Propagation of Initial Mass Uncertainty in Aircraft Cruise Flight

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The propagation of initial mass uncertainty in cruise flight is studied. Two cruise conditions are analyzed: one with given cruise fuel load and the other with given cruise range. Two different distributions of initial mass are considered: uniform and gamma type. The generalized polynomial chaos method is used to study the evolution of mean and variance of the aircraft mass. To compute the mass distribution function as a function of time, two approximate methods are developed. These methods are also applied to study the distribution functions of the flight time (in the case of given fuel load) and of the fuel consumption (in the case of given range). The dynamics of mass evolution in cruise flight is defined by a nonlinear equation, which can be solved analytically; this exact solution is used to assess the accuracy of the proposed methods. Comparison of the numerical results with the exact analytical solutions shows an excellent agreement in all cases, hence verifying the methods developed in this work.

I. Introduction

The air traffic management (ATM) system is a very complex system, which contains a large number of heterogeneous components, such as airports, aircraft, navigation systems, flight management systems (FMSs), traffic controllers, and weather (see Kim et al. [1]). Correspondingly, its performance is affected by numerous factors. Within the trajectory-based operations concept of SESAR and NextGen, aircraft trajectories are key to study ATM operations, which are subject to many uncertainties. Sources of uncertainty for aircraft trajectories include wind and severe weather, navigational errors, aircraft performance inaccuracies, or errors in the FMS, among others. The analysis of the impact of uncertainties in aircraft trajectories and its propagation through the flight segments is of great interest, because it might help to understand how sensitive the system is to the lack of precise data and measurement errors and, therefore, aid in the design of a more robust ATM system, with improved safety levels.

Among those sources, weather uncertainty has perhaps the greatest impact on ATM operations, being responsible for much of the delays. Its analysis has been addressed by many authors using different methods, for example, the following: Nilim et al. [2] consider a trajectory-based air traffic management scenario to minimize delays under weather uncertainty, in which the weather processes are modeled as stationary Markov chains. Pepper et al. [3] present a method, based on Bayesian decision networks, of accounting for uncertain weather information in air traffic flow management. Clarke et al. [4] develop a methodology to study airspace capacity in the presence of weather uncertainty and formulate a stochastic dynamic programming algorithm for traffic flow management. Zheng and Zhao [5] develop a statistical model of wind uncertainties and apply it to stochastic trajectory prediction in the case of straight, level aircraft flight trajectories.

The framework for this work is the analysis of uncertainty propagation in aircraft trajectories and, eventually, its effect on the ATM system. In this paper, several tools are presented to analyze uncertainty propagation in a nonlinear problem and they are applied to study the effect of initial aircraft mass uncertainty and its propagation through the cruise flight phase. The relevance of this problem resides in two facts: first, the initial mass is an important source of uncertainty in trajectory prediction, which determines mass evolution and, therefore, fuel consumption and flight cost; and, second, cruise uncertainties have a large impact on the overall flight because the cruise phase is the largest portion of the flight (at least for long-haul routes). In the applications, two cruise conditions are studied: one with given cruise fuel load and the other with given cruise range.

Several methods have been proposed to study uncertainty propagation in dynamic systems, beyond the classical Monte Carlo methods (which can be very expensive computationally). Halder and Bhattacharya [6] classify those methods in two categories: parametric...
(in which one evolves the statistical moments) and nonparametric (in which the probability density function is evolved). They address the problem of uncertainty propagation in planetary entry, descent, and landing, using a nonparametric method that reduces to solving the stochastic Liouville equation.

In this paper, the evolution in time of the mean and the variance of the aircraft mass is studied using the generalized polynomial chaos (GPC) method (a parametric method according to [6]). The GPC representation was introduced by Wiener [7] and is based on the fact that any second-order process (i.e., a process with finite second-order moments) can be represented as a Fourier-type series, with time-dependent coefficients, and using orthogonal polynomials as GPC basis functions in terms of random variables. A general introduction to GPC can be found in Xiu and Karniadakis [8] and in Schoutens [9], and details in numerical computations are studied in Debusschere et al. [10]. The method of polynomial chaos is used in the works of Prabhakar et al. [11] and Dutta and Bhattacharya [12] to study, respectively, uncertainty propagation and trajectory estimation, for hypersonic flight dynamics with uncertain initial data, and by Fisher and Bhattacharya [13] in the problem of optimal trajectory generation in the context of stochastic optimal control.

Also, the distribution function of the aircraft mass is analyzed using two approximate methods developed in this paper (nonparametric methods according to [6]). One method is based on the resolution of the variational equation for the sensitivity function with respect to the initial condition, and the other is based on the computation of the probability measure of the random variable as a function of time. These two methods are also applied, first, to the analysis of the distribution function of the flight time, in the case of a given fuel load and, second, to the analysis of the distribution function of the fuel consumption, in the case of a given range. In this way, the effect of the initial mass uncertainty in flight properties other than mass is studied as well.

In this paper, the case of cruise at constant altitude and constant speed is considered (cruise segments defined by these two flight constraints are commonly flown by commercial aircraft, according to air traffic control procedures). In this case, the evolution of aircraft mass is defined by a nonlinear equation that can be solved analytically. Results are presented for two different distributions of initial mass (uniform and gamma type). The analytical solutions represent benchmark solutions that are used to assess the accuracy of the proposed methods. Comparison with the exact analytical results is made, showing an excellent agreement in all cases.

This paper is organized as follows. First, the problem of mass evolution in cruise flight is solved. Then, in Sec. III, the two initial mass distributions considered are described. In Sec. IV, mean and variance of the mass distribution are analyzed using the GPC method. In Sec. V, the two nonparametric methods developed to study the evolution of distribution functions are presented and are applied to the mass distribution function. These two methods are used, in Sec. VI, to study the distribution functions of flight time and fuel consumption. Some numerical results are presented in Sec. VII, and some conclusions are drawn in Sec. VIII. Finally, the exact analytical solutions are presented in the Appendix.

II. Mass Evolution in Cruise Flight

The equations of motion for symmetric flight in a vertical plane (constant heading), using a flat Earth model, for constant altitude and constant speed are (see [14])

\[
\frac{dx}{dt} = V, \quad \frac{dm}{dt} = -cT \quad T = D, \quad L = mg \quad (1)
\]

where \(x\) is the horizontal distance, \(t\) is the time, \(V\) is the speed, \(T\) is the thrust, \(D\) is the aerodynamic drag, \(L\) is the lift, \(m\) is the aircraft mass, \(g\) is the acceleration of gravity, and \(c\) is the specific fuel consumption, which can be taken as a function of altitude and speed, and it is therefore constant under the given cruise condition.

The drag can be written as

\[
D = \frac{1}{2} \rho V^2 SC_D
\]

where \(\rho\) is the density, \(S\) is the wing surface area, and the drag coefficient \(C_D\) is modeled by a parabolic polar \(C_D = C_{D_0} + \frac{C_{D_1}}{\rho V^2 S}\)

where \(C_L\) is the lift coefficient given by

\[
C_L = \frac{2L}{\rho V^2 S}
\]

and the coefficients \(C_{D_0}\) and \(C_{D_1}\) are constant under the given cruise condition. Using these definitions and Eq. (1), an autonomous equation for the mass evolution is obtained:

\[
\frac{dm}{dt} = -c \left( \frac{1}{2} \rho V^2 SC_D + m^2 \frac{2C_{D_1}g^2}{\rho V^2 S} \right) \quad (2)
\]

Thus, one can write

\[
\frac{dm}{dt} = -(A + Bm^2) \quad (3)
\]

where the constants \(A\) and \(B\) are defined as

\[
A = c \left( \frac{1}{2} \rho V^2 SC_{D_0} \right) \\
B = c \left( \frac{2C_{D_1}g^2}{\rho V^2 S} \right)
\]

Note that \(A, B > 0\). Equation (3) is a nonlinear equation describing the evolution of mass during cruise flight, to be solved with the initial condition

\[
m(0) = m_0 \quad (4)
\]

To emphasize the dependence of the mass \(m(t)\) on the initial condition, the mass is written as \(m(t; m_0)\), even though it is often just denoted as \(m\) for the sake of simplicity. The explicit solution of Eqs. (3) and (4) is

\[
m(t; m_0) = \sqrt{\frac{A}{B}} \left( m_0 - \sqrt{(A/B) \tan(\sqrt{AB}t)} \right) \quad (5)
\]

A. Cruise with Given Fuel Load

For the case in which the cruise fuel load is given, denoting the given mass of fuel as \(m_F < m_0\), the solution obtained by Eq. (5) is valid in the time interval \(t \in [0, t_f(m_0)]\), where \(t_f(m_0)\) (the flight time) is obtained from \(m(t_f(m_0); m_0) = m_0 - m_f\). From Eq. (5), one can directly compute this time as

\[
t_f(m_0) = 1 \sqrt{\frac{A}{B}} \arctan \left( \frac{\sqrt{ABm_f}}{A + Bm_0(m_0 - m_f)} \right) \quad (6)
\]

Note that \(t_f\) is a monotonically decreasing function of \(m_0\). Thus, for a given amount of fuel, the larger \(m_0\), the smaller \(t_f\) and, as a consequence, the smaller the distance traveled by the aircraft. The initial mass \(m_0\) is unbounded and has a lower limit equal to \(m_f\) (although these limits are not physically meaningful). Thus, for \(m_0 \in (m_f, \infty)\) one obtains from Eq. (6)

\[
t_f \in \left( 0, \frac{1}{\sqrt{AB}} \arctan \left( \frac{B}{A m_f} \right) \right)
\]
Also, since \(m(t_f; m_0) = m_0 - m_F\), the final value of the aircraft mass satisfies \(m(t_f; m_0) \in (0, \infty)\).

In the next sections, the evolution of mass and the behavior of the flight time are studied for an uncertain value of the initial mass, whereas the rest of the parameters (some of them embedded in the constants \(A\) and \(B\)) have a fixed value.

B. Cruise with Given Range

For the case in which the cruise range is given, taking \(x\) as the independent variable, one has

\[
\frac{dm}{dx} = -\frac{1}{V}(A + Bm^2) \tag{7}
\]

and the same initial condition (4). The explicit solution of Eqs. (4) and (7) is

\[
m(x; m_0) = \frac{m_0 - \sqrt{(A/B)\tan((1/V)\sqrt{ABx})}}{\sqrt{(A/B) + m_0 \tan((1/V)\sqrt{ABx})}} \tag{8}
\]

If the given cruise range is \(x_f\), then the final value of the aircraft mass \(m(x_f; m_0)\) is given by Eq. (8) particularized for \(x = x_f\), and the fuel consumption during the cruise is

\[
m_F(m_0) = m_0 - m(x_f; m_0) = \frac{m_0^2 + (A/B)\tan((1/V)\sqrt{ABx_f})}{\sqrt{(A/B) + m_0 \tan((1/V)\sqrt{ABx_f})}} \tag{9}
\]

Note that \(m_F\) is a monotonically increasing function of \(m_0\); the larger \(m_0\), the larger the fuel consumption. As before, \(m_0\) is unbounded and, to have \(m(x_f; m_0) > 0\), it has a lower limit equal to

\[
\frac{\sqrt{A}}{B} \tan \left( \frac{1}{V} \sqrt{ABx_f} \right)
\]

Thus for

\[
m_0 \in \left( \frac{\sqrt{A}}{B} \tan \left( \frac{1}{V} \sqrt{ABx_f} \right), \infty \right)
\]

one obtains from Eq. (9) that

\[
m_F \in \left( \frac{\sqrt{A}}{B} \tan \left( \frac{1}{V} \sqrt{ABx_f} \right), \infty \right)
\]

Also, from Eq. (8), the final value of the aircraft mass satisfies

\[
m(x_f; m_0) \in \left( 0, \frac{\sqrt{A}}{B} \tan \left( \frac{1}{V} \sqrt{ABx_f} \right) \right)^{-1}
\]

In the next sections, the behavior of the fuel consumption is studied for an uncertain value of the initial mass, whereas, as before, the rest of the parameters have a fixed value. In this case, the flight time is known, trivially given by

\[
t_f = \frac{x_f}{V}
\]

III. Initial Mass Distribution

It is realistic to consider that the initial mass \(m_0\) is not a deterministic variable which is known a priori, but rather a random variable which is not known. Then, the solution given by Eq. (5) is still valid but in a probabilistic sense [i.e., \(m(t; m_0)\) is a random process]. If the distribution of \(m_0\) is known, it is possible to study the time evolution of the distribution of the aircraft mass \(m(t; m_0)\), as well as its statistical properties (mean, variance, typical deviation).

In this work, to analyze mass evolution, two probabilistic models for \(m_0\) are considered: uniform and gamma distributions, which are described next. Note that a Gaussian distribution representing the initial mass uncertainty would be nonphysical, because it would allow (with small but nonzero probability) negative initial mass and, therefore, it is not considered in this paper.

A. Uniform Distribution

First, it is considered that \(m_0\) is distributed as a uniform continuous variable whose probability density function is

\[
f_{m_0}(m_0) = \frac{1}{2\delta_m}
\]

in the interval \([\tilde{m}_0 - \delta_m, \tilde{m}_0 + \delta_m]\), and zero otherwise, where \(\tilde{m}_0\) is the mean and \(\delta_m\) is the width of the uniform distribution, as shown in Fig. 1.

Denoting by \(\Delta\) the standardized uniform distribution taking values in the interval \([-1, 1]\), one has that \(m_0 = \tilde{m}_0 + \delta_m \Delta\). The mean of \(m_0\) is

\[
E[m_0] = \int_0^\infty m_0 f_{m_0}(m_0) \, dm_0 = \tilde{m}_0
\]

where \(E[\cdot]\) is the mathematical expectation, and the variance of \(m_0\) is

\[
Var[m_0] = E[m_0^2] - (E[m_0])^2 = \frac{\delta_m^2}{3}
\]

B. Gamma Distribution

The gamma distribution (see [15]) represents a continuous nonnegative random variable and is denoted by \(G(k, \theta)\), where \(k > 0\) is the shape parameter and \(\theta > 0\) is the scale parameter. It is known that \(E[G(k, \theta)] = k\theta\) and \(Var[G(k, \theta)] = k\theta^2\) and that the probability density function of \(G(k, \theta)\) is

\[
f(x; k, \theta) = x^{k-1} e^{-x/\theta} \theta^k \Gamma(k) \tag{10}
\]

for \(x \geq 0\) (and zero otherwise), where \(\Gamma\) is the Euler gamma function. Using the property that, for \(\theta > 0\), one has \(G(k, \theta) = \theta G(k, 1)\), the value \(\theta = 1\) is considered in this paper without loss of generality.

To represent the initial mass distribution, let

\[
m_0 = \tilde{m}_0 + \frac{\delta_m}{\sqrt{3}k} G(k, 1 - k)
\]

where \(\tilde{m}_0\) and \(\delta_m\) are the same values chosen for the uniform distribution. Hence, only the values \(m_0 \geq M_0\) have nonzero probability, where \(M_0\) (the minimum possible value of mass for the given values of \(\tilde{m}_0\) and \(\delta_m\)) is obtained making \(G(k, 1) = 0\) and it is given by

\[
f_{m_0}(m_0) = \frac{1}{2\delta_m}
\]

Fig. 1 Shape of the probability density functions of the initial mass.
Thus, one has the following probability density function
\[
f_{m_0}(m_0) = \begin{cases} 
(m_0 - M_0)^{k-1} e^{-\frac{(m_0 - M_0)^2}{\delta_m^2}} / (\delta_m^2 / 3)^{1/2} \Gamma(k), & m_0 \geq M_0 \\
0, & \text{otherwise}
\end{cases}
\]  
and zero otherwise. In this way, one has \( \mathbb{E}[m_0] = \bar{m}_0 \) and
\[
\text{Var}[m_0] = \frac{\delta_m^2}{3} 
\]
(independently of \( k \)), as for the previously chosen uniform distribution.

Note that for \( k \to \infty \), one has
\[
\frac{G(k, 1) - k}{\sqrt{k}} \to N(0, 1) 
\]
which implies
\[
m_0 \to N\left(\bar{m}_0, \frac{\delta_m^2}{3}\right) 
\]
(i.e., for large \( k \) the gamma distribution resembles a Gaussian distribution). However, the maximum value of \( k \) is limited by the fact that \( M_0 \) should be greater than zero. Therefore, the value of \( k \) must be chosen taking into account Eq. (10).

In Fig. 1, the shape of the probability density function of \( m_0 \) is plotted for different values of \( k \) and compared with the uniform distribution.

### IV. Analysis of Mass Mean and Variance

To compute the mean and variance of the mass, the GPC method is used (see [2]), in which the process is represented as a Fourier-type series, with time-dependent coefficients, and orthogonal polynomials in terms of random variables are used as basis functions. The orthogonal polynomials used in GPC are chosen from the Askey scheme (a way of organizing certain orthogonal polynomials into a hierarchy, see [16]). If one chooses a family of polynomials that are orthogonal, the convergence of the series is exponential. The orthogonality property implies that, when taking expectation with respect to the random variable for two polynomials of the family \( \phi_i \) and \( \phi_j \), then \( \mathbb{E}[\phi_i \phi_j] = \delta_{ij} \mathbb{E}[\phi_i^2] \), where \( \delta_{ij} \) is the Kronecker delta. For the uniform distribution \( \Delta \), the adequate orthogonal polynomials are the Legendre polynomials \( L_m(\Delta) \), whereas, for the gamma distribution \( G(k, 1) \), one must use the generalized Laguerre polynomials \( \phi_k^{\infty-1}(G) \).

To apply the GPC method, one must first write the initial mass distribution \( m_0 \) in terms of the orthogonal polynomials. For the uniform distribution, one can write
\[
m_0 = \bar{m}_0 L_0(\Delta) + \delta_m L_1(\Delta), \quad \text{whereas, for the gamma distribution, it follows that}
\]
\[
m_0 = \bar{m}_0 \phi_0^{\infty-1}(G) - \frac{\delta_m}{\sqrt{3k}} \phi_1^{\infty-1}(G)
\]
In the following, the uniform distribution case is considered (the gamma distribution is handled analogously). It is assumed that \( m(t; m_0) \) can be written as
\[
m(t; m_0) = \sum_{i=0}^{P} \hat{h}_i(t) L_i(\Delta) 
\]  
(12)
where the coefficients \( \hat{h}_i \) are to be found using the mass equation (3), and \( P \) is the order of the approximation, which is to be taken sufficiently large. Substituting Eq. (12) in Eq. (3), the following equation is obtained
\[
\sum_{i=0}^{P} \hat{h}_i(t) L_i(\Delta) = -A - B \sum_{i=0}^{P} \hat{h}_i(t) L_i(\Delta) L_i(\Delta) 
\]
(13)
Now, multiplying Eq. (13) by \( L_j(\Delta) \) for \( l = 0, \ldots, P \), taking expectation with respect to \( \Delta \), and using the orthogonality property of the \( L_i \) polynomials, one obtains \( P + 1 \) equations
\[
\hat{h}_i(t) \mathbb{E}[L_i^2(\Delta)] = -A \delta_i 0 \\
-B \sum_{i=0}^{P} \hat{h}_i(t) \mathbb{E}[L_i(\Delta) L_j(\Delta) L_i(\Delta)] 
\]
(14)
and calling
\[
C_{i,j} = \frac{\mathbb{E}[L_i L_j L_1]}{\mathbb{E}[L_1^2]} 
\]
(which is a number that can be exactly computed because the involved expectations are just integrals of polynomials), it follows that
\[
\hat{h}_i = -A \delta_{i0} - B \sum_{j=0}^{P} \hat{h}_j(t) C_{i,j} 
\]
(15)
which is a system of \( P + 1 \) nonlinear coupled ordinary differential equations. The same result is reached for the gamma distribution case, with the corresponding \( C_{i,j} \) coefficients. The initial conditions of Eq. (15) depend on the initial mass distribution. For the uniform distribution case, they are
\[
h_0(0) = \bar{m}_0, \quad h_i(0) = \delta_m, \quad h_i(0) = 0, \quad \text{for } l = 2, \ldots, P 
\]
(16)
whereas, for the gamma distribution, they are given by
\[
h_0(0) = \bar{m}_0, \quad h_1(0) = -\frac{\delta_m}{\sqrt{3k}}, \quad h_j(0) = 0, \quad \text{for } l = 2, \ldots, P 
\]
(17)
The advantage of the GPC method is that a small or moderate value of \( P \) is enough to get good results, thus resulting in a method that is not very intensive computationally.

Once the coefficients \( \hat{h}_i \) are found, it is possible to compute from Eq. (12) approximate values for quantities of interest such as mean and variance. For the uniform distribution, taking into account Eq. (12) and \( L_0(\Delta) = 1 \), it follows that
\[
\mathbb{E}[m(t; m_0)] = \sum_{i=0}^{P} \hat{h}_i(t) \mathbb{E}[L_i(\Delta)] = \sum_{i=0}^{P} \hat{h}_i(t) \mathbb{E}[L_i(\Delta)L_0(\Delta)] 
\]
\[
= \hat{h}_0(t) \mathbb{E}[L_0^2(\Delta)] = h_0(t) 
\]
(18)
To compute the variance
\[
\text{Var}[m(t; m_0)] = \mathbb{E}[m^2(t; m_0)] - \mathbb{E}[m(t; m_0)]^2 
\]
\[
= \sum_{i=0}^{P} \sum_{j=0}^{P} \hat{h}_i(t) \hat{h}_j(t) \mathbb{E}[L_i(\Delta)L_j(\Delta)] - h_0^2 = \sum_{i=0}^{P} \hat{h}_i^2(t) \mathbb{E}[L_i^2(\Delta)] 
\]
(19)
For the gamma distribution, similar results hold:
\[
\mathbb{E}[m(t; m_0)] = h_0(t) 
\]
(20)
\[ \text{Var}[m(t; m_0)] = \sum_{i=1}^{P} h_i^2(t)[\varphi_r^{-1}(G)]^2 \]  \tag{21}

V. Analysis of the Evolution of the Mass Distribution Function

In this section, because the GPC method cannot be used to obtain distribution functions (see [11]), two original approximate methods to obtain the distribution function of the mass (which evolves in time) are developed.

Recall first that, given a random variable \( x \) with distribution function \( f_x(x) \), if one defines another random variable \( y \) using a transformation \( g \) such that \( y = g(x) \), then it is known that the distribution function \( f_y(y) \) of \( y \) is given by (see [15])

\[ f_y(y) = f_x(g^{-1}(y)) \left| \frac{g'(g^{-1}(y))}{g'(y)} \right| \] \tag{22}

with expression (22) valid only if the function \( g(x) \) is invertible in the domain of \( x \).

Denoting \( m = m(t; m_0) = \varphi_r(m_0) \) as the solution of the differential equation (3) with initial condition (4), it follows from standard uniqueness results in differential equations (see [17]) that the function relating \( m \) and \( m_0 \) (for a given time \( t \)) is always monotonous. Indeed, if it were not monotonous, there would be values of mass (for a given time \( t \)) that could be reached from two different initial conditions, which would contradict uniqueness. Because it is monotonous, it is therefore invertible. Thus, it is possible to write

\[ f_m(m, t) = \frac{f_m(m_0, \varphi_r^{-1}(m))}{\varphi_r'(m_0)} \] \tag{23}

where \( f_m \) is the distribution of the initial mass and \( f_m(m, t) \) is the distribution of the mass at time \( t \).

A. Approximate Method 1

The objective is to numerically approximate Eq. (23). For that, take \( n \) consecutive points from the domain of \( m_0 \), denoted as \( m_i^0 \), \( i = 1, \ldots, n \), so that \( m_0^0 < m_1^0 < \ldots < m_n^0 \). Now, fix a time \( t > 0 \); solving the mass equation (3) for each \( i \) with \( m_i^0 \) as the initial condition, one can compute the value of mass at time \( \tau \), \( m^\tau_i = \varphi_r(m_i^0) \). The numerator of Eq. (23) is computed for each \( i \) as \( f_m(m_i^0) \). To compute the denominator of Eq. (23), the theory of differential equations is used. Noting that

\[ \varphi_r'(m_0) = \frac{\partial m}{\partial m_0}(t) \]

is the value of the derivative of the solution \( m \) with respect to \( m_0 \) (also known as the sensitivity function with respect to the initial condition), a differential equation can be written for \( \varphi_r'(m_0) \):

\[ \frac{d}{dt} \varphi_r'(m_0) = \frac{d}{dt} \left( \frac{\partial m}{\partial m_0} \right) \neg Bm \frac{\partial m}{\partial m_0} = -2Bm \frac{\partial m}{\partial m_0} \] \tag{24}

with initial condition [obtained from Eq. (4)]

\[ \varphi_r'(m_0) = 1 \] \tag{25}

This is the so-called variational equation, which is linear, and its solution is given by

\[ \varphi_r'(m_0) = \exp(-2B \int_0^t m(t; m_0) \, dt) \] \tag{26}

Numerically solving Eq. (26) to find \( \varphi_r'(m_i^0) \) at time \( t = \tau \), the denominator of Eq. (23) is computed for each \( i \).

Thus, for a fixed time \( \tau \), one finds the value of \( f_m(m, \tau) \) at the \( n \) points \( m^\tau_i = \varphi_r(m_i^0), i = 1, \ldots, n \), as

\[ f_m(m^\tau_i, \tau) = \frac{f_m(m_i^0, \varphi_r^{-1}(m^\tau_i))}{\varphi_r'(m_i^0)} \] \tag{27}

B. Approximate Method 2

Now, another method that avoids having to solve the differential equation for the sensitivity function (24) is formulated. As in the previous method, take \( n \) consecutive points from the domain of \( m_0 \), \( m_0^0 < m_1^0 < \ldots < m_n^0 \), fix a time \( \tau > 0 \), and solve the mass equation (3) to compute the value of mass at time \( t = \tau \), \( m^\tau = \varphi_r(m_i^0) \). To find the value of \( f_m(m, \tau) \) at these points, the intermediate value theorem for integrals is used:

\[ Pr(m^0 \leq m \leq m^{i+1}) = \int_{m^0}^{m^{i+1}} f_m(m, \tau) \, dm = (m^{i+1} - m^i) f_m(\xi^i, \tau) \] \tag{28}

where \( Pr \) is the probability measure and \( \xi^i \in [m^i, m^{i+1}] \), for \( i = 1, \ldots, n - 1 \).

Given the uniqueness of the solution, intervals in the initial condition are univocally mapped into intervals in the solution (as illustrated in Fig. 2), thus the probability of the mass \( m \) being in the interval \( [m^i, m^{i+1}] \) is the same as the probability of the initial mass \( m_0 \) being in the interval \( (m_0^i, m_0^{i+1}) \), that is, \( Pr(m^0 \leq m \leq m^{i+1}) = Pr(m_0^i \leq m_0 \leq m_0^{i+1}) \). These probabilities can be computed (numerically or analytically) from the distribution function of \( m_0 \). Thus, one has

\[ f_m(\xi^i, \tau) = \frac{Pr(m_0^i \leq m_0 \leq m_0^{i+1})}{m^{i+1} - m^i}, \quad i = 1, \ldots, n - 1 \] \tag{29}

Taking

\[ f_m(m^i, \tau) = f_m(\xi^i, \tau) \]

\[ f_m(m^{i+1}, \tau) = f_m(\xi^{i+1}, \tau) + f_m(\xi^i, \tau), \quad i = 2, \ldots, n - 1 \]

\[ f_m(m^n, \tau) = f_m(\xi^{n-1}, \tau) \] \tag{30}

an approximation of \( f_m \) is obtained at \( n \) points.

VI. Analysis of the Distribution Function of the Flight Time and the Fuel Consumption

In this section, the distribution functions of the flight time \( t_f \) (in the case of a given fuel load) and of the fuel consumption \( m_f \) (in the case of a given range) are analyzed using the approximate methods developed in Sec. V.

A. Distribution Function of the Flight Time

The flight time \( t_f \) is defined explicitly by Eq. (6), where it can be seen that it is a function of the initial mass and hence a random variable itself. Calling \( t_f = \varphi(m_0) \), one has that

\[ m(\varphi(m_0); m_0) = m_0 - m_f \] \tag{31}

The distribution function of \( t_f \) is given, similarly to Eq. (23), by

\[ f_{t_f}(t_f) = \frac{f_m(m_0, \varphi^{-1}(t_f))}{|\varphi'(\varphi^{-1}(t_f))|} \] \tag{32}

Fig. 2 Evolution of the initial mass intervals in time.
if \( \psi \) is invertible. To see that this is the case, take the derivative with respect to \( m_0 \) in Eq. (31),

\[
\frac{dm}{dt}(t_j; m_0) \psi'(m_0) = \frac{dm}{dm_0}(t_j; m_0) = 1 \tag{33}
\]

Note that

\[
\frac{dm}{dt}(t_j; m_0) = \dot{m}(t_j)
\]

thus using the mass equation (3), it is found that

\[
\frac{dm}{dt}(t_j; m_0) = -(A + Bm^2(t_j)) = -(A + B(m_0 - m_F)^2) < 0 \tag{34}
\]

Thus, from Eqs. (33) and (34), one has

\[
\psi'(m_0) = \frac{1 - (\frac{dm}{dm_0}(t_j; m_0)}{A + B(m_0 - m_F)^2} \tag{35}
\]

On the other hand, \( \frac{dm}{dm_0} \) satisfies the differential equation (24), hence, one has from Eq. (26) that \( \frac{dm}{dm_0}(t_j; m_0) < 1 \) for \( t > 0 \) and, in particular, \( \frac{dm}{dm_0}(t_j; m_0) = 1 \) Thus, \( \psi'(m_0) < 0 \) and it follows that \( t_j = \psi(m_0) \) is monotonically decreasing with \( m_0 \) and hence invertible. Therefore Eq. (32) is a valid equation to compute \( f_i(t_j) \).

1. **Approximate Method 1**

Take \( n \) consecutive points from the domain of \( m_0 \), denoted as \( m^i_0 \), \( i = 1, \ldots, n \), so that \( m^1_0 < m^2_0 < \ldots < m^n_0 \). Each of these points determine a value \( t^{i}_j \) by solving the mass equation (3) with initial condition \( m^i_0 \) and stopping when \( m = m^i_0 - m_F \). Then, combining Eqs. (32) and (35),

\[
f_i(t^i_j) = \frac{f_{m_0}(\psi^{-1}(t^i_j))}{\psi'(\psi^{-1}(t^i_j))} = \frac{f_{m_0}(m^i_0)}{\psi'(\psi^{-1}(t^i_j))} \tag{36}
\]

where \( \frac{dm}{dm_0}(t^i_j; m^i_0) \) is obtained by computing Eq. (26) for \( t = t^i_j \) and \( m = m^i_0 \). Thus, the value of \( f_i(t_j) \) at \( n \) points is obtained.

2. **Approximate Method 2**

Take \( n \) consecutive points from the domain of \( m_0 \), as before, each of which determines a value \( t^i_j \). It has to be noted that, because it was found before that \( \psi'(m_0) \) is continuous, the intermediate value theorem for integrals can be applied to find

\[
Pr_t (t^{i+1}_j \leq t_j \leq t^i_j) = \int_{t^{i+1}_j}^{t^i_j} f_i(t_j; m_0) \, d\mu = (t^i_j - t^{i+1}_j) f_i(\xi^i) \tag{37}
\]

where \( \xi^i \in [t^{i+1}_j, t^i_j] \), for \( i = 1, \ldots, n - 1 \).

Reasoning as in Sec. VB, it can be seen that intervals in the initial condition \( m_0 \) are univocally mapped into intervals of \( t_j \). However, noting that increasing values of \( m_0 \) produce decreasing values of \( t_j \), one has that the interval \( (m^i_0, m^{i+1}_0) \) is mapped into the interval \( (t^{i+1}_j, t^i_j) \). Thus, it is deduced that \( Pr_t (t^{i+1}_j \leq t_j \leq t^i_j) = Pr(m^i_0 \leq m_0 \leq m^{i+1}_0) \), hence

\[
f_i(\xi^i) = \frac{Pr(m^i_0 \leq m_0 \leq m^{i+1}_0)}{t^i_j - t^{i+1}_j}, \quad i = 1, \ldots, n - 1 \tag{38}
\]

Taking

\[
f_i(t^i_j) = f_i(\xi^i)
\]

\[
f_i(t^{i+1}_j) = f_i(\xi^{i+1}) + f_i(\xi^i), \quad i = 2, \ldots, n - 1
\]

\[
f_i(t^1_j) = f_i(\xi^1)
\]

an approximation of \( f_i(t_j) \) is obtained at \( n \) points.

**B. Distribution Function of the Fuel Consumption**

The fuel consumption \( m_F \) is defined explicitly by Eq. (9) as a function of the initial mass; thus \( m_F \) is a random variable itself. Calling this function \( m_F = \psi(m_0) \), the distribution function of \( m_F \) is given, similar to Eq. (23), by

\[
f_{m_F}(m_F) = \frac{f_{m_0}(\psi^{-1}(m_F))}{\psi'(\psi^{-1}(m_F))} \tag{40}
\]

if \( \psi \) is invertible. To prove that this is the case, notice from Eq. (9) that

\[
\psi'(m_0) = \frac{dm_F}{dm_0}(m_0) = 1 - \frac{dm}{dm_0}(x_j; m_0) \tag{41}
\]

Similar to Eq. (24), the variable \( \frac{dm}{dm_0} \) satisfies now a differential equation with respect to distance

\[
\frac{d}{dx} \left( \frac{dm}{dm_0} \right) = -\frac{2Bm}{V} \frac{dm}{dm_0} \tag{42}
\]

with initial condition [from Eq. (4)]

\[
\frac{dm}{dm_0}(0) = 1 \tag{43}
\]

whose solution is given by

\[
\frac{dm}{dm_0}(x; m_0) = \exp \left( -\frac{2B}{V} \int_0^x m(x; m_0) \, dx \right) \tag{44}
\]

Thus, from Eq. (41), one has \( \psi'(m_0) > 0 \) for \( x_j > 0 \) which implies invertibility of \( \psi(m_0) \). Hence, Eq. (40) is a valid equation to compute \( f_{m_F}(m_F) \).

1. **Approximate Method 1**

Take \( n \) consecutive points from the domain of \( m_0 \), denoted as \( m^i_0 \), \( i = 1, \ldots, n \), so that \( m^1_0 \leq m^2_0 \leq \ldots \leq m^n_0 \). Each of these points determines a value \( m^i_F = m^i_0 - m(x_j; m^i_0) \) by solving the mass equation (2) with initial condition \( m^i_0 \) and stopping when \( x = x_j \). Then, using Eq. (41),

\[
f_{m_F}(m^i_F) = \frac{f_{m_0}(\psi^{-1}(m^i_F))}{\psi'(\psi^{-1}(m^i_F))} = \frac{f_{m_0}(m^i_0)}{1 - (\frac{dm}{dm_0})(x_j; m^i_0)} \tag{45}
\]

where \( \frac{dm}{dm_0}(x_j; m^i_0) \) is obtained by computing Eq. (44) for \( x = x_j \) and \( m_0 = m^i_0 \). Thus, the value of \( f_{m_F} \) at \( n \) points is obtained.

2. **Approximate Method 2**

Take \( n \) consecutive points from the domain of \( m_0 \), as before, each of which determines a value \( m^i_F \). Because it was found before that \( \psi'(m_0) > 0 \), increasing values of \( m_0 \) produce increasing values of \( m_F \). As it was done for the distribution of the mass, the intermediate value theorem for integrals can be applied to find

\[
Pr(m^i_F \leq m_F \leq m^{i+1}_F) = \int_{m^i_F}^{m^{i+1}_F} f_i(t_j; m_0) \, d\mu = (m^{i+1}_F - m^i_F) f_{m_F}(\xi^i) \tag{46}
\]

where \( \xi^i \in [m^i_F, m^{i+1}_F] \), for \( i = 1, \ldots, n - 1 \).
Reasoning as in Sec. V.B, it can be seen that intervals in the initial condition \( m_0 \) are univocally mapped into intervals of \( m_F \). Thus, it is deduced that 
\[
Pr(m_F^i \leq m_F \leq m_F^{i+1}) = Pr(m_0^i \leq m_0 \leq m_0^{i+1}),
\]

\[
f_{m_F}(\xi^i) = \frac{Pr(m_0^i \leq m_0 \leq m_0^{i+1})}{m_F^{i+1} - m_F^i}, \quad i = 1, \ldots, n - 1
\]  

(47)

Taking 
\[
f_{m_F}(m_F^1) = f_{m_F}(\xi^1)
\]
\[
f_{m_F}(m_F^i) = \frac{f_{m_F}(\xi^{i-1}) + f_{m_F}(\xi^i)}{2}, \quad i = 2, \ldots, n - 1
\]
\[
f_{m_F}(m_F^n) = f_{m_F}(\xi^{n-1})
\]

(48)

an approximation of \( f_{m_F} \) is obtained at \( n \) points.

## VII. Results

Now, the methods presented in previous sections are applied to the two initial mass distributions defined in Sec. III. The numerical resolution of the different problems is performed using the MATLAB environment. The numerical results are compared with the exact results of the problem, so that their accuracy can be assessed; these exact results are presented in the Appendix.

For the numerical application, the following values are used: \( C_d = 0.015, \ C_l = 0.042, \ \rho = 0.5 \rho_0, \ \rho_0 = 1.225 \text{ kg/m}^3, \ V = 200 \text{ m/s}, \ c = 5 \cdot 10^{-5} \text{ s/m}, \ S = 150 \text{ m}^2, \ g = 9.8 \text{ m/s}^2, \ m_k = 25,000 \text{ kg} \) in the case of a given fuel load, and \( x_f = 2500 \text{ km} \) in the case of a given range. For the initial mass distributions, the nominal values chosen for mean and width are \( \bar{m}_0 = 81,633 \text{ kg} \) and \( \delta_m = 5000 \text{ kg} \), which yields a typical deviation 
\[
\sigma[m_0] = \sqrt{\text{Var}[m_0]} = \frac{\delta_m}{\sqrt{3}} = 2887 \text{ kg}
\]

and for the gamma distribution, the nominal value \( k = 8.5 \) is considered. A parametric study as a function of \( \delta_m \) and \( k \) is also presented. For the nominal values, the two initial mass distributions are shown in Fig. 3. For the uniform distribution, the values of \( m_0 \) with nonzero probability are in the interval \([\bar{m}_0 - \delta_m, \bar{m}_0 + \delta_m] = \]

![Fig. 3 Probability density functions of the initial mass (\( \bar{m}_0 = 81,633 \text{ kg}, \delta_m = 5000 \text{ kg}, \text{and} \ k = 8.5 \)).](image)

![Fig. 4 GPC coefficients for the uniform distribution case.](image)

![Fig. 5 Evolution of mass mean and typical deviation for the uniform distribution case.](image)
A. Mass Mean and Variance

1. Uniform Distribution of the Initial Mass

To find the mean and variance using GPC, the value $P = 3$ is chosen in the GPC expansion of the mass [Eq. (12)], which, as already mentioned, is enough to obtain a good representation of $m$. The coefficients of the GPC expansion are shown in Fig. 4. Note the fast decrease of their order of magnitude (six orders of magnitude from $h_0$ to $h_1$).

The evolution of mean and variance is negligible; the absolute error is less than $4 \cdot 10^{-5}$ for the mean and less than $2 \cdot 10^{-3}$ for the typical deviation. Again, both the mean and the standard deviation decrease with time.

Although the fact that the mean mass decreases with time is to be expected (because fuel mass in consumed), it is remarkable that the standard deviation of the mass also decreases with time. Thus, the dispersion of the distribution function and, therefore, the uncertainty decreases with time. This result can be explained by noting that the larger the aircraft mass, the larger its rate of decrease (which is given at each instant by $\Delta m = \Delta m(t)$). Thus, if one computes the solution $m(t)$ given by Eq. (5) for $m_0 = \bar{m}_0 \pm \delta m$, say, $m_+(t) = m(t; \bar{m}_0 + \delta m)$ and $m_-(t) = m(t; \bar{m}_0 - \delta m)$, the distance $\Delta m(t) = m_+(t) - m_-(t)$ decreases with time: For example, at $t = 0$, one has $\Delta m = 25\Delta m_0 = 10,000$ kg and at $t = 1.2 \times 10^4$ s, $\Delta m = 8321$ kg.

B. Distribution Function of the Mass

1. Uniform Distribution of the Initial Mass

The mass distribution is represented at several time instants in Fig. 8. Both approximate methods developed in Sec. V to approximate Eq. (23) show excellent agreement with the exact analytical results [Eqs. (A12)] and are indistinguishable from them. The results in Fig. 8 show that, as time increases (and $m$ decreases), the width of the distribution function decreases, whereas the probability density increases. Thus, uncertainty decreases with time (as it was seen in Fig. 5). Note also that the uniform shape is approximately maintained.

2. Gamma Distribution of the Initial Mass

In this case, the mass distribution is represented at several time instants in Fig. 9. Again, both numerical methods developed in Sec. V

### Table 1: Values of mass mean and typical deviation at selected times for the uniform distribution case

<table>
<thead>
<tr>
<th>Time, s</th>
<th>$E[m(t; m_0)]$, kg</th>
<th>$\sigma[m(t; m_0)]$, kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \times 10^3$</td>
<td>77,485</td>
<td>2787</td>
</tr>
<tr>
<td>$4 \times 10^3$</td>
<td>73,477</td>
<td>2696</td>
</tr>
<tr>
<td>$6 \times 10^3$</td>
<td>69,596</td>
<td>2613</td>
</tr>
<tr>
<td>$8 \times 10^3$</td>
<td>65,831</td>
<td>2536</td>
</tr>
<tr>
<td>$10^4$</td>
<td>62,175</td>
<td>2467</td>
</tr>
<tr>
<td>$1.2 \times 10^4$</td>
<td>58,616</td>
<td>2402</td>
</tr>
</tbody>
</table>

[76, 633, 86, 633] in kilograms, and for the gamma distribution, they are in $[m, \infty) = [73, 217, \infty)$ in kilograms.

For the GPC method, the number of terms used in the expansions is $P = 3$, which turns out to be enough to obtain a good representation of $m$. In the computation of the distribution functions, the number of discretization points considered is $n = 1000$, which has proven to be good enough. All the integrations have been performed using the MATLAB environment.

In Sec. VII.A, the GPC method is applied to obtain the evolution of mass mean and variance. The distribution function of the mass and its evolution in time are analyzed in Sec. VII.B. The distribution function of the flight time in the case of a given fuel load is studied in Sec. VII.C and that of the fuel consumption in the case of a given range is studied in Sec. VII.D.

### Fig. 6 GPC coefficients for the gamma distribution case ($k = 8.5$).
Fig. 7 Evolution of mass mean and typical deviation for the gamma distribution case \((k = 8.5)\).

to approximate Eq. (23) show excellent agreement with the exact analytical results \[\text{Eq. (A14)}\] and are indistinguishable from them. As in Figs. 8 and 9 show that uncertainty decreases as time increases. Also, the shape of the distribution function is approximately of gamma type at all times.

C. Distribution Function of the Flight Time

1. Uniform Distribution of the Initial Mass

The distribution function of the flight time is represented in Fig. 10. Note that it looks approximately uniform, similar to the initial mass distribution. As in the computation of the mass distribution function, both approximate methods developed in Sec. V to approximate Eq. (32) show excellent agreement with the exact analytical result \[\text{Eq. (A19)}\]. The values of \(t_f\) with nonzero probability are those in the interval \([T_1, T_2] = [12.625, 13.664]\) in seconds, for values of \(m_0\) with nonzero probability in \([\bar{m}_0 - \delta_m, \bar{m}_0 + \delta_m] = [76.633, 86.633]\) in kilograms, where, as shown in the Appendix,

\[
T_1 = \frac{1}{\sqrt{AB}} \arctan \left( \frac{\sqrt{ABm_0}}{A + B(\bar{m}_0 + \delta_m)(\bar{m}_0 + \delta_m - m_f)} \right)
\]

\[
T_2 = \frac{1}{\sqrt{AB}} \arctan \left( \frac{\sqrt{ABm_0}}{A + B(\bar{m}_0 - \delta_m)(\bar{m}_0 - \delta_m - m_f)} \right)
\]

The mean and the typical deviation of the flight time are obtained using the distribution function, computed numerically from

\[
E[t_f] = \int_0^\infty t_f f_{t_f}(t_f) \, dt_f
\]

\[
\sigma[t_f] = \left( \int_0^\infty t_f^2 f_{t_f}(t_f) \, dt_f - (E[t_f])^2 \right)^{\frac{1}{2}}
\]

The results are given in Table 3.

2. Gamma Distribution of the Initial Mass

The distribution function in this case is represented in Fig. 10. Note that this distribution function is somewhat different from a gamma distribution, because the values of \(t_f\) with nonzero probability are

**Table 2** Values of mass mean and typical deviation at selected times for the gamma distribution case \((k = 8.5)\)

<table>
<thead>
<tr>
<th>Time, s</th>
<th>(E[m(t; m_0)]), kg</th>
<th>(\sigma[m(t; m_0)]), kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 \times 10^4</td>
<td>77,485</td>
<td>2786</td>
</tr>
<tr>
<td>4 \times 10^4</td>
<td>73,477</td>
<td>2695</td>
</tr>
<tr>
<td>6 \times 10^4</td>
<td>69,596</td>
<td>2610</td>
</tr>
<tr>
<td>8 \times 10^4</td>
<td>65,831</td>
<td>2533</td>
</tr>
<tr>
<td>10^5</td>
<td>62,175</td>
<td>2462</td>
</tr>
<tr>
<td>1.2 \times 10^5</td>
<td>58,616</td>
<td>2397</td>
</tr>
</tbody>
</table>
those in the finite interval \((0, T] = (0, 1.4019)\) in seconds, for values of \(m_0\) with nonzero probability in \([M_0, \infty) = [73, 217, \infty)\) in kilograms, where, as shown in the Appendix,

\[
T = \frac{1}{\sqrt{AB}} \arctan \left( \frac{\sqrt{ABm_F}}{A + BM_0(M_0 - m_F)} \right) \tag{53}
\]

Moreover, because \(t_f\) decreases when \(m_0\) increases, the bell of the distribution is sort of inverted (with respect to the bell of the initial mass distribution).

As in the computation of the mass distribution function, both numerical methods developed in Sec. V to approximate Eq. (32) show excellent agreement with the exact analytical result [Eq. (A21)].

Again, the mean and the typical deviation are computed numerically from Eqs. (51) and (52) using the distribution function. The results are given in Table 4.

Note that these results of mean and standard deviation are very close to the ones obtained before for the uniform distribution (especially for the exact distribution functions), showing again that the initial mass distribution chosen affects the results very weakly.

Now the effect of \(k\) on the results is analyzed. Values of \(\sigma(t_f)\) for different values of \(k\) and \(\delta_m\) (obtained using the exact solution) are given in Fig. 12, in which it is seen that there is no significant effect from changing \(k\) and, as in the uniform distribution case, there is a proportionality between the values of \(\sigma(t_f)\) and \(\delta_m\). The values of \(E(t_f)\) are not significantly affected by changing \(k\) or \(\delta_m\).

D. Distribution Function of the Fuel Consumption

1. Uniform Distribution of the Initial Mass

The distribution function of the fuel consumption is represented in Fig. 13. Note that it looks approximately uniform, similar to the initial mass distribution, although smaller values of \(m_F\) show a slightly higher probability. As before, both approximate methods developed in Sec. V to approximate Eq. (40) show excellent agreement with the exact analytical results [Eq. (A25)].

The values of \(m_F\) with nonzero probability are those in the interval \([M_1, M_2] = [23, 1043, 24, 775]\) in kilograms, for values of \(m_0\) with nonzero probability in \([\bar{m}_0 - \delta_m, \bar{m}_0 + \delta_m] = [76, 633, 86, 633]\) in kilograms, where, as shown in the Appendix,

\[
M_1 = \frac{(\bar{m}_0 - \delta_m)^2 + \frac{x_2}{2} \tan(\frac{1}{2} \sqrt{ABx_f})}{\frac{1}{2} + (\bar{m}_0 - \delta_m) \tan(\frac{1}{2} \sqrt{ABx_f})} \tag{54}
\]

As for the flight time, the mean and the typical deviation of the fuel consumption are obtained using the distribution function, computed numerically from

\[
E(m_F) = \int_0^\infty m_f f_{m_F}(m_F) \, dm_F \tag{56}
\]

\[
(\sigma(m_F))^2 = \int_0^\infty m_f^2 f_{m_F}(m_F) \, dm_F - (E(m_F))^2 \tag{57}
\]

The results are given in Table 5.

Now the effect of \(\delta_m\) on the results is analyzed. Values of \(\sigma(m_F)\) for different values of \(\delta_m\) (obtained using the exact solution) are given in Fig. 14, in which one can see that there is a proportionality between both parameters. The values of \(E(m_F)\) are not significantly affected by changing \(\delta_m\).

2. Gamma Distribution of the Initial Mass

The distribution function in this case is represented in Fig. 13 for \(k = 8.5\). Note that its shape is approximately gamma type, as the

![Fig. 11 Typical deviation of the flight time vs \(\delta_m\) in the uniform distribution case.](image-url)

![Fig. 13 Distribution functions of the fuel consumption: a) uniform distribution case and b) gamma distribution case (\(k = 8.5\)).](image-url)
initial mass distribution. The values of \( m_F \) with nonzero probability are those in the interval \([M, \infty) = [22, 499, \infty)\) in kilograms, for values of \( m_0 \) with nonzero probability in \([M_0, \infty) = [73, 217, \infty)\) in kilograms, where, as shown in the Appendix,

\[
M = \frac{(M_0^2 + \frac{\delta}{2}) \tan(\frac{1}{2} \sqrt{ABx_F})}{\sqrt{\pi} + M_0 \tan(\frac{1}{2} \sqrt{ABx_F})} \tag{58}
\]

As before, the approximate methods developed in Sec. V to approximate Eq. (40) show excellent agreement with the exact analytical result (Eq. (A27)).

Again, the mean and the typical deviation are computed numerically from Eqs. (56) and (57) using the distribution function. The results are given in Table 6. Note that these results of mean and standard deviation are very close to the ones obtained before for the uniform distribution, showing again that the initial mass distribution chosen affects very weakly the results.

Now the effect of \( k \) on the results is analyzed. Values of \( \sigma[m_F] \) for different values of \( k \) and \( \delta_m \) (obtained using the exact solution) are given in Fig. 15, in which one can see that the influence of \( k \) in \( \sigma[m_F] \) is negligible. Also, as in the uniform distribution case, there is a proportionality between the values of \( \sigma[m_F] \) and \( \delta_m \). The values of \( E[m_F] \) are not significantly affected by changing \( k \) or \( \delta_m \).

VIII. Conclusions

The problem of propagation of initial mass uncertainty in cruise flight has been studied, using a nonlinear model that has known analytical solution. To study the evolution of mean and variance of the aircraft mass, the generalized polynomial chaos (GPC) method has been used, in which an expansion with just four terms has proven to be accurate enough. The study of the evolution of the mass distribution function has also been considered and two approximate methods have been developed. These two methods are applicable to problems in which there is just one random variable and for the analysis of distribution functions of functions of the random variable which are invertible. Using these methods, the distribution functions of the flight time in the case of a given fuel load, and of the fuel consumption in the case of a given range, have been also studied. The results obtained with these methods have been compared with the exact analytical results, showing an excellent agreement in all cases; thus, the accuracy of the methods has been assessed and, therefore, they are proposed as accurate and computationally efficient candidates to study uncertainty propagation.

The results presented in this work show that both mass mean and standard deviation decrease with time, with the distribution function getting narrower and more concentrated around the mean; thus, an important conclusion of this analysis is that uncertainty (represented by the dispersion of the distribution function) decreases with time. On the other hand, the shape of the distribution function of the mass is fundamentally unchanged from its initial shape. The results also show that the values of both mean and standard deviation are very weakly affected by the specific distribution function chosen for the initial mass (at least in the uniform and gamma cases).

The distribution functions of other flight properties different from mass (flight time and fuel consumption) have been analyzed as well, and their main statistical properties have been computed. Again, it has been shown that the results are affected very weakly by the choice of the initial mass distribution. The influence of the parameters of the initial mass distributions (\( \delta_m \) and \( k \)) has been studied: The mean is not significantly affected by changing \( \delta_m \) or \( k \), and the typical deviation varies almost linearly with \( \delta_m \) and is not affected by \( k \). In these cases, the mean and variance have been obtained directly using the known distribution functions (and not the GPC method, as in the case of the mass distribution).

The approximate methods developed in this paper can be applied to other flight phases defined by more complicated flight conditions, and they can be extended to consider other sources of uncertainty, not only in the initial conditions but, for example, persistently affecting the system, such as wind. The analysis of these problems is left for future work.

Appendix: Exact Results

A1. Mean and Typical Deviation of the Mass

In this Appendix, the different analytic expressions used for comparison purposes throughout the paper are presented, and their derivation is briefly explained. To simplify the notation, the following parameters are defined: \( c_1(t) = \tan(\sqrt{AB}) \geq 0 \),

\[
c_2 = \sqrt{\frac{A}{B}} > 0
\]

and

\[
\begin{array}{|c|c|c|c|}
\hline
\delta_m, \text{ kg} & 23,891 & 23,891 & 23,894 \\
\hline
\sigma[m_F], \text{ kg} & 506.46 & 506.37 & 506.46 \\
\hline
\end{array}
\]
\[ c_1 = \tan \left( \frac{1}{\sqrt{A}} \right) > 0 \]

A. Uniform Distribution of the Initial Mass

The analytical value of the mean is computed directly from Eq. (5), obtaining

\[
E[m(t; m_0)] = \frac{1}{\Delta m} \int_{m_0-\delta_m}^{m_0+\delta_m} m(t; m_0) \, dm_0
\]

\[
= \frac{c_2}{\Delta m} \int_{m_0-\delta_m}^{m_0+\delta_m} (c_2 + (m_0 + \delta_m)c_1(t)) \, dm_0
\]

\[
= \frac{c_2}{c_1(t)} \left( 1 - \frac{c_2}{c_1(t)} \right) \Delta m \left[ (m_0 + \delta_m)c_1(t) \right]
\]

(A1)

Similarly, the computation of the variance of \(m(t)\) gives

\[
\text{Var}[m(t; m_0)] = E[m^2(t; m_0)] - (E[m(t; m_0)])^2
\]

\[
= \frac{1}{\Delta m} \int_{m_0-\delta_m}^{m_0+\delta_m} m^2(t; m_0) \, dm_0 - (E[m(t; m_0)])^2
\]

\[
= \frac{c_2^2}{\Delta m} \int_{m_0-\delta_m}^{m_0+\delta_m} (c_2 + (m_0 + \delta_m)c_1(t))^2 \, dm_0
\]

\[
= \frac{c_2^2}{c_1(t)} \left[ (c_1(t) + 1)^2 \Delta m \left[ (m_0 + \delta_m)c_1(t) \right] - \frac{4c_4}{3c_2} \delta_m \delta_{\delta_m} \right]
\]

(A2)

Expressions (A1) and (A2) are both indeterminate for \(t = 0\) (which implies \(c_1 = 0\)). For numerical purposes, it is convenient to approximate both expressions up to order 3 in \(1/c_1\) using the asymptotic series

\[ \Gamma(s, x) = x^{s-1}e^{-x} \left( 1 + (s-1) \frac{1}{x} + (s-1)(s-2) \frac{1}{x^2} + \ldots \right) \]

valid for \(x \to \infty\) (see [18]). It follows that

\[
E[m(t; m_0)] \approx \frac{c_2^3 m_0 - c_2^2 c_1}{c_1 M_0 + c_2} + \frac{\delta_0^2}{3} \left( c_1^2 + \frac{\delta_0^2}{3} \right) c_1 M_0 + c_2 \]

(A3)

B. Gamma Distribution of the Initial Mass

For the gamma distribution, the exact value of the mean obtained from Eq. (5) is

\[
E[m(t; m_0)] = c_2 \int_0^\infty m_0 - c_1 c_2 (m_0 - M_0)^k e^{-\left( (m_0 - M_0)^{\frac{1}{1+k}} \right)} \, dm_0
\]

\[
= \frac{c_2}{c_1(t)} \left[ \frac{c_1(t)(1+1)}{c_1(t)} e^{\frac{\delta_0^2}{\Delta m} M_0 + \frac{\delta_0^2}{\Delta m}} \right]
\]

\[
\times \left[ \frac{M_0 + \frac{\delta_0^2}{\Delta m}}{\frac{\delta_0^2}{\Delta m}} \right] \Gamma \left( 1 - k, \frac{M_0 + \frac{\delta_0^2}{\Delta m}}{\frac{\delta_0^2}{\Delta m}} \right)
\]

(A5)

where \(M_0\) is defined by Eq. (10) and \(\Gamma(s, x)\) is the upper incomplete Euler gamma function defined as

\[ \Gamma(s, x) = \int_x^\infty t^{s-1}e^{-t} \, dt \]

(see [18]).

The variance of \(m(t)\) is as follows:

\[
\text{Var}[m(t; m_0)] = E[m^2(t; m_0)] - (E[m(t; m_0)])^2
\]

\[
= \frac{c_2^3}{c_1(t)} \int_0^\infty \left( m_0 - c_1 c_2 \right)^2 (m_0 - M_0)^k e^{\frac{\delta_0^2}{\Delta m} M_0 + \frac{\delta_0^2}{\Delta m}} \, dm_0
\]

\[
\times \left[ \frac{M_0 + \frac{\delta_0^2}{\Delta m}}{\frac{\delta_0^2}{\Delta m}} \right] \Gamma \left( 1 - k, \frac{M_0 + \frac{\delta_0^2}{\Delta m}}{\frac{\delta_0^2}{\Delta m}} \right)
\]

\[
\times \left[ \frac{M_0 + \frac{\delta_0^2}{\Delta m}}{\frac{\delta_0^2}{\Delta m}} \right] \Gamma \left( 1 - k, \frac{M_0 + \frac{\delta_0^2}{\Delta m}}{\frac{\delta_0^2}{\Delta m}} \right)
\]

(A6)

Expressions (A5) and (A6) are both indeterminate for \(t = 0\) (which implies \(c_1 = 0\)). For numerical purposes, it is convenient to approximate both expressions up to order 3 in \(1/c_1\) using the asymptotic series

\[ \Gamma(s, x) = x^{s-1}e^{-x} \left( 1 + (s-1) \frac{1}{x} + (s-1)(s-2) \frac{1}{x^2} + \ldots \right) \]

valid for \(x \to \infty\) (see [18]). It follows that

\[
E[m(t; m_0)] \approx \frac{c_2^3 M_0 - c_2^2 c_1}{c_1 M_0 + c_2} + k \frac{c_2^3 (c_1^2 + 1)}{c_1 M_0 + c_2} \left( \frac{\delta_0}{\Delta m} \right)^2 \left( \frac{c_1}{c_1 M_0 + c_2} \right)^2
\]

(A7)

B2. Distribution Function of the Mass

To compute the distribution function of the mass, note that \(\frac{\partial m}{\partial m_0}\) can be exactly computed from Eq. (5) as

\[
\frac{\partial m}{\partial m_0} (t; m_0) = \frac{A}{B} \left( 1 + \tan^2(\sqrt{AB}) \right) \frac{c_2^2}{c_2^2 + c_1 M_0 + c_2}
\]

(A9)

Also from Eq. (5), \(m_0\) can be written in terms of \(m(t)\) as follows:
Thus \( \frac{\partial m}{\partial m_0} \) in terms of \( m \) is written as

\[
\frac{\partial m}{\partial m_0}(t; m) = \frac{(c_2 - c_1(t)m)^2}{c_2^2(1 + c_1(t)m)}
\]  

(A11)

A. Uniform Distribution of the Initial Mass

Since

\[
f_{m_0} = \frac{1}{2\delta_m}
\]

using Eqs. (23) and (A11), the exact distribution function of the mass as a function of time is

\[
f_m(m, t) = \frac{c_2^2(1 + c_1(t)m)^2}{2\delta_m(c_2 - c_1(t)m)^2}
\]  

(A12)

if

\[
m \in \left[ \frac{c_2(m_0 - \delta_0) - c_2^2c_1(t)}{c_2 - c_1(t)m}, \frac{c_2(m_0 + \delta_0) - c_2^2c_1(t)}{c_2 + c_1(t)(m_0 + \delta_0)} \right] 
\]

and zero otherwise. The limit points in the interval have been found from Eq. (5) evaluated at the limit points in the initial mass distribution \((m_0 - \delta_0)\) and \((m_0 + \delta_0)\).

B. Gamma Distribution of the Initial Mass

In this case, the distribution function \(f_{m_0}\), given by Eq. (11), has to be written in terms of \( m(t) \) using Eq. (A10) as follows:

\[
f_{m_0}(m, t) = \left( \frac{c_2m + c_2^2c_1(t)}{c_2 - c_1(t)m} - M_0 \right)^{k-1} e^{\frac{-\sqrt{\frac{m}{\delta_m} - c_2^2c_1(t)m}}{c_2 - c_1(t)m}} \quad (A13)
\]

Then, using Eqs. (23), (A11), and (A13), the exact distribution function of the mass as a function of time is

\[
f_m(m, t) = \left( \frac{c_2m + c_2^2c_1(t)}{c_2 - c_1(t)m} - M_0 \right)^{k-1} e^{\frac{-\sqrt{\frac{m}{\delta_m} - c_2^2c_1(t)m}}{c_2 - c_1(t)m}} \times \frac{c_2^2(1 + c_1(t)m)^2}{(c_2 - c_1(t)m)^2}
\]  

(A14)

if

\[
m \geq \frac{c_2M_0 - c_2^2c_1(t)}{c_2 - c_1(t)m}
\]

and zero otherwise. The lower limit is found evaluating Eq. (5) at \( m_0 = M_0 \).

A3. Distribution Function of the Flight Time

To compute Eq. (32), using Eq. (6), the values of \( \varphi \) and its inverse can be explicitly obtained as

\[
t_f = \varphi(m_0) = \frac{1}{\sqrt{AB}} \arctan \left( \frac{\sqrt{AB}m_F}{A + Bm_0(m_0 - m_F)} \right)
\]  

(A15)

\[
m = \varphi^{-1}(t_f) = \frac{m_F}{2} + \frac{m_F^2}{4} - \frac{\sqrt{\frac{\delta_m}{B}m_F}}{\tan(\sqrt{AB}t_f)}
\]  

(A16)

where

\[
\Phi(t_f) = \sqrt{\frac{\delta_m}{B}m_F} \tan(\sqrt{AB}t_f)
\]

is defined to simplify the expressions. Also \( \varphi'(m_0) \) is given by

\[
\varphi'(m_0) = \frac{-Bm_F(2m_0 - m_F)}{(A + Bm_0(m_0 - m_F))^2 + ABm_F^2}
\]  

(A17)

Hence,

\[
|\varphi'(\varphi^{-1}(t_f))| = \frac{2\sin^2(\sqrt{AB}t_f)}{Am_F} \Phi(t_f)
\]  

(A18)

These results are now used to derive an explicit expression for \( t_f \), for the two initial mass distributions under consideration.

A. Uniform Distribution of the Initial Mass

From Eq. (32), using

\[
f_{m_0} = \frac{1}{2\delta_m}
\]

and Eq. (A18), the resulting expression for the exact distribution function of the flight time is

\[
f_{t_f}(t_f) = \frac{Am_F}{4\delta_m \sin^2(\sqrt{AB}t_f)} \Phi(t_f)
\]  

(A19)

if \( t_f \in [T_1, T_2] \), and zero otherwise, where the endpoints of this interval are found evaluating Eq. (A15) at the endpoints of the uniform distribution of \( m_0 \) (namely, \( m_0 - \delta_0 \) and \( m_0 + \delta_0 \)) and are given by

\[
T_1 = \frac{1}{\sqrt{AB}} \arctan \left( \frac{\sqrt{AB}m_F}{A + B(m_0 + \delta_0)(m_0 + \delta_0 - m_F)} \right)
\]

and

\[
T_2 = \frac{1}{\sqrt{AB}} \arctan \left( \frac{\sqrt{AB}m_F}{A + B(m_0 - \delta_0)(m_0 - \delta_0 - m_F)} \right)
\]

B. Gamma Distribution of the Initial Mass

To find \( t_f \), now, the distribution function \( f_{m_0} \) for the gamma case [Eq. (11)] has to be written in terms of \( t_f \) using Eq. (A16) as follows:

\[
f_{m_0}(t_f) = \left( \frac{m_F}{2} + \Phi(t_f) - M_0 \right)^{k-1} e^{\frac{-\sqrt{\frac{m}{\delta_m} + \Phi(t_f) - M_0}}{c_2 - c_1(t)m}} \quad (A20)
\]

Then, from Eq. (32), using Eqs. (A24) and (A20), the resulting expression for the exact distribution function of the flight time is

\[
f_{t_f}(t_f) = \left( \frac{m_F}{2} + \Phi(t_f) - M_0 \right)^{k-1} \frac{Am_F}{2\sin^2(\sqrt{AB}t_f) (c_2 - c_1(t)m)^2} \Phi(t_f)
\]  

(A21)
for
\[ t_f \in \left( 0, \frac{1}{\sqrt{A}} \arctan \left( \frac{\sqrt{ABm_F}}{A + BM_0(M_0 - m_f)} \right) \right) \]
and zero otherwise. The upper limit value is found evaluating Eq. (A15) at \( m_0 = M_0 \).

A4. Distribution Function of the Fuel Consumption

To compute Eq. (40), the inverse of \( \psi'(m_0) \) is necessary. For that, one has to solve for \( m_0 \) in Eq. (9), finding
\[ m_0 = \psi^{-1}(m_F) = \frac{m_F}{2} + \frac{\sqrt{m_F^2 + m_Fc_2}}{c_3} - c_3^2 = \frac{m_F}{2} + \Psi(m_F) \tag{A22} \]
where
\[ \Psi(m_F) = \frac{\sqrt{m_F^2 + m_Fc_2}}{c_3} - c_3^2 \]
has been defined.

Also, taking the derivative with respect to \( m_0 \) in Eq. (9), the value of \( \psi'(m_0) \) is found as
\[ \psi'(m_0) = c_3 \left( \frac{c_3^2}{(c_2 + m(F)c_3)^2} \right) \tag{A23} \]
For Eq. (40), it is necessary to explicitly compute \( |\psi'(\psi^{-1}(m_F))| \) using Eqs. (A22) and (A23), finding
\[ |\psi'(\psi^{-1}(m_F))| = \left| \frac{m_F + \Psi(m_F)}{c_3} \right| \tag{A24} \]
This result is now used to derive an explicit expression for \( f_{m_0} \), for the two initial mass distributions under consideration.

A. Uniform Distribution of the Initial Mass

From Eq. (40), using
\[ f_{m_0} = \frac{1}{2\delta_m} \]
and Eq. (A24), the resulting expression for the exact distribution function of the flight time is
\[ f_{m_0}(m_F) = \frac{1}{2\delta_m} \left| c_3 + 3 \left( \frac{m_F}{2} + \Psi(m_F) \right) \right|^2 \tag{A25} \]
if
\[ m_F \in \left[ \left( \left( \tilde{m}_0 - \delta_m \right)^2 + c_3^2 \right)c_3, \left( \left( \tilde{m}_0 + \delta_m \right)^2 + c_3^2 \right)c_3 \right] \]
and zero otherwise. The endpoints of this interval are found by evaluating Eq. (9) at the endpoints of the uniform distribution of \( m_0 \) (\( m_0 = \tilde{m}_0 - \delta_m \) and \( m_0 = \tilde{m}_0 + \delta_m \)).

B. Gamma Distribution of the Initial Mass

To find \( f_{m_0} \) now, the distribution function \( f_{m_0} \) for the gamma case [Eq. (11)] has to be written in terms of \( m_F \) using Eq. (A22) as follows:
\[ f_{m_0}(m_F) = \left( \frac{m_F}{2} + \Psi(m_F) - M_0 \right)^{k-1} e^{-\left( \frac{m_F}{2} + \Psi(m_F) - M_0 \right)\delta_m} \left( \frac{\delta_m}{\sqrt{\pi}} \right)^3 \Gamma(k) \tag{A26} \]
Then, from Eq. (40), using Eqs. (A24) and (A26), the resulting expression for the exact distribution function of the fuel consumption is
\[ f_{m_0}(m_F) = \frac{e^{-\left( \frac{m_F}{2} + \Psi(m_F) - M_0 \right)\delta_m} \left( \frac{m_F}{2} + \Psi(m_F) - M_0 \right)^{k-1} \left( c_2 + c_3 \left( \frac{m_F}{2} + \Psi(m_F) \right) \right)^2 \left( \frac{\delta_m}{\sqrt{\pi}} \right)^3 \Gamma(k) c_3}{\left( \frac{m_F}{2} + \Psi(m_F) \right)^2 c_3 + 2 \left( \frac{m_F}{2} + \Psi(m_F) \right) c_2 - c_3 c_2^3} \tag{A27} \]
for
\[ m_F \geq \left( \frac{M_0^2 + c_3^2}{c_2 + M_0 c_3} \right) \]
and zero otherwise. The lower limit is found evaluating Eq. (9) at \( m_0 = M_0 \).

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