International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: http://www.tandfonline.com/loi/tcon20

Boundary control and estimation of reaction–diffusion equations on the sphere under revolution symmetry conditions

Rafael Vazquez & Miroslav Krstic

INTERNATIONAL JOURNAL OF CONTROL, 2017
http://dx.doi.org/10.1080/00207179.2017.1286691

Taylor & Francis
Taylor & Francis Group

Boundary control and estimation of reaction–diffusion equations on the sphere under revolution symmetry conditions

Rafael Vazquez[a] and Miroslav Krstic[b]

[a]Department of Aerospace Engineering, Universidad de Sevilla, Camino de los Descubrimiento s.n., Sevilla, Spain; [b]Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA, USA

ABSTRACT
Recently, the problem of designing boundary controllers and observers for unstable linear constant-coefficient reaction–diffusion equation on N-balls has been solved by means of the backstepping method. However, the extension of these results to spatially varying coefficients is far from trivial. This work deals with radially varying reaction coefficients under revolution symmetry conditions on a sphere (the three-dimensional case). Under these conditions, the equations become singular in the radius. When applying the backstepping method, a similar type of singularity appears in the backstepping control and observer kernel equations. However, with a simple scaling transformation, we are able to reduce the singular equation to a regular equation, which turns out to be the same kernel equations appearing when using the one-dimensional backstepping method. In addition, the scaling transformation allows us to prove stability in the H^1 space.

ARTICLE HISTORY
Received 8 February 2016
Accepted 22 January 2017

KEYWORDS
Distributed parameter systems; infinite-dimensional backstepping; boundary control; boundary observer; reaction–diffusion system

1. Introduction

In a series of previous results, the problem of designing boundary controllers and observers for unstable linear constant-coefficient reaction–diffusion equation on n-balls has been solved. In particular, Vazquez and Krstic (2014b) and Vazquez and Krstic (2015) describe, respectively, the (full-state) control design for the particular case of a two-dimensional (2D) disk and a general n-ball; that same design, augmented with an observer, is applied (following the ideas of Meurer & Krstic, 2011) in Qi, Vazquez, and Krstic (2015) to multi-agent deployment in three-dimensional (3D) space, with the agents distributed on a disk-shaped grid and commanded by leader agents located at the boundary. The output-feedback generalisation to n-balls is presented in Vazquez and Krstic (2016a). Older, related results that use backstepping include the design an output feedback law for a convection problem on an annular domain (see Li & Xie, 2010; Vazquez & Krstic, 2010), or observer and controller designs on cuboid domains (see, respectively, Jadachowski, Meurer, & Kugi, 2015; Meurer, 2013).

This work, together with a similar result on a disk presented in Vazquez and Krstic (2016b), can be seen as a first step towards extending this family of previous results to the non-constant coefficient case, by assuming a certain symmetry for the initial conditions, which simplifies the problem. There have been specific results on disk- or spherical-shaped domains (see e.g. Bribiesca Argomedo, Prieur Witrant, & Bremond, 2013; Moura, Chaturvedi, & Krstic, 2012), which have assumed these same symmetry conditions, which are of interest in engineering applications; for instance, the cited works consider diverse applications such as fusion reactors and batteries. The simmetry condition is equivalent to only considering the 0-th order harmonic (i.e. the mean) in the general design presented in Vazquez and Krstic (2015).

Based on the domain shape, we use spherical coordinates, and based on the symmetry of the initial conditions and imposing an equally symmetric controller (and observer), the system can be written as a single one-dimensional (1D) system with singular terms. We design a feedback law and an observer for this system using the backstepping method (see Krstic & Smyshlyaev, 2008a, for the basis of the method and several applications). The backstepping method has proved itself to be an ubiquitous method for control and estimation of partial differential equations (PDEs), with many other applications including, among others, flow control (see for instance Vazquez & Krstic, 2008a; Vazquez, Trelat, & Coron, 2008), parabolic systems (Baccoli, Pisano, & Orlov, 2015), nonlinear PDEs (Vazquez & Krstic, 2008b), hyperbolic 1D systems (see e.g. Coron, Vazquez, Krstic, & Bastin, 2013; Di Meglio, Vazquez, & Krstic, 2013; Krstic & Smyshlyaev, 2008b; Vazquez & Krstic, 2014a), adaptive control (Smyshlyaev & Krstic, 2010), wave equations

CONTACT Rafael Vazquez rvazquez1@us.es

(Smyshlyaev, Cerpa, & Krstic, 2010), tracking (Meurer & Kugi, 2009), and delays (Krstic, 2009). The main idea of backstepping is finding an invertible transformation that maps the system into a stable *target* system which needs to be chosen judiciously. To find the transformation, a hyperbolic partial differential equation (called the *kernel equation*) needs to be solved. Typically, the well-posedness of the kernel equation is studied by transforming it into an integral equation and then applying successive approximations to construct a solution. The convergence of the successive approximation series guarantees that a solution always exists, it is unique, and it is bounded. In both the observer and controller problems posed in this paper, one obtains a *singular* kernel equation. However, there is a simple scaling transformation that allows us to reduce the kernel equation to the usual PDE found in the application of backstepping to 1D parabolic systems, which is known to be well-posed. The same transformation can be shown to be regular enough, so that H^1 stability can be easily shown.

Interestingly, the result presented in this paper for the sphere contrasts with the more complicated 2D case (disk-shaped domain), outlined in Vazquez and Krstic (2016b), where the singularity cannot be eliminated and a special method of proof is required (based on a combinatorial sequence of integers, the Catalan numbers).

The structure of the paper is as follows. In Section 2, we introduce the problem and state our main result. We explain our design method and find the control and observer kernel equations in Section 3, showing its well-posedness. Next, we show closed-loop stability and well-posedness in Section 4. We conclude the paper with some remarks in Section 5.

2. Reaction–diffusion system on a ball under revolution symmetry conditions

Consider the following spatially varying coefficient reaction–diffusion system

$$u_t = \epsilon \triangle u + \lambda(\vec{x})u = \epsilon \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) + \lambda(\vec{x})u, \tag{1}$$

with $\varepsilon > 0$, with $\vec{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, where $u = u(t, \vec{x})$ is the state variable, evolving for $t > 0$ on a ball domain of radius R, denoted as $B(R) = \{\vec{x} : \|\vec{x}\| < R\}$. In addition, the coefficient λ is assumed to be differentiable in $B(R)$.

The boundary conditions of (1) are over the sphere of radius R, denoted as $S(R) = \{\vec{x} : \|\vec{x}\| = R\}$,

$$u(t, \vec{x})\Big|_{S(R)} = U(t, \vec{x}), \tag{2}$$

where $U(t, \vec{x})$ is the actuation. The measurement $Y(t, \vec{x})$ is the normal (outer) derivative at the spherical boundary, i.e.

$$Y(t, x_1, x_2, x_3) = \vec{\nabla} u(t, x_1, x_2, x_3) \cdot \vec{n}|_{S(R)}, \tag{3}$$

where $\vec{n}$ is the normal vector at the spherical boundary, $\vec{n} = \frac{\vec{x}}{\|\vec{x}\|}$. Note that a weighted spatial average of the state could also be considered (as shown in Tsubakino & Hara, 2015).

When $U = 0$ and λ has large positive values, (1)–(2) becomes unstable. Our objective is to find an *output-feedback* control law for $U(t)$ using the measurement $Y(t)$ so that the origin of the system (1)–(2) becomes stable in some appropriate norm.

Denote simply by L^2 (resp. H^1) the space of square-integrable functions (resp. of functions with square-integrable gradient defined in the weak sense, see e.g. Brezis, 2011) over $B(R)$. Finally, denote by H_0^1 the space of H^1 functions vanishing at the boundary $S(R)$ in the usual sense of traces, see e.g. Evans (1998, p.259). Then for $f \in L^2$ and $f \in H^1$, its respective norms are defined as

$$\|f\|_{L^2}^2 = \iiint_{B(R)} f^2(\vec{x}) d\vec{x}, \tag{4}$$

$$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \iiint_{B(R)} \|\vec{\nabla} f(\vec{x})\|^2 d\vec{x}, \tag{5}$$

where the volume integrals are extended to the ball of radius R.

The problem posed here is solved in Vazquez and Krstic (2016a) for general dimension but for constant coefficient λ, obtaining in particular the following result for the sphere

Theorem 2.1 (Vazquez & Krstic, 2016a)**:** *Consider* (1)–(2) *with constant $\lambda > 0$, with initial conditions $u_0(\vec{x})$, and the the following* (*explicit*) *feedback law for U:*

$$U(t, \vec{x}) = -\frac{1}{4\pi}\sqrt{\frac{\lambda}{\epsilon}} \iiint_{B(R)} \mathrm{I}_1 \left[\sqrt{\frac{\lambda}{\epsilon}(R^2 - \|\vec{\xi}\|^2)} \right] \times \frac{\sqrt{R^2 - \|\vec{\xi}\|^2}}{\|\vec{x} - \vec{\xi}\|^3} \hat{u}(t, \vec{\xi}) d\vec{\xi}, \tag{6}$$

where $\hat{u}$ is the state of an observer (*which approximates u*)*, computed as the solution of the following PDE*

$$\hat{u}_t = \epsilon \triangle \hat{u} + \lambda \hat{u} - \frac{\sqrt{\lambda\epsilon}}{4\pi} \mathrm{I}_1 \left[\sqrt{\frac{\lambda}{\epsilon}(R^2 - \|\vec{x}\|^2)} \right] \times \sqrt{R^2 - \|\vec{x}\|^2} \iint_{S(R)} \frac{Y(t, \vec{\xi}) - \hat{u}_r(t, \vec{\xi})}{\|\vec{x} - \vec{\xi}\|^3} d\vec{\xi}, \tag{7}$$

where the surface integral is extended to the sphere of radius R, with boundary conditions

$$\hat{u}(t,\vec{x})\Big|_{\vec{x}\in S(R)} = U(t,\vec{x}). \tag{8}$$

and initial conditions $\hat{u}_0$. Assume in addition that $u_0 \in H_0^1$ and $\hat{u}_0 \equiv 0$. Then the augmented system $(u, \hat{u})$ has a unique H^1 solution, and the equilibrium profile $u, \hat{u} \equiv 0$ is exponentially stable in the H^1 norm, i.e. there exists c_1, $c_2 > 0$ such that

$$\|u(t,\cdot)\|_{H^1} + \|\hat{u}(t,\cdot)\|_{H^1} \le c_1 e^{-c_2 t}\|u_0\|_{H^1}. \tag{9}$$

Extending this result to the case of spatially varying coefficients is challenging. To be able to solve the problem at least in a particular case, we state the main assumption of the paper.

Assumption 2.1 (Revolution symmetry conditions): *Consider that the initial conditions $u_0(\vec{x})$ only depend on the distance to the origin (the radius, $r = \|\vec{x}\|$). Similarly, $\lambda(x_1, x_2, x_3)$ is assumed to depend only on the radius, i.e.*

$$\lambda(x_1,x_2,x_2) = \lambda(\|\vec{x}\|). \tag{10}$$

In addition, we will design a feedback control law that is uniform on the surface of the sphere, i.e. $U(t, x_1, x_2, x_3) = U(t)$.

The revolution symmetry condition is frequently used in engineering models. For instance, in Bribiesca et al. (2013), it is used to model the evolution of the magnetic flux profile in a tokamak. Another example is the work by Moura et al. (2012), where it is used to model an electrochemical cell. Note that, under the framework established in Vazquez and Krstic (2016a), this assumption is in fact equivalent to only considering $l = 0$ in the spherical harmonics expansion (i.e. only the mean value over the ball).

To make Assumption 2.1 explicit, Equation (1) can be written in spherical coordinates (r, θ_1, θ_2), where $r \in [0, R]$ is the radial coordinate, and the angular coordinates are $\theta_1 \in [0, \pi]$ and $\theta_2 \in [0, 2\pi]$, so that

$$x_1 = r\cos\theta_2\sin\theta_1, \tag{11}$$
$$x_2 = r\sin\theta_2\sin\theta_1, \tag{12}$$
$$x_3 = r\cos\theta_1. \tag{13}$$

In these coordinates:

$$u_t = \epsilon\left(u_{rr} + 2\frac{u_r}{r}\right) + \frac{\epsilon}{r^2} \times \left(\frac{\partial^2 u}{\partial\theta_1^2} + \frac{1}{\tan\theta_1}\frac{\partial u}{\partial\theta_1} + \frac{1}{\sin^2\theta_1}\frac{\partial^2 u}{\partial\theta_2^2}\right) + \lambda(r)u, \tag{14}$$

with boundary conditions

$$u(t,R,\theta_1,\theta_2) = U(t,\theta_1,\theta_2). \tag{15}$$

Using Assumption 2.1, the state u only depends on time and radius, thus the system dynamics is described by the following 1D equation

$$u_t = \epsilon\left(u_{rr} + 2\frac{u_r}{r}\right) + \lambda(r)u, \tag{16}$$

for $r \in [0, R)$, with boundary conditions

$$u(t,R) = U(t), \quad u_r(t,0) = 0, \tag{17}$$

where the second boundary condition takes into account that symmetry imposes a zero-flux condition at the centre of the sphere, and is necessary for the equation to be well-posed.

The measurement Y becomes then a single value (independent of where it is measured in the boundary) and can be written as

$$Y(t) = \frac{\partial u}{\partial r}(t,R). \tag{18}$$

Notice that under Assumption 2.1, for $f \in L^2$ (resp. $f \in H^1$) with revolution symmetry conditions (i.e. $f = f(r)$, so $\nabla f = f_r\vec{r}$), the L^2 norm (resp. H^1 norm) can be redefined (to avoid a 4π factor in many expressions) as the following equivalent norm

$$\|f\|_{L^2}^2 = \int_0^R f^2(r)r^2 dr, \tag{19}$$
$$\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \int_0^R f_r^2(r)r^2 dr. \tag{20}$$

Using the description in radial coordinates, the main result of this paper extends Theorem 2.1 to spatially vaying $\lambda(r)$ in spherical domains under Assumption 2.1 as follows.

Theorem 2.2: *Consider (16)–(17), with initial conditions $u_0(r)$, and the the following feedback law for U:*

$$U(t) = \int_0^R k(\rho)\hat{u}(t,\rho)d\rho, \tag{21}$$

where $\hat{u}(t,r)$ is the state of an observer (which approximates u), computed as the solution of the following PDE

$$\hat{u}_t = \epsilon\left(\hat{u}_{rr} + 2\frac{\hat{u}_r}{r}\right) + \lambda(r)u + p(r)\left[Y(t,r) - \hat{u}_r(t,R)\right], \tag{22}$$

with boundary conditions

$$\hat{u}(t,R) = U(t),\ \hat{u}_r(t,0) = 0, \tag{23}$$

and initial conditions $\hat{u}_0$*. The controller and observer gains, respectively,* $k(\rho)$ *in* (21) *and* $p(r)$ *in* (22)*, are computed from the unique solution* $K(r,\rho)$ *to the kernel equation*

$$K_{rr} + 2\frac{K_r}{r} - K_{\rho\rho} + 2\frac{K_\rho}{\rho} - 2\frac{K}{\rho^2} = \frac{\lambda(\rho)+c}{\epsilon}K \tag{24}$$

for $c > 0$*, with boundary conditions*

$$K(r,0) = 0, \tag{25}$$

$$K_\rho(r,0) = 0, \tag{26}$$

$$K(r,r) = -\int_0^r \frac{c+\lambda(\rho)}{2\epsilon}d\rho, \tag{27}$$

in the domain $\mathcal{T} = \{(r,\rho) : 0 \le \rho \le r \le R\}$*, and then setting*

$$k(\rho) = K(R,\rho), \quad p(r) = \epsilon\frac{R^2}{r^2}K(R,r). \tag{28}$$

Assume in addition that $u_0 \in H_0^1$ *and* $\hat{u}_0 \equiv 0$*. Then the augmented system* $(u,\hat{u})$ *has an unique* H^1 *solution, and the equilibrium profile* $u, \hat{u} \equiv 0$ *is exponentially stable in the* H^1 *norm, i.e. there exists* $c_1 > 0$ *such that*

$$\|u(t,\cdot)\|_{H^1} + \|\hat{u}(t,\cdot)\|_{H^1} \le c_1 \mathrm{e}^{-2ct}\|u_0\|_{H^1}. \tag{29}$$

It must be noted that the parameter $c > 0$ is chosen by the designer and controls the rate of convergence of Theorem 2.2. Its size will determine the value of the kernel K, as seen from (24)–(27), and therefore the size of controller and output injection gains.

In the next sections, we prove Theorem 2.2. First, Section 3 applies the backstepping method, finding a feedback law and an observer gain whose kernel is the solution of the singular hyperbolic PDE (24)–(27), which is shown well-posed. Next, Section 4 deals with well-posedness and stability of the closed-loop system.

3. Boundary controller and boundary observer design

In this section, we use the backstepping method to design the control law (21), which is detailed in Section 3.1 and the observer (22), which is given in Section 3.2. The well-posedness of the kernel equations is analysed for both controller and observer in Section 3.3. Both designs are deduced separately, later in Section 4 it is shown that combining the observer and the controller stabilises the closed-loop system.

3.1 Design of boundary feedback control law

Assume for the moment that the full state $u(t,r)$ is known. To design a feedback control law, the backstepping method is used, whose main idea is to use a transformation of the form

$$w(t,r) = u(t,r) - \int_0^r K(r,\rho)u(t,\rho)d\rho, \tag{30}$$

which maps (16) into

$$w_t = \epsilon\left(w_{rr} + 2\frac{w_r}{r}\right) - cw, \tag{31}$$

an stable reaction–diffusion equation for $c > 0$, with boundary conditions

$$w(t,R) = 0,\ w_r(t,0) = 0. \tag{32}$$

To find the conditions that $K(r,\rho)$ must verify (the *control kernel equations*), both the original system (16) and the target system (31) are substituted into the transformation (30). After a rather tedious but straightforward calculation (see e.g. Krstic & Smyshlyaev, 2008a), in this case one obtains

$$K_{rr} + 2\frac{K_r}{r} - K_{\rho\rho} + 2\frac{K_\rho}{\rho} - 2\frac{K}{\rho^2} = \frac{\lambda(\rho)+c}{\epsilon}K \tag{33}$$

with boundary conditions

$$K(r,0) = 0, \tag{34}$$

$$K_\rho(r,0) = 0, \tag{35}$$

$$K(r,r) = -\int_0^r \frac{c+\lambda(\rho)}{2\epsilon}d\rho, \tag{36}$$

in the domain $\mathcal{T} = \{(r,\rho) : 0 \le \rho \le r \le R\}$. Assuming (33)–(36) has a solution, then substituting the boundary conditions (17) and (32) one finds the control law as

$$U(t) = \int_0^R K(R,\rho)u(t,\rho)d\rho, \tag{37}$$

as stated in (28).

3.2 Design of boundary observer

If the full state $u(t,r)$ is unknown, then an observer is needed. Based on the structure of the system, the following observer is proposed

$$\hat{u}_t = \epsilon\left(\hat{u}_{rr} + 2\frac{\hat{u}_r}{r}\right) + \lambda(r)\hat{u} + p(r)\left[Y(t) - \hat{u}_r(t,R)\right], \tag{38}$$

with boundary conditions

$$\hat{u}(t, R) = U(t), \hat{u}_r(t, 0) = 0, \tag{39}$$

where $p(r)$ is the observer gain, to be determined. Defining the observer error as $\tilde{u} = u - \hat{u}$, we obtain that $\hat{u}$ verifies

$$\tilde{u}_t = \epsilon\left(\tilde{u}_{rr} + 2\frac{\tilde{u}_r}{r}\right) + \lambda(r)\tilde{u} - p(r)\tilde{u}_r(t, R), \tag{40}$$

with boundary conditions

$$\tilde{u}(t, R) = 0, \tilde{u}_r(t, 0) = 0. \tag{41}$$

To design the observer gain kernel, so that (40)–(41) becomes stable, the backstepping method is used again. Now we use a transformation

$$\tilde{u}(t, r) = \tilde{w}(t, r) - \int_r^R P(r, \rho)\tilde{w}(t, \rho)d\rho, \tag{42}$$

which maps (40) into

$$\tilde{w}_t = \epsilon\left(\tilde{w}_{rr} + 2\frac{\tilde{w}_r}{r}\right) - c\tilde{w}, \tag{43}$$

a stable reaction–diffusion equation for $c > 0$, with boundary conditions

$$\tilde{w}(t, R) = 0, \tilde{w}_r(t, 0) = 0. \tag{44}$$

The procedure to obtain the conditions that $P(r, \rho)$ must verify (the *observer kernel equations*) is very similar to the method used for the control law design. Both the original error system (40) and the target error system (43) are substituted into the transformation (42). After a similarly tedious calculation (see again Krstic & Smyshlyaev, 2008a), now one obtains

$$P_{rr} + 2\frac{P_r}{r} - P_{\rho\rho} + 2\frac{P_\rho}{\rho} - 2\frac{P}{\rho^2} = -\frac{\lambda(\rho) + c}{\epsilon}P \tag{45}$$

with boundary conditions

$$P_r(0, \rho) = 0, \tag{46}$$

$$P(r, r) = -\int_0^r \frac{c + \lambda(\rho)}{2\epsilon}d\rho, \tag{47}$$

in the domain $\mathcal{T}' = \{(r, \rho) : 0 \le r \le \rho \le R\}$, and the additional condition $p(r) = \varepsilon P(r, R)$. It can be shown that, if one defines $\check{P}(r, \rho) = \frac{\rho^2}{r^2}P(\rho, r)$, it can be verified that $\check{P}$ satisfies exactly (33)–(36). Thus, assuming (33)–(36) is solvable, one obtains $P(r, \rho) = \frac{\rho^2}{r^2}K(\rho, r)$. Thus, one finds the observer kernel gain as

$$p(r) = \epsilon\frac{R^2}{r^2}K(R, r), \tag{48}$$

as stated in (28). In next section, we verify that (42) does not result into singularities.

3.3 Well-posedness of the kernel equations

Both controller and observer kernels can be computed from (33)–(36). To solve these equations, consider a scaling transformation, namely

$$K(r, \rho) = \frac{\rho}{r}\check{K}(r, \rho), \tag{49}$$

in the domain $\mathcal{T}$. Since $\rho \le r$, (49) is in principle well-defined. Computing the derivatives that appear in (33), we obtain

$$K_r(r, \rho) = \frac{\rho}{r}\check{K}_r(r, \rho) - \frac{\rho}{r^2}\check{K}(r, \rho), \tag{50}$$

$$K_{rr}(r, \rho) = \frac{\rho}{r}\check{K}_{rr}(r, \rho) - 2\frac{\rho}{r^2}\check{K}_r(r, \rho) + 2\frac{\rho}{r^3}\check{K}(r, \rho), \tag{51}$$

$$K_\rho(r, \rho) = \frac{\rho}{r}\check{K}_\rho(r, \rho) + \frac{1}{r}\check{K}(r, \rho), \tag{52}$$

$$K_{\rho\rho}(r, \rho) = \frac{\rho}{r}\check{K}_{\rho\rho}(r, \rho) + 2\frac{1}{r}\check{K}_\rho(r, \rho), \tag{53}$$

and replacing in the left-hand side of (33), we obtain

$$\begin{aligned} &K_{rr} + 2\frac{K_r}{r} - K_{\rho\rho} + 2\frac{K_\rho}{\rho} - 2\frac{K}{\rho^2} \\ &= \frac{\rho}{r}\Big(\check{K}_{rr} - 2\frac{1}{r}\check{K}_r + 2\frac{1}{r^2}\check{K} + 2\frac{1}{r}\check{K}_r - 2\frac{1}{r^2}\check{K} - \check{K}_{\rho\rho} \\ &\quad - 2\frac{1}{\rho}\check{K}_\rho + 2\frac{1}{\rho}\check{K}_\rho + 2\frac{1}{\rho^2}\check{K} - 2\frac{1}{\rho^2}\check{K}\Big) \\ &= \frac{\rho}{r}(\check{K}_{rr} - \check{K}_{\rho\rho}), \end{aligned} \tag{54}$$

thus finally we arrive at a equation for $\check{K}$ as follows:

$$\check{K}_{rr} - \check{K}_{\rho\rho} = \frac{\lambda(\rho) + c}{\epsilon}\check{K} \tag{55}$$

with boundary conditions obtained by replacing the scaling transformation in (34)–(36), namely

$$\check{K}(r, 0) = 0, \tag{56}$$

$$\check{K}(r, r) = -\int_0^r \frac{c + \lambda(\rho)}{2\epsilon}d\rho. \tag{57}$$

The kernel equations for $\check{K}$ are identical to those appearing in Krstic and Smyshlyaev (2008a) for 1D reaction–diffusion systems, where it is proved (transforming the kernel equations into an integral equation and using the successive approximations method) that $\check{K}$ is a continuous and differentiable function in the domain $\mathcal{T}$. Since $\rho \le r$, from (49) we obtain that K is continuous in $\mathcal{T}$. In addition, using boundary condition (56), we also obtain K differentiable in $\mathcal{T}$.

Similarly, since we have shown that the observer kernel verifies $P(r, \rho) = \frac{\rho^2}{r^2} K(\rho, r)$, we obtain $P(r, \rho) = \frac{\rho}{r}\check{K}(\rho, r)$. In $\mathcal{T}'$, one has $r \le \rho$, thus the analysis is a bit different. From boundary condition (56), we obtain that $\check{K}(r, \rho) = \rho\psi(r, \rho)$ where ψ is continuous and differentiable. Thus, $P(r, \rho) = \rho\psi(\rho, r)$, and one has that both P and P_r and continuous and differentiable in $\mathcal{T}'$. Finally, that boundary condition (48) is satisfied can also be verified. One has that $P_r(0, \rho) = \rho\psi_\rho(\rho, 0)$. From the differential Equation (55), when $\rho \to 0$, one obtains $\check{K}_{\rho\rho}(r, 0) = 0$, by using boundary conditions (57) and (56) differentiated twice. Since $\check{K}(r, \rho) = \rho\psi(r, \rho)$, one gets $\check{K}_{\rho\rho}(r, \rho) = 2\psi_\rho(r, \rho) + \rho\psi_{\rho\rho}(r, \rho)$, thus one obtains $\psi_\rho(r, 0) = 0$. Thus, it can be concluded $P_r(0, \rho) = 0$.

4. Closed-loop well-posedness and stability

We now prove Theorem 2.2. We first remark that the stability of the augmented $(u, \hat{u})$ system is equivalent to the stability of the augmented $(\tilde{u}, \hat{u})$ system. To obtain the stability result of Theorem 2.2, we need three elements. We begin by obtaining the existence of an inverse transformation (for both control and observer transformations) that allows us to recover the original variables from the transformed variables. We follow by showing that both transformations (or any transformation with a similar structure) are invertible maps from H^1 into H^1 (Proposition 4.1). We continue by stating a well-posedness and stability result for the augmented $(\tilde{w}, \hat{u})$ system in physical space (Proposition 4.2). Combining the two propositions, it is straightforward to construct the proof of Theorem 2.2 by mapping the results for the target augmented system to the original augmented system with the observer and kernel transformations (see for instance Vazquez & Krstic, 2016a).

In what follows, C (with subscript) will denote generic positive constants.

4.1 Invertibility of the transformations

We start with the control transformation. We pose an inverse transform as follows

$$u(t, r) = w(t, r) + \int_0^r L(r, \rho)w(t, \rho)d\rho, \quad (58)$$

and proceeding in the same fashion of Section 3.1, we find the following kernel equations for L:

$$L_{rr} + \frac{L_r}{r} - L_{\rho\rho} + \frac{L_\rho}{\rho} - \frac{L}{\rho^2} = -\frac{\lambda + c}{\epsilon} L. \quad (59)$$

with boundary conditions

$$L(r, 0) = 0, \quad (60)$$

$$L_\rho(r, 0) = 0, \quad (61)$$

$$L(r, r) = -\int_0^r \frac{\lambda(\rho) + c}{2\epsilon} d\rho, \quad (62)$$

in the domain $\mathcal{T}$. These equations are identical to (33)–(36) but substituting λ by $-\lambda$ and changing the sign of the kernel. Thus, the same logic of Section 3.3 applies, and it can be easily shown that $L(r, \rho) = \frac{\rho}{r}\check{L}(r, \rho)$ where $\check{L}$ is a continuous and differentiable function in the domain $\mathcal{T}$, and as before it can be concluded that both L and L_r are continuous in $\mathcal{T}$. Note than alternative method of proof would be substituting (58) into (30) and equating the resulting operator to the identity in L^2 (or H^1). With this procedure one obtains an integral equation for L that can be solved using successive approximations.

It is obvious that a very similar result can be achieved for the observer inverse transformation, which is defined as

$$\tilde{w}(t, r) = \tilde{u}(t, r) + \int_r^R R(r, \rho)\tilde{u}(t, \rho)d\rho, \quad (63)$$

with the kernel R being very similar in structure to L. As in Section 3.3, it can be concluded that R is continuous and differentiable in $\mathcal{T}'$.

4.2 The control and observer transformation as maps between functional spaces

We next show that both the direct and inverse control and observer transformation transform L^2 (resp. H^1) functions back into L^2 (resp. H^1) functions. We use the functional structure of the transformations that was found in Section 3.3.

Proposition 4.1: *Assume that the function $g(r)$ is related to the function $f(r)$ by means of the transformation $g = f - \int_0^r \frac{\rho}{r} F(r, \rho) f(\rho) d\rho$ where F is continuous and differentiable in the domain $\mathcal{T}$. Then:*

$$\|g\|_{L^2} \le C_1\|f\|_{L^2}, \; \|g\|_{H^1} \le C_2\|f\|_{H^1}, \quad (64)$$

where the constants C_1, C_2 are positive and do not depend on f.

Similarly, assume that the function $g(r)$ is related to the function $f(r)$ by means of the transformation

$g = f - \int_r^R \rho F(r,\rho) f(\rho) d\rho$ where F is continuous and differentiable in the domain $\mathcal{T}'$. Then:

$$\|g\|_{L^2} \le C_3 \|f\|_{L^2}, \ \|g\|_{H^1} \le C_4 \|f\|_{H^1}, \tag{65}$$

where the constants C_3, C_4 are positive and do not depend on f.

Proof: We show only the first part of the result, since the second is easily obtained from the functional structure of the transformation by applying the same methods.

First, note that, for $0 \le \rho \le r \le R$ we have

$$|F(r,\rho)| \le C_I, \ |F_r(r,\rho)| + |F_\rho(r,\rho)| \le C_J \tag{66}$$

for $C_I, C_J > 0$.

We obtain

$$\begin{aligned} |g|^2 &= \left| f - \int_0^r \frac{\rho}{r} F(r,\rho) f(\rho) d\rho \right|^2 \\ &\le 2|f|^2 + 2\left(\int_0^r \frac{\rho}{r} |F(r,\rho) f(\rho)| d\rho \right)^2 \\ &\le 2|f|^2 + \frac{2}{r^2} \left(\int_0^r \rho |F(r,\rho) f(\rho)| d\rho \right)^2 \\ &\le 2|f|^2 + \frac{2}{r} C_I^2 \int_0^R |f(\rho)|^2 \rho^2 d\rho \end{aligned} \tag{67}$$

where we have used the Cauchy–Schwarz inequality, given that f is square-integrable. Therefore,

$$\|g\|_{L^2}^2 = \int_0^R |g(r)|^2 r^2 dr \le 2\left(1 + \frac{R^2}{2} C_I^2\right) \|f\|_{L^2}^2 = C_1 \|f\|_{L^2}^2. \tag{68}$$

This shows the L^2 part of the proposition. To prove the H^1 part, note that

$$\begin{aligned} \frac{dg}{dr} &= \frac{df}{dr} - F(r,r) f(r) - \int_0^r \frac{\rho}{r} \frac{\partial F}{\partial r} f(\rho) d\rho \\ &\quad + \int_0^r \frac{\rho}{r^2} F(r,\rho) f(\rho) d\rho \end{aligned} \tag{69}$$

Integrating by parts in the last expression

$$\begin{aligned} \frac{dg}{dr} &= \frac{df}{dr} - \frac{1}{2} F(r,r) f(r) - \int_0^r \frac{\rho}{r} \frac{\partial F}{\partial r} f(\rho) d\rho \\ &\quad - \int_0^r \frac{\rho^2}{2r^2} F_\rho(r,\rho) f(\rho) d\rho \\ &\quad - \int_0^r \frac{\rho^2}{2r^2} F(r,\rho) \frac{df}{d\rho}(\rho) d\rho \end{aligned} \tag{70}$$

Thus

$$\begin{aligned} \left|\frac{dg}{dr}\right|^2 &\le 5\left|\frac{df}{dr}\right|^2 + \frac{5}{4} C_I^2 |f|^2 + \frac{5}{r} C_J^2 \int_0^R |f(\rho)|^2 \rho^2 d\rho \\ &\quad + \frac{5}{12r} C_J^2 \int_0^R |f(\rho)|^2 \rho^2 d\rho \\ &\quad + \frac{5}{12r} C_I^2 \int_0^R \left| \frac{df}{d\rho}(\rho) \right|^2 \rho^2 d\rho, \end{aligned} \tag{71}$$

and therefore,

$$\begin{aligned} \int_0^R \left| \frac{dg}{dr}(r) \right|^2 r^2 dr &\le 5\left(1 + \frac{R^2}{24} C_I^2\right) \int_0^R \left| \frac{dg}{dr}(r) \right|^2 r^2 dr \\ &\quad + 5\left(\frac{1}{4} C_I^2 + \frac{R^2 C_J^2}{2} + \frac{R^2}{24} C_J^2 \right) \int_0^R |f(r)|^2 r^2 dr, \end{aligned} \tag{72}$$

which gives the H^1 part of the proposition. ■

4.3 Stability of the target system

Consider first the $(\tilde{w}, \hat{w})$ system with control law (21), where $\tilde{w}$ is defined by transformation (63) and $\hat{w}$ is defined by applying the control transformation (30) to $\tilde{u}$.

The PDEs verified by $(\tilde{w}, \hat{w})$ are

$$\tilde{w}_t = \frac{\epsilon}{r^2} \left(r^2 \tilde{w}_r \right)_r - c\tilde{w}, \tag{73}$$

$$\hat{w}_t = \frac{\epsilon}{r^2} \left(r^2 \hat{w}_r \right)_r - c\hat{w} - H(r) \tilde{w}_r(t,R) \tag{74}$$

with boundary conditions

$$\tilde{w}(t,R) = 0, \ \tilde{w}_r(t,0) = 0, \tag{75}$$

$$\hat{w}(t,R) = 0, \ \hat{w}_r(t,0) = 0, \tag{76}$$

where

$$H(r) = p(r) - \int_0^r K(r,\rho) p(\rho) d\rho \tag{77}$$

and with $\hat{w}_0 = 0$, $\tilde{w}_0$ obtained from applying the observer transformation to $\tilde{u}_0$ (thus $\tilde{w}_0 \in H_0^1$). Notice that the PDE system is actually a *cascade* system; $\tilde{w}$ verifies an autonomous PDE and its solution (or more specifically, a certain trace of the solution on the boundary) drives the PDE $\hat{w}$. The following result holds.

Proposition 4.2: *Consider the system (73)–(76) with initial conditions $\tilde{w}_0 \in H_0^1$, $\hat{w}_0 = 0$. Then, $\tilde{w}, \hat{w} \in \mathcal{C}\left[[0,\infty), H_0^1\right] \cap L^2\left[(0,\infty), H^2\right]$ and also $\partial_t \tilde{w}, \partial_t \hat{w} \in L^2\left[(0,\infty), L^2\right]$. Moreover, the following bounds are verified*

$$\|\tilde{w}(t,\cdot)\|_{H^1} \le D_1 e^{-2ct} \|\tilde{w}_0\|_{H^1}, \tag{78}$$

$$\|\tilde{w}(t,\cdot)\|_{H^1} + \|\hat{w}(t,\cdot)\|_{H^1} \le D_2 e^{-2ct}\|\tilde{w}_0\|_{H^1}, \quad (79)$$

where D_1 and D_2 are positive constants.

The well-posedness part of the result can be deduced by applying standard results on the sphere, for the $\tilde{w}$ system (see for instance Brezis, 2011, p. 328), and then applying Assumption 2.1. For the $\hat{w}$ system, notice that, given the regularity of $\hat{w}$, the trace of $\hat{w}_r$ is an L^2 function and we obtain the same regularity result (see for instance Brezis, 2011, p. 357).

The stability estimate is obtained using a Lyapunov argument. Define $V_1(t) = \frac{1}{2}\|\tilde{w}(t,\cdot)\|^2_{L^2}$ and $V_2(t) = \frac{1}{2}\|\tilde{w}_r(t,\cdot)\|^2_{L^2}$. Then, first

$$\dot{V}_1 = \int_0^R \tilde{w}(t,r)\tilde{w}_t(t,r)r^2dr = \epsilon\int_0^R \tilde{w}(t,r)\left(r^2\tilde{w}_r\right)_r dr - c\int_0^R \tilde{w}^2(t,r)r^2dr, \quad (80)$$

and integrating by parts,

$$\dot{V}_1 = -\epsilon\int_0^R \tilde{w}_r^2(t,r)r^2dr - c\int_0^R \tilde{w}^2(t,r)r^2dr, \quad (81)$$

To bound the first integral, we use a Poincaré-type inequality (derived from Brezis, 2011, p. 290, Corollary 9.19, with $p=2$ and using (19) and the fact that $\nabla f = f_r\vec{r}$ under the revolution symmetry condition),

$$\int_0^R \tilde{w}^2(t,r)r^2dr \le C_p\int_0^R \tilde{w}_r^2(t,r)r^2dr, \quad (82)$$

which also implies $V_1 \le C_pV_2$. thus we reach $\dot{V}_1 = -2\epsilon V_2 - 2cV_1 \le -2(C_p\epsilon + c)V_1$. On the other hand,

$$\dot{V}_2 = \int_0^R \tilde{w}_r\tilde{w}_{rt}r^2dr. \quad (83)$$

Integrating by parts

$$\dot{V}_2 = -\int_0^R \left(r^2\tilde{w}_r\right)_r\tilde{w}_t dr = -\int_0^R \frac{\epsilon}{r^2}\left(\left(r^2\tilde{w}_r\right)_r\right)^2 dr + c\int_0^R \left(r^2\tilde{w}_r\right)_r\tilde{w}dr \quad (84)$$

and therefore, we obtain that

$$\dot{V}_2 = -\epsilon\int_0^R \left(2\tilde{w}_r + r\tilde{w}_{rr}\right)^2 dr - c\int_0^R \tilde{w}_r^2r^2 \quad (85)$$

thus,

$$\dot{V}_1 + \dot{V}_2 \le -2c(V_1+V_2) - \epsilon\left\|\tilde{w}_{rr} + 2\frac{\tilde{w}_r}{r}\right\|^2_{L^2},$$

and applying Gronwall's Inequality we obtain the stability result (78). Note that in the previous computations, in principle more regularity than $\tilde{w}_t \in L^2\left[(0,\infty), L^2\right]$ is required. However, the result can be concluded by using the same argument as in the proof of Theorem 10.2 in (Brezis, 2011, p. 328), which uses the smoothing property of the heat equation (Brezis, 2011, Theorem 10.1). This property guarantees higher regularity of solutions for $t > 0$.

To obtain (79), define now $V_3(t) = \frac{1}{2}\|\hat{w}(t,\cdot)\|^2_{L^2}$ and $V_4(t) = \frac{1}{2}\|\hat{w}_r(t,\cdot)\|^2_{L^2}$. We obtain the same results as before with additional terms due to the forcing function in (74), namely

$$\dot{V}_3 = -2\epsilon V_4 - 2cV_3 - \tilde{w}_r(t,R)\int_0^R H(r)\hat{w}(t,r)r^2dr, \quad (86)$$

$$\dot{V}_4 = -\epsilon\int_0^R \left(2\hat{w}_r + r\hat{w}_{rr}\right)^2 dr - c\int_0^R \hat{w}_r^2r^2 + \tilde{w}_r(t,R)\int_0^R \left(r^2\hat{w}_r\right)_r H(r)dr. \quad (87)$$

The functions $H(r)$ and H_r are bounded, so that $H(r) \le C_H$ and $H_r(r) \le D_H$. By virtue of the trace theorem (Evans, 1998, p. 258)

$$\tilde{w}_r^2(t,R) \le K_T\|\tilde{w}_r\|^2_{H^1} \quad (88)$$

and since $\tilde{w}$ vanishes at the boundary $\|\tilde{w}_r\|^2_{H^1} \le K_V\left\|\tilde{w}_{rr} + 2\frac{\tilde{w}_r}{r}\right\|^2_{L^2}$. Thus, we reach

$$\dot{V}_3 \le -2cV_3 + \beta_1\|\tilde{w}_{rr} + 2\frac{\tilde{w}_r}{r}\|^2_{L^2}, \quad (89)$$

$$\dot{V}_4 \le -2cV_4 + \beta_2\left\|\tilde{w}_{rr} + 2\frac{\tilde{w}_r}{r}\right\|^2_{L^2}, \quad (90)$$

for some $\beta_1, \beta_2 > 0$. We obtain then

$$\dot{V}_3 + \dot{V}_4 \le -2c(V_3+V_4) + \beta_3\left\|\tilde{w}_{rr} + 2\frac{\tilde{w}_r}{r}\right\|^2_{L^2}, \quad (91)$$

for β_3 positive. Then, defining $V_5 = V_1 + V_2 + \frac{\epsilon}{\beta_3}(V_3 + V_4)$, we obtain $\dot{V}_5 \le -2cV_5$, and applying Gronwall's Inequality and taking into account $\hat{w}_0 = 0$, we obtain the final result (79).

5. Conclusion

This paper is a first step towards extending boundary stabilisation results for constant-coefficient reaction–diffusion equations in spheres to radially varying coefficients. An assumption of revolution symmetry conditions has been made to simplify the equations, which become singular in the radius, complicating the design. The traditional backstepping method can be applied and the well-posedness of the kernel equation can be shown with a simple scaling transformation. The contrast with the 2D case, which is much more complicated (as shown in Vazquez & Krstic, 2016b), is striking.

There are several open problems that still need to be addressed. First, dropping the revolution symmetry conditions would make the problem truly 3D, but unfortunately the scaling transformation proposed in this work does not directly extend to spherical harmonics of order higher than the mean. In addition, it does not seem to be possible to define similar transformations to prove well-posedness of the kernel equations for higher order harmonics. Thus, a different strategy would need to be devised . In addition, if the coefficients also vary with the angle, then spherical harmonics cannot be used and the kernel equation becomes an ultra-hyperbolic equation in four dimensions, which seems quite difficult to address.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

Ministerio de Economía y Competitividad [grant number MTM2015-65608-P].

References

Baccoli, A., Pisano, A., & Orlov, Y. (2015). Boundary control of coupled reaction-diffusion processes with constant parameters. *Automatica, 54*(4), 80–90.

Brezis, H. (2011). *Functional analysis, sobolev spaces and partial differential equations*. New York, NY: Springer.

Bribiesca Argomedo, F., Prieur C., Witrant, E., & Bremond, S. (2013). A strict control Lyapunov function for a diffusion equation with time-varying distributed coefficients. *IEEE Transactions on Automatic Control, 58*(2), 290–303.

Coron, J.-M., Vazquez, R., Krstic, M., & Bastin, G. (2013). Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping. *SIAM Journal on Control and Optimization, 51*(3), 2005–2035.

Di Meglio, F., Vazquez, R., & Krstic, M. (2013). Stabilization of a system of n+1 coupled first-order hyperbolic linear PDEs with a single boundary input. *IEEE Transactions on Automatic Control, 58*(12), 3097–3111.

Evans, L.C. (1998). *Partial differential equations*. Providence: American Mathematical Society.

Jadachowski, L., Meurer, T., & Kugi, A. (2015). Backstepping observers for linear PDEs on higher-dimensional spatial domains. *Automatica, 51*(1), 85–97.

Krstic, M. (2009). *Delay compensation for nonlinear, adaptive, and PDE systems*. Boston, MA: Birkhäuser.

Krstic, M., & Smyshlyaev, A. (2008a). *Boundary control of PDEs: A course on backstepping design*. Philadelphia, PA: SIAM.

Krstic, M., & Smyshlyaev, A. (2008b). Backstepping boundary control for first order hyperbolic PDEs and application to systems with actuator and sensor delays. *Systems & Control Letters, 57*(9), 750–758.

Li, G., , & Xie, C. (2010). Feedback stabilization of reaction-diffusion equation in a two-dimensional region. In *Proceedings of the 2010 IEEE Conference on Decision and Control (CDC)* (pp. 2985–2989), Atlanta, GA: IEEE.

Meurer, T., & Krstic, M. (2011). Finite-time multi-agent deployment: A nonlinear PDE motion planning approach. *Automatica, 47*(11), 2534–2542.

Meurer, T. (2013). *Control of higher-dimensional PDEs: Flatness and backstepping designs*. New York, NY: Springer.

Meurer, T., & Kugi, A. (2009). Tracking control for boundary controlled parabolic PDEs with varying parameters: Combining backstepping , and differential flatness. *Automatica, 45*(5), 1182–1194.

Moura, S.J., Chaturvedi, N.A., & Krstic, M. (2012). PDE estimation techniques for advanced battery management systems—Part I: SOC estimation. In *Proceedings of the 2012 American Control Conference* (pp. 559–565), Montreal, QC: IEEE.

Qi, J., Vazquez, R., & Krstic, M. (2015). Multi-agent deployment in 3-D via PDE control. *IEEE Transactions on Automatic Control, 60*(4), 891–906.

Smyshlyaev, A., Cerpa, E., & Krstic, M. (2010). Boundary stabilization of a 1-D wave equation with in-domain antidamping. *SIAM Journal on Control and Optimization, 48*(6), 4014–4031.

Smyshlyaev, A., & Krstic, M. (2010). *Adaptive control of parabolic PDEs*. Princeton: Princeton University Press.

Tsubakino, D., & Hara, S. (2015). Backstepping observer design for parabolic PDEs with measurement of weighted spatial averages. *Automatica, 53*(3), 179–187.

Vazquez, R., & Krstic, M. (2008a). *Control of turbulent and magnetohydrodynamic channel flow*. Boston, MA: Birkhäuser.

Vazquez, R., & Krstic, M. (2008b). Control of 1-D parabolic PDEs with volterra nonlinearities — Part I: Design. *Automatica, 44*(11), 2778–2790.

Vazquez, R., & Krstic, M. (2010). Boundary observer for output-feedback stabilization of thermal convection loop. *IEEE Transactions on Control Systems Technology, 18*(4), 789–797.

Vazquez, R., & Krstic, M. (2014a). Marcum Q-functions and explicit kernels for stabilization of linear hyperbolic systems with constant coefficients. *Systems & Control Letters, 68*(6), 33–42.

Vazquez, R., & Krstic, M. (2014b). Explicit boundary control of a reaction-diffusion equation on a disk. In *Proceedings of the 2014 IFAC World Congress* (pp. 1562–1567), Cape Town: IFAC.

Vazquez, R., & Krstic, M. (2015). Explicit boundary control of reaction-diffusion PDEs on arbitrary-dimensional balls. In

Proceedings of the 2015 European Control Conference (pp. 879–884), Linz: IEEE.

Vazquez, R., & Krstic, M. (2016a). Explicit output-feedback boundary control of reaction-diffusion PDEs on arbitrary-dimensional balls. *ESAIM: Control, Optimisation and Calculus of Variations*. *22*(4), 1078–1096.

Vazquez, R., & Krstic, M. (2016b). Boundary control of a singular reaction-diffusion equation on a disk. In Proceedings of the 2nd IFAC Workshop on Control of Systems Governed by Partial DIfferential Equations (pp. 74–79), Bertinoro: IFAC.

Vazquez, R., Trelat, E., & Coron, J.-M. (2008). Control for fast and stable laminar-to-high-Reynolds-numbers transfer in a 2D Navier-Stokes channel flow. *Discrete and Continuous Dynamical Systems - Series B, 10*(4), 925–956.