alpha_3}\right)\right)\right] \\ &\quad -V_3\left[\underline{\epsilon}-\left(\frac{\alpha_2\bar{\epsilon}'}{2}+\frac{p\alpha_3}{2}+\frac{g\alpha_4}{2}\right)\right] \\ &\quad +\frac{g}{2}\left(\frac{1}{\alpha_4}+g\right)\int_0^1\frac{w_x^TL^TQLw_x+w_{xx}^TL^TQLw_{xx}}{2}dx \\ &\quad +\frac{\bar{q}}{2}\left((1+\bar{c})+\bar{\epsilon}\right)\left(|\dot{b}(t)|+|b(t)|\right)\sqrt{\frac{2}{\underline{q}}(V_2+V_3)}, \quad (56)\end{aligned}$$

Choose now $\alpha_2 = \frac{\epsilon}{3\bar{\epsilon}'}, \alpha_3 = \frac{\epsilon}{3\bar{p}}, \alpha_4 = \frac{\epsilon}{3g}$ so that $\left(\frac{\alpha_2\bar{\epsilon}'}{2}+\frac{p\alpha_3}{2}+\frac{g\alpha_4}{2}\right) < \underline{\epsilon}/2$. Call $K_5 = 2p+1$, $K_6 = \left(2p+\frac{\epsilon}{4}+\frac{3}{\epsilon}\left(\bar{\epsilon}'^2+p^2\right)\right) - 2\underline{\epsilon}$, $K_7 = \frac{\bar{q}^2}{2\underline{\epsilon}\underline{q}}\left((1+\bar{c})+\bar{\epsilon}\right)^2$, $K_8 = \frac{g^2}{2}\left(\frac{1}{3\underline{\epsilon}}+1\right)$. Then:

$$\begin{aligned}\dot{V} &\leq -V_1\left[2\underline{c}-K_5\right]-V_2\left[2\underline{c}-K_6\right]-\frac{\epsilon}{4}V_3 \\ &\quad +K_7\left(|\dot{b}(t)|^2+|b(t)|^2\right) \\ &\quad +K_8\int_0^1\frac{w_x^TL^TQLw_x+w_{xx}^TL^TQLw_{xx}}{2}dx, \quad (57)\end{aligned}$$

and defining $c^* = \frac{1}{2}\max\{K_5, K_6+\frac{\epsilon}{4}\}$ (which only depends on the bounds of $\Sigma(x)$ and $\Phi(x)$), we get that if $\underline{c} \geq c^*+\delta$, we obtain

$$\begin{aligned}\dot{V} &\leq -2\delta V + K_7\left(|\dot{b}(t)|^2+|b(t)|^2\right) \\ &\quad -\int_0^1\frac{w_x^TRw_x+w_{xx}^TRw_{xx}}{2}dx, \quad (58)\end{aligned}$$

where $R = \frac{\epsilon}{4}Q - K_8L^TQL$ and D_2 can be set as large as desired. Assume for the moment that R is definite positive. Then, applying Gronwall's inequality, we obtain

$$\begin{aligned}V &\leq V(0)\quad \mathrm{e}^{-2\delta t}+K_7\int_0^t \mathrm{e}^{-2\delta(t-\tau)}\left(|\dot{b}(\tau)|^2+|b(\tau)|^2\right)d\tau \\ &\leq \quad V(0)\mathrm{e}^{-2\delta t}+K_7\left(\|b\|_{L^2}+\|\dot{b}\|_{L^2}\right), \quad (59)\end{aligned}$$

and then the proposition is proved. It only remains to prove that R can be made a positive definite matrix by adequately choosing the coefficients of Q. To see if this is possible, let us check what is L^TQL. First notice that $(QL)_{ij} = q_i$ if $j < i$ and zero otherwise. Then

$$(L^TQL)_{ij} = \sum_{l=1}^n L_{li}(QL)_{lj} = \sum_{l=i+1}^n (QL)_{lj} = \sum_{\max\{i,j\}+1}^n q_l, \quad (60)$$

where the sum is considered to be zero if $i = n$ and/or $j = n$. Let us now prove by induction on the dimension n that $R(Q) = \frac{\epsilon}{4}Q - K_8L^TQL$ can always be made positive definite. Call Q_n the matrix that we will find for each dimension, and $M_n = L_n^TQ_nL_n$. For $n = 1$, since $Q_1 = q_1 > 0$ and $M_1 = 0$, the result is obvious and q_1 can be chosen arbitrarily. For $n > 1$, we can construct both Q_n and M_n from the previous Q_{n-1} and M_{n-1} as follows

$$Q_n = \begin{bmatrix} Q_{n-1} & 0 \\ 0 & q_n \end{bmatrix}, \quad M_n = \begin{bmatrix} M_{n-1}+q_n\mathbf{J}_{n-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad (61)$$

where $\mathbf{J}_{n-1}$ is a square matrix of dimension $n-1$ full of ones. Assume now that $R(Q_{n-1})$ is positive definite. In particular this means that all the eigenvalues of $R(Q_{n-1})$ are positive, and since R is symmetric, they are also real. Call $\lambda_{\min}$ the smallest eigenvalue of a square matrix. Denote $\mu_{n-1} = \lambda_{\min}(R(Q_{n-1})) > 0$. Choosing $q_n = \frac{\mu_{n-1}}{2K_8(n-1)}$, we obtain

$$R(Q_n) = \begin{bmatrix} R(Q_{n-1}) - \frac{\mu_{n-1}}{2(n-1)}\mathbf{J}_{n-1} & 0 \\ 0 & \frac{\epsilon\mu_{n-1}}{8K_8(n-1)} \end{bmatrix}, \quad (62)$$

and computing the eigenvalues of $R(Q_n)$, we obtain one eigenvalue equal to $\frac{\epsilon\mu_{n-1}}{8K_8(n-1)} > 0$, plus the eigenvalues of $R(Q_{n-1}) - \frac{\mu_{n-1}}{2(n-1)}\mathbf{J}_{n-1}$. Now, from Weyl's inequality [10, p. 239] we then have that

$$\begin{aligned}&\lambda_{\min}\left(R(Q_{n-1})-\frac{\mu_{n-1}}{2(n-1)}\mathbf{J}_{n-1}\right) \\ \leq\ &\lambda_{\min}\left(R(Q_{n-1})\right)-\lambda_{\min}\left(\frac{\mu_{n-1}}{2(n-1)}\mathbf{J}_{n-1}\right) \\ =\ &\frac{\mu_{n-1}}{2} > 0, \quad (63)\end{aligned}$$

where we have used that the eigenvalues of $\mathbf{J}_{n-1}$ are 0 (repeated $n-2$ times) and $n-1$ [10, p. 65]. Therefore the newly formed $R(Q_n) > 0$, and the proposition is proved. □

D. Proof of Theorem 1

Assume for the moment that Theorem 2 holds and there is a solution to the kernel equations such that the transformation (24) is invertible and both the transformation and its inverse map H^1 functions into H^1 functions. Consider now the target system equation (19) with boundary conditions (21–22) and initial conditions $w_0(x)$ given by applying the backstepping transformation (24) to the initial conditions of u, $u_0(x)$, i.e.,

$$w_0(x) = u_0(x) - \int_0^x K(x,\xi)u_0(\xi)d\xi. \quad (64)$$

Then, since $u_0 \in H^2$, we have w_0 in H^2. In addition, given the definition of $b(t)$ (see Equation 16), we have $w_0(1) = b(0)$, and $b, \dot{b} \in L^2([0,\infty)]$. Thus the conditions for well-posedness of Assumption 1 are fulfilled and we obtain well-posedness for u in H^1 given the properties of the transformation. This shows that the assumption of closed-loop well-posedness in Theorem 1 is equivalent to Assumption 1. In addition it is obvious that $\|b\|_{L^2} + \|\dot{b}\|_{L^2} \leq D_4\mathrm{e}^{-\alpha_1 t}\|u_0\|_{H^1}$. Using Proposition 1 we then obtain, if $c_i \geq c^*+\delta$,

$$\begin{aligned}\|u(\cdot,t)\|_{H^1} &\leq K_2\|w(\cdot,t)\|_{H^1} \\ &\leq K_2\left(D_1\mathrm{e}^{-2\delta t}\|w_0\|_{H^1}+D_3\left(\|b\|_{L^2}+\|\dot{b}\|_{L^2}\right)\right) \\ &\leq K_2(K_1D_1+D_4)\mathrm{e}^{-D_5 t}\|u_0\|_{H^1}, \quad (65)\end{aligned}$$

where $D_5 = \min\{\alpha_1, 2\delta\}$. Thus Theorem 1 is proved. □

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IV. Well-posedness of the kernel equations (proof of Theorem 2)

To prove Theorem 2, we are going to write the kernel equations (12)–(15) in a different form. Then, we can use Theorem A.1 of [12].

Define first

$$\begin{aligned} L(x,\xi) &= \sqrt{\Sigma}(x)K_x(x,\xi)+K_\xi(x,\xi)\sqrt{\Sigma}(\xi) \\ &\quad +F_1(x,\xi)K(x,\xi)+K(x,\xi)F_2(x,\xi) \quad (66) \end{aligned}$$

where the functions F_1 and F_2 are to be found.

Now, we compute $\sqrt{\Sigma}(x)L_x - L_\xi\sqrt{\Sigma}(\xi)$ using (66). It is worth noticing that the cross-derivatives of K cancel out and the differential operator of (12) appear. Replacing its value from (12), we obtain

$$\begin{aligned} &\sqrt{\Sigma}(x)L_x - L_\xi\sqrt{\Sigma}(\xi) \\ =\ & K(\Lambda(\xi)-\Phi'(\xi)-F_{2\xi}\sqrt{\Sigma}(\xi)-F_2^2) \\ &+(C+\sqrt{\Sigma}(x)F_{1x}+F_1^2)K \\ &-\left(\frac{\Sigma'(x)}{2}+\Phi(x)-\sqrt{\Sigma}(x)F_1-F_1\sqrt{\Sigma}(x)\right)K_x \\ &+K_\xi\left(\frac{\Sigma'(\xi)}{2}-\Phi(\xi)-F_2\sqrt{\Sigma}(\xi)-\sqrt{\Sigma}(\xi)F_2\right) \\ &+\sqrt{\Sigma}(x)KF_{2x}-F_{1\xi}K\sqrt{\Sigma}(\xi) \\ &-F_1L+LF_2 \quad (67) \end{aligned}$$

Now, F_1 and F_2 are chosen so that the second and third lines of (67) cancel out. This is always possible [13], by defining

$$(F_1)_{ij} = \frac{\delta_{ij}\frac{\epsilon_i'(x)}{2}+\phi_{ij}(x)}{\sqrt{\epsilon_i}(x)+\sqrt{\epsilon_j}(x)}, \quad (68)$$

$$(F_2)_{ij} = \frac{\delta_{ij}\frac{\epsilon_i'(\xi)}{2}-\phi_{ij}(\xi)}{\sqrt{\epsilon_i}(\xi)+\sqrt{\epsilon_j}(\xi)}, \quad (69)$$

and noticing that F_1 only depends on x and F_2 only depends on ξ, the fourth line of (67) is also zero. Thus our original $n\times n$ system (12) is replaced by a $n^2\times n^2$ system of first-order hyperbolic equation on the same domain $\mathcal{T}$, namely

$$\begin{aligned} \sqrt{\Sigma}(x)K_x+K_\xi\sqrt{\Sigma}(\xi) &= L-F_1(x)K-KF_2(\xi), \quad (70) \\ \sqrt{\Sigma}(x)L_x-L_\xi\sqrt{\Sigma}(\xi) &= KF_3(\xi)+F_4(x)K \\ &\quad -F_1(x)L+LF_2(\xi), \quad (71) \end{aligned}$$

where $F_3(\xi)=\Lambda(\xi)-\Phi'(\xi)-F_{2\xi}\sqrt{\Sigma}(\xi)-F_2^2$, $F_4(x)=C+\sqrt{\Sigma}(x)F_{1x}+F_1^2$, which are virtually identical to the kernel equations appearing in [12] and [11] (there are some differences in the right-hand side coefficients, but they do not affect the proofs). It remains to be seen if the boundary conditions are the same.

To find the boundary conditions for L, we need to analyze separate cases depending on the position of each coefficient L_{ij} and K_{ij} in the kernel matrices L and K. First, (35), namely $K(x,x)\Sigma(x)=\Sigma(x)K(x,x)$ can be written as $K_{ij}(x,x)(\epsilon_i(x)-\epsilon_j(x))=0$. This condition is automatically verified if $i=j$, otherwise $K_{ij}(x,x)=0$. This allows us to write (34) as

$$\begin{aligned} 0 &= \phi_{ij}(x)K_{jj}(x,x)-\phi_{ij}(x)K_{ii}(x,x)+\lambda_{ij}(x)+\delta_{ij}c_i \\ &\quad +K_{ij\xi}(x,x)\epsilon_j(x)+\epsilon_i(x)K_{ijx}(x,x) \\ &\quad +\frac{d}{dx}\left(\epsilon_i(x)K_{ij}(x,x)\right), \quad (72) \end{aligned}$$

and similarly, we can solve for $L_{ij}(x,x)$ in (70), finding

$$\begin{aligned} L_{ij}(x,x) &= \sqrt{\epsilon_i}(x)K_{ijx}(x,x)+\sqrt{\epsilon_j}(\xi)K_{ij\xi}(x,x) \\ &\quad +F_{ij1}(x)K_{jj}(x,x)+F_{ij2}(x)K_{ii}(x,x), \quad (73) \end{aligned}$$

If $i=j$, then (72) reduces to

$$\begin{aligned} 0 &= \lambda_{ii}(x)+c_i \\ &\quad +\epsilon_i'(x)K_{ii}(x,x)+2\epsilon_i(x)\frac{d}{dx}\left(K_{ii}(x,x)\right), \quad (74) \end{aligned}$$

which integrates (combined with (15)) to

$$K_{ii} = \frac{-1}{\sqrt{\epsilon(x)}}\int_0^x \frac{\lambda_{ii}(\xi)+c_i}{2\sqrt{\epsilon(\xi)}}d\xi \quad (75)$$

In addition, (73) reduces to

$$\begin{aligned} L_{ii}(x,x) &= \sqrt{\epsilon_i}(x)\frac{d}{dx}K_{ii}(x,x) \\ &\quad +(F_{ii1}(x)+F_{ii2}(x))K_{ii}(x,x) \\ &= \sqrt{\epsilon_i}(x)\frac{d}{dx}K_{ii}(x,x) \\ &\quad +\frac{\epsilon_i'(x)}{2\sqrt{\epsilon_i}(x)}K_{ii}(x,x) \\ &= -\frac{\lambda_{ii}(x)+c_i}{2\sqrt{\epsilon_i}(x)}, \quad (76) \end{aligned}$$

If $i\neq j$, then since $K_{ij}(x,x)=0$, we get $K_{ijx}(x,x)=-K_{ij\xi}(x,x)$. Therefore we obtain, from (72),

$$\begin{aligned} 0 &= \lambda_{ij}(x)+\phi_{ij}(x)\left(K_{jj}(x,x)-K_{ii}(x,x)\right) \\ &\quad +K_{ijx}(x,x)(\epsilon_i(x)-\epsilon_j(x)) \quad (77) \end{aligned}$$

and from (73),

$$\begin{aligned} L_{ij}(x,x) &= K_{ijx}(x,x)(\sqrt{\epsilon_i(x)}-\sqrt{\epsilon_j(x)}) \\ &\quad +F_{ij1}(x)K_{jj}(x,x)+F_{ij2}(x)K_{ii}, \quad (78) \end{aligned}$$

which combined gives us

$$\begin{aligned} L_{ij}(x,x) &= \frac{K_{ijx}(x,x)(\epsilon_i(x)-\epsilon_j(x))}{\sqrt{\epsilon_i(x)}+\sqrt{\epsilon_j(x)}} \\ &\quad +F_{ij1}(x)K_{jj}(x,x)+F_{ij2}(x)K_{ii} \\ &= -\frac{\lambda_{ij}+\phi_{ij}(x)\left(K_{jj}(x,x)-K_{ii}(x,x)\right)}{\sqrt{\epsilon_i(x)}+\sqrt{\epsilon_j(x)}} \\ &\quad +F_{ij1}(x)K_{jj}(x,x)+F_{ij2}(x)K_{ii} \\ &= -\frac{\lambda_{ij}}{\sqrt{\epsilon_i(x)}+\sqrt{\epsilon_j(x)}}, \quad (79) \end{aligned}$$

when introducing the definitions of F_1 and F_2. Thus we are finally led to the following combination of boundary conditions

- If $i = j$, then simply

$$L_{ii}(x,x) = -\frac{\lambda_{ii}(x)+c_i}{2\sqrt{\epsilon_i(x)}}, \quad (80)$$

$$K_{ii}(x,0) = 0, \quad (81)$$

- If $i < j$ then

$$K_{ij}(x,x) = K_{ij}(x,0) = 0, \quad (82)$$

$$L_{ij}(x,x) = -\frac{\lambda_{ij}(x)}{\sqrt{\epsilon_i(x)}+\sqrt{\epsilon_j(x)}}, \quad (83)$$

$$(84)$$

- Finally if $i > j$ and $\epsilon_i \neq \epsilon_j$ then

$$K_{ij}(x,x) = 0, \quad (85)$$

$$K_{ij}(1,\xi) = l_{ij}(\xi), \quad (86)$$

$$L_{ij}(x,x) = -\frac{\lambda_{ij}(x)}{\sqrt{\epsilon_i(x)}+\sqrt{\epsilon_j(x)}}, \quad (87)$$

and the additional condition $g_{ij}(x) = -K_{ij}(x,0)\epsilon_j(0)$.

It must be noticed that (86) are additional arbitrary conditions that are introduced for the kernel equations to be well-posed. These functions $l_{ij}(\xi)$ cannot be arbitrary, but need to verify certain compatibility conditions in the corner $\xi = 1$ for the kernels to be piecewise differentiable (see [12] for details).

Comparing these boundary conditions with those verified by the kernels in [12] and [11], we can see that they are exactly the same (it must be noted than in the second of these papers the nomenclature for K and L is the opposite). Thus the results in these papers apply, and we obtain a piecewise differentiable and invertible kernel which can be readily verified to transform H^1 functions into H^1 functions. □

V. Conclusion

This paper presents an extension of the backstepping method to coupled parabolic systems with advection terms and spatially-varying coefficients. The result is more general than a recently published extension that only considered constant-coefficient coupled reaction-diffusion systems.

Interestingly, the basis of the result is finding an equivalence between the kernel equations for this case and the kernel equations for general hyperbolic 1-D coupled systems, which have recently been established to be well-posed and piecewise differentiable. Thus, this paper unveils a direct connection between backstepping controllers for parabolic and hyperbolic systems. In [11] the resulting kernels are shown to have lines of discontinuity, thus an additional challenge is to pose stable numerical schemes that allow computing approximation of the kernels to use in practice.

Future work includes considering Neumann or Robin boundary conditions, which leads to slightly different kernel equations, and observer design, which will allow to consider output-feedback controllers, as well as a complete proof of Assumption 1.

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References

[1] H. Anfinsen and O.M. Aamo, "Disturbance Rejection in the Interior Domain of Linear 2 2 Hyperbolic Systems," *IEEE Transactions on Automatic Control*, vol. 60, pp.186–191, 2015.

[2] A. Baccoli, A. Pisano, Y. Orlov, "Boundary control of coupled reaction-diffusion processes with constant parameters," *Automatica*, vol. 54, pp. 80–90, 2015.

[3] V. Barbu, "Boundary Stabilization of Equilibrium Solutions to Parabolic Equations," *IEEE Transactions on Automatic Control*, vol. 58, pp. 2416–2420, 2013.

[4] H. Brezis, *Functional analysis, Sobolev spaces and Partial Differential Equations*, Springer, 2011.

[5] E. Cerpa and J.-M. Coron, "Rapid stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition," *IEEE Trans. Automat. Control*, Vol. 58, pp. 1688–1695, 2013.

[6] J.-M. Coron, R. Vazquez, M. Krstic, and G. Bastin, "Local Exponential H^2 Stabilization of a 2×2 Quasilinear Hyperbolic System using Backstepping," *SIAM J. Control Optim.*, vol. 51, pp. 2005–2035, 2013.

[7] F. Di Meglio, R. Vazquez, and M. Krstic, "Stabilization of a system of n+1 coupled first-order hyperbolic linear PDEs with a single boundary input," *IEEE Transactions on Automatic Control*, PP, 2013.

[8] L.C. Evans, *Partial Differential Equations*, AMS, Providence, Rhode Island, 1998.

[9] A. Hasan, "Disturbance Attenuation of $n{+}1$ Coupled Hyperbolic PDEs," *Systems & Control Letters*, *2014 IEEE Conference on Decision and Control*, 2014.

[10] R.A. Horn and C.R. Johnson, *Matrix Analysis*, 2nd Edition, Cambridge University Press, New York, 2013.

[11] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstic,"Control of Homodirectional and General Heterodirectional Linear Coupled Hyperbolic PDEs," Preprint, available at http://arxiv.org/abs/1504.07491,2015.

[12] L. Hu, R. Vazquez, F. Di Meglio, and M. Krstic,"Boundary exponential stabilization of 1-D inhomogeneous quasilinear hyperbolic systems," Preprint, available at http://arxiv.org/abs/1512.03539,2015.

[13] A. Jameson, *SIAM J. Appl. Math.*, "Solution of equation $AX{+}XB = C$ by inversion of an $M \times M$ or $N \times N$ matrix," Vol. 16, No. 5, pp. 1020–1023, 1968.

[14] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs*, SIAM, 2008

[15] M. Krstic, *Delay Compensation for nonlinear, Adaptive, and PDE Systems*, Birkhauser, 2009.

[16] M. Krstic and A. Smyshlyaev, "Backstepping boundary control for first order hyperbolic PDEs and application to systems with actuator and sensor delays," *Syst. Contr. Lett.*, vol. 57, pp. 750–758, 2008.

[17] W. Liu, *Elementary feedback stabilization of the linear reaction-convection-diffusion equation and the wave equation*, Springer, 2009.

[18] T. Meurer and A. Kugi, "Tracking control for boundary controlled parabolic PDEs with varying parameters: Combining backstepping and differential flatness," *Automatica*, vol. 45, pp. 1182–1194, 2009.

[19] S.J. Moura and H.K. Fathy, "Optimal Boundary Control of Reaction-Diffusion Partial Differential Equations via Weak Variations," *Journal of Dynamic Systems, Measurement, and Control*, vol 135, 034501, 2013.

[20] J. Qi, R. Vazquez and M. Krstic, "Multi-Agent Deployment in 3-D via PDE Control," *IEEE Transactions on Automatic Control*, in Press, 2015.

[21] A. Smyshlyaev, E. Cerpa, and M. Krstic, "Boundary stabilization of a 1-D wave equation with in-domain antidamping," *SIAM J. Control Optim.*, vol. 48, pp. 4014–4031, 2010.

[22] A. Smyshlyaev and M. Krstic, *Adaptive Control of Parabolic PDEs*, Princeton University Press, 2010.

[23] R. Vazquez and M. Krstic, *Control of Turbulent and Magnetohydrodynamic Channel Flow*. Birkhauser, 2008.

[24] R. Vazquez and M. Krstic, "Control of 1-D parabolic PDEs with Volterra nonlinearities — Part I: Design," *Automatica*, vol. 44, pp. 2778–2790, 2008.

[25] R. Vazquez and M. Krstic, "Boundary observer for output-feedback stabilization of thermal convection loop," *IEEE Trans. Control Syst. Technol.*, vol.18, pp. 789–797, 2010.

[26] R. Vazquez and M. Krstic. Marcum Q-functions and explicit kernels for stabilization of linear hyperbolic systems with constant coefficients. *Systems & Control Letters* , 68:33–42, 2014

[27] R. Vazquez and M. Krstic, "Boundary Control of Reaction-Diffusion PDEs on Balls in Spaces of Arbitrary Dimensions," *Preprint*, available online at http://arxiv.org/abs/1511.06641, 2015.

[28] R. Vazquez, E. Trelat and J.-M. Coron, "Control for fast and stable laminar-to-high-Reynolds-numbers transfer in a 2D navier-Stokes channel flow," *Disc. Cont. Dyn. Syst. Ser. B*, vol. 10, pp. 925–956, 2008.