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Propagation of Initial Mass Uncertainty in Aircraft Cruise Flight

Rafael Vazquez* and Damián Rivas†
Universidad de Sevilla, 41092 Seville, Spain

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The propagation of initial mass uncertainty in cruise flight is studied. Two cruise conditions are analyzed: one with given cruise fuel load and the other with given cruise range. Two different distributions of initial mass are considered: uniform and gamma type. The generalized polynomial chaos method is used to study the evolution of mean and variance of the aircraft mass. To compute the mass distribution function as a function of time, two approximate methods are developed. These methods are also applied to study the distribution functions of the flight time (in the case of given fuel load) and of the fuel consumption (in the case of given range). The dynamics of mass evolution in cruise flight is defined by a nonlinear equation, which can be solved analytically; this exact solution is used to assess the accuracy of the proposed methods. Comparison of the numerical results with the exact analytical solutions shows an excellent agreement in all cases, hence verifying the methods developed in this work.

Nomenclature

A, B	=	constants of the problem
C_D, C_L	=	drag and lift coefficients
C_{D_0}, C_{D_2}	=	coefficients of the drag polar
c	=	specific fuel consumption
D	=	aerodynamic drag
$E[.]$	=	expectation
f_x	=	probability density function of random variable x
$G(k;1)$	=	gamma distribution with scale parameter equal to one
g	=	gravity acceleration
h_i	=	coefficients of the generalized polynomial chaos expansion
k	=	shape parameter of the gamma distribution
L	=	lift
L_n	=	Legendre polynomials
M_0	=	minimum value of m_0 with nonzero probability for the gamma distribution
m	=	aircraft mass
m_F	=	fuel load
m_0	=	initial aircraft mass
$\bar{m}_0$	=	mean of the initial mass distribution
S	=	wing surface area
T	=	thrust
t	=	time
t_f	=	flight time
V	=	aircraft speed
Var[.]	=	variance
x	=	horizontal distance
x_f	=	range
$\Gamma(a)$	=	Euler gamma function
$\Gamma(a,b)$	=	incomplete Euler gamma function
Δ	=	standard uniform distribution
δ_m	=	width of the uniform distribution
$\sigma[.]$	=	typical deviation
ϕ_n^{k-1}	=	generalized Laguerre polynomials

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*Associate Professor, Department of Aerospace Engineering, Escuela Superior de Ingenieros; rvazquez1@us.es.

†Professor, Department of Aerospace Engineering, Escuela Superior de Ingenieros; drivas@us.es.

I. Introduction

THE air traffic management (ATM) system is a very complex system, which contains a large number of heterogeneous components, such as airports, aircraft, navigation systems, flight management systems (FMSs), traffic controllers, and weather (see Kim et al. [1]). Correspondingly, its performance is affected by numerous factors. Within the trajectory-based operations concept of SESAR and NextGen, aircraft trajectories are key to study ATM operations, which are subject to many uncertainties. Sources of uncertainty for aircraft trajectories include wind and severe weather, navigational errors, aircraft performance inaccuracies, or errors in the FMS, among others. The analysis of the impact of uncertainties in aircraft trajectories and its propagation through the flight segments is of great interest, because it might help to understand how sensitive the system is to the lack of precise data and measurement errors and, therefore, aid in the design of a more robust ATM system, with improved safety levels.

Among those sources, weather uncertainty has perhaps the greatest impact on ATM operations, being responsible for much of the delays. Its analysis has been addressed by many authors using different methods, for example, the following: Nilim et al. [2] consider a trajectory-based air traffic management scenario to minimize delays under weather uncertainty, in which the weather processes are modeled as stationary Markov chains. Pepper et al. [3] present a method, based on Bayesian decision networks, of accounting for uncertain weather information in air traffic flow management. Clarke et al. [4] develop a methodology to study airspace capacity in the presence of weather uncertainty and formulate a stochastic dynamic programming algorithm for traffic flow management. Zheng and Zhao [5] develop a statistical model of wind uncertainties and apply it to stochastic trajectory prediction in the case of straight, level aircraft flight trajectories.

The framework for this work is the analysis of uncertainty propagation in aircraft trajectories and, eventually, its effect on the ATM system. In this paper, several tools are presented to analyze uncertainty propagation in a nonlinear problem and they are applied to study the effect of initial aircraft mass uncertainty and its propagation through the cruise flight phase. The relevance of this problem resides in two facts: first, the initial mass is an important source of uncertainty in trajectory prediction, which determines mass evolution and, therefore, fuel consumption and flight cost; and, second, cruise uncertainties have a large impact on the overall flight because the cruise phase is the largest portion of the flight (at least for long-haul routes). In the applications, two cruise conditions are studied: one with given cruise fuel load and the other with given cruise range.

Several methods have been proposed to study uncertainty propagation in dynamic systems, beyond the classical Monte Carlo methods (which can be very expensive computationally). Halder and Bhattacharya [6] classify those methods in two categories: parametric (in which one evolves the statistical moments) and nonparametric (in which the probability density function is evolved). They address the problem of uncertainty propagation in planetary entry, descent, and landing, using a nonparametric method that reduces to solving the stochastic Liouville equation.

In this paper, the evolution in time of the mean and the variance of the aircraft mass is studied using the generalized polynomial chaos (GPC) method (a parametric method according to [6]). The GPC representation was introduced by Wiener [7] and is based on the fact that any second-order process (i.e., a process with finite second-order moments) can be represented as a Fourier-type series, with time-dependent coefficients, and using orthogonal polynomials as GPC basis functions in terms of random variables. A general introduction to GPC can be found in Xiu and Karniadakis [8] and in Schoutens [9], and details in numerical computations are studied in Debusschere et al. [10]. The method of polynomial chaos is used in the works of Prabhakar et al. [11] and Dutta and Bhattacharya [12] to study, respectively, uncertainty propagation and trajectory estimation, for hypersonic flight dynamics with uncertain initial data, and by Fisher and Bhattacharya [13] in the problem of optimal trajectory generation in the context of stochastic optimal control.

Also, the distribution function of the aircraft mass is analyzed using two approximate methods developed in this paper (nonparametric methods according to [6]). One method is based on the resolution of the variational equation for the sensitivity function with respect to the initial condition, and the other is based on the computation of the probability measure of the random variable as a function of time. These two methods are also applied, first, to the analysis of the distribution function of the flight time, in the case of a given fuel load and, second, to the analysis of the distribution function of the fuel consumption, in the case of a given range. In this way, the effect of the initial mass uncertainty in flight properties other than mass is studied as well.

In this paper, the case of cruise at constant altitude and constant speed is considered (cruise segments defined by these two flight constraints are commonly flown by commercial aircraft, according to air traffic control procedures). In this case, the evolution of aircraft mass is defined by a nonlinear equation that can be solved analytically. Results are presented for two different distributions of initial mass (uniform and gamma type). The analytical solutions represent benchmark solutions that are used to assess the accuracy of the proposed methods. Comparison with the exact analytical results is made, showing an excellent agreement in all cases.

This paper is organized as follows. First, the problem of mass evolution in cruise flight is solved. Then, in Sec. III, the two initial mass distributions considered are described. In Sec. IV, mean and variance of the mass distribution are analyzed using the GPC method. In Sec. V, the two nonparametric methods developed to study the evolution of distribution functions are presented and are applied to the mass distribution function. These two methods are used, in Sec. VI, to study the distribution functions of flight time and fuel consumption. Some numerical results are presented in Sec. VII, and some conclusions are drawn in Sec VIII. Finally, the exact analytical solutions are presented in the Appendix.

II. Mass Evolution in Cruise Flight

The equations of motion for symmetric flight in a vertical plane (constant heading), using a flat Earth model, for constant altitude and constant speed are (see [14])

$$\frac{\mathrm{d}x}{\mathrm{d}t} = V, \qquad \frac{\mathrm{d}m}{\mathrm{d}t} = -cT \quad T = D, \qquad L = mg \tag{1}$$

where x is the horizontal distance, t is the time, V is the speed, T is the thrust, D is the aerodynamic drag, L is the lift, m is the aircraft mass, g is the acceleration of gravity, and c is the specific fuel consumption, which can be taken as a function of altitude and speed, and it is therefore constant under the given cruise condition.

The drag can be written as

$$D = \frac{1}{2}\rho V^2 S C_D$$

where ρ is the density, S is the wing surface area, and the drag coefficient C_D is modeled by a parabolic polar $C_D = C_{D_0} + C_{D_2} C_L^2$, where C_L is the lift coefficient given by

$$C_L = \frac{2L}{\rho V^2 S}$$

and the coefficients C_{D_0} and C_{D_2} are constant under the given cruise condition. Using these definitions and Eq. (1), an autonomous equation for the mass evolution is obtained:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -c\left(\frac{1}{2}\rho V^2 S C_{D_0} + m^2 \frac{2C_{D_2} g^2}{\rho V^2 S}\right) \tag{2}$$

Thus, one can write

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -(A + Bm^2) \tag{3}$$

where the constants A and B are defined as

$$A = c\frac{1}{2}\rho V^2 S C_{D_0}$$

and

$$B = c\frac{2C_{D_2} g^2}{\rho V^2 S}$$

Note that $A, B > 0$. Equation (3) is a nonlinear equation describing the evolution of mass during cruise flight, to be solved with the initial condition

$$m(0) = m_0 \tag{4}$$

To emphasize the dependence of the mass $m(t)$ on the initial condition, the mass is written as $m(t; m_0)$, even though it is often just denoted as m for the sake of simplicity. The explicit solution of Eqs. (3) and (4) is

$$m(t; m_0) = \sqrt{\frac{A}{B}}\,\frac{m_0 - \sqrt{(A/B)}\tan(\sqrt{AB}t)}{\sqrt{(A/B)} + m_0 \tan(\sqrt{AB}t)} \tag{5}$$

A. Cruise with Given Fuel Load

For the case in which the cruise fuel load is given, denoting the given mass of fuel as $m_F < m_0$, the solution obtained by Eq. (5) is valid in the time interval $t \in [0, t_f(m_0)]$, where $t_f(m_0)$ (the flight time) is obtained from $m(t_f(m_0); m_0) = m_0 - m_F$. From Eq. (5), one can directly compute this time as

$$t_f(m_0) = \frac{1}{\sqrt{AB}}\arctan\left(\frac{\sqrt{AB}m_F}{A + Bm_0(m_0 - m_F)}\right) \tag{6}$$

Note that t_f is a monotonically decreasing function of m_0. Thus, for a given amount of fuel, the larger m_0, the smaller t_f and, as a consequence, the smaller the distance traveled by the aircraft. The initial mass m_0 is unbounded and has a lower limit equal to m_F (although these limits are not physically meaningful). Thus, for $m_0 \in (m_F, \infty)$ one obtains from Eq. (6)

$$t_f \in \left(0, \frac{1}{\sqrt{AB}}\arctan\left(\sqrt{\frac{B}{A}}m_F\right)\right)$$

Also, since $m(t_f; m_0) = m_0 - m_F$, the final value of the aircraft mass satisfies $m(t_f; m_0) \in (0, \infty)$.

In the next sections, the evolution of mass and the behavior of the flight time are studied for an uncertain value of the initial mass, whereas the rest of the parameters (some of them embedded in the constants A and B) have a fixed value.

B. Cruise with Given Range

For the case in which the cruise range is given, taking x as the independent variable, one has

$$\frac{\mathrm{d}m}{\mathrm{d}x} = -\frac{1}{V}(A + Bm^2) \tag{7}$$

and the same initial condition (4). The explicit solution of Eqs. (4) and (7) is

$$m(x; m_0) = \sqrt{\frac{A}{B}}\frac{m_0 - \sqrt{(A/B)}\tan[(1/V)\sqrt{AB}x]}{\sqrt{(A/B)} + m_0\tan[(1/V)\sqrt{AB}x]} \tag{8}$$

If the given cruise range is x_f, then the final value of the aircraft mass $m(x_f; m_0)$ is given by Eq. (8) particularized for $x = x_f$, and the fuel consumption during the cruise is

$$m_F(m_0) = m_0 - m(x_f; m_0) = \frac{[m_0^2 + (A/B)]\tan[(1/V)\sqrt{AB}x_f]}{\sqrt{(A/B)} + m_0\tan[(1/V)\sqrt{AB}x_f]} \tag{9}$$

Note that m_F is a monotonically increasing function of m_0: the larger m_0, the larger the fuel consumption. As before, m_0 is unbounded and, to have $m(x_f; m_0) > 0$, it has a lower limit equal to

$$\sqrt{\frac{A}{B}}\tan\left(\frac{1}{V}\sqrt{AB}x_f\right)$$

Thus for

$$m_0 \in \left(\sqrt{\frac{A}{B}}\tan\left(\frac{1}{V}\sqrt{AB}x_f\right), \infty\right)$$

one obtains from Eq. (9) that

$$m_F \in \left(\sqrt{\frac{A}{B}}\tan\left(\frac{1}{V}\sqrt{AB}x_f\right), \infty\right)$$

Also, from Eq. (8), the final value of the aircraft mass satisfies

$$m(x_f; m_0) \in \left(0, \sqrt{\frac{A}{B}}\left(\tan\left(\frac{1}{V}\sqrt{AB}x_f\right)\right)^{-1}\right)$$

In the next sections, the behavior of the fuel consumption is studied for an uncertain value of the initial mass, whereas, as before, the rest of the parameters have a fixed value. In this case, the flight time is known, trivially given by

$$t_f = \frac{x_f}{V}$$

III. Initial Mass Distribution

It is realistic to consider that the initial mass m_0 is not a deterministic variable which is known a priori, but rather a random variable which is not known. Then, the solution given by Eq. (5) is still valid but in a probabilistic sense [i.e., $m(t; m_0)$ is a random process]. If the distribution of m_0 is known, it is possible to study the time evolution of the distribution of the aircraft mass $m(t; m_0)$, as well as its statistical properties (mean, variance, typical deviation).

In this work, to analyze mass evolution, two probabilistic models for m_0 are considered: uniform and gamma distributions, which are described next. Note that a Gaussian distribution representing the initial mass uncertainty would be nonphysical, because it would allow (with small but nonzero probability) *negative* initial mass and, therefore, it is not considered in this paper.

A. Uniform Distribution

First, it is considered that m_0 is distributed as a uniform continuous variable whose probability density function is

$$f_{m_0}(m_0) = \frac{1}{2\delta_m}$$

in the interval $[\bar{m}_0 - \delta_m, \bar{m}_0 + \delta_m]$, and zero otherwise, where $\bar{m}_0$ is the mean and δ_m is the width of the uniform distribution, as shown in Fig. 1.

Denoting by Δ the standardized uniform distribution taking values in the interval $[-1, 1]$, one has that $m_0 = \bar{m}_0 + \delta_m\Delta$. The mean of m_0 is

$$E[m_0] = \int_0^\infty m_0 f_{m_0}(m_0)\,\mathrm{d}m_0 = \bar{m}_0$$

where $E[\cdot]$ is the mathematical expectation, and the variance of m_0 is

$$\mathrm{Var}[m_0] = E[m_0^2] - (E[m_0])^2 = \frac{\delta_m^2}{3}$$

B. Gamma Distribution

The gamma distribution (see [15]) represents a continuous nonnegative random variable and is denoted by $G(k, \theta)$, where $k > 0$ is the shape parameter and $\theta > 0$ is the scale parameter. It is known that $E[G(k, \theta)] = k\theta$ and $\mathrm{Var}[G(k, \theta)] = k\theta^2$ and that the probability density function of $G(k, \theta)$ is

$$f(x; k, \theta) = x^{k-1}\frac{e^{-x/\theta}}{\theta^k\Gamma(k)}$$

for $x \geq 0$ (and zero otherwise), where Γ is the Euler gamma function. Using the property that, for $\theta > 0$, one has $G(k, \theta) = \theta G(k, 1)$, the value $\theta = 1$ is considered in this paper without loss of generality.

To represent the initial mass distribution, let

$$m_0 = \bar{m}_0 + \frac{\delta_m}{\sqrt{3k}}(G(k, 1) - k)$$

where $\bar{m}_0$ and δ_m are the same values chosen for the uniform distribution. Hence, only the values $m_0 \geq M_0$ have nonzero probability, where M_0 (the minimum possible value of mass for the given values of $\bar{m}_0$ and δ_m) is obtained making $G(k, 1) = 0$ and it is given by

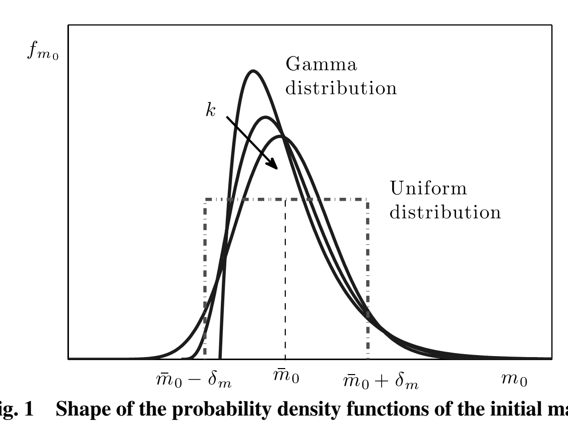

Fig. 1 Shape of the probability density functions of the initial mass.

$$M_0 = \bar{m}_0 - \frac{\delta_m}{\sqrt{3}}\sqrt{k} \tag{10}$$

Thus, one has the following probability density function

$$f_{m_0}(m_0) = (m_0 - M_0)^{k-1}\frac{e^{-(m_0-M_0)(\sqrt{3k}/\delta_m)}}{(\delta_m/\sqrt{3k})^k\Gamma(k)}, \qquad m_0 \geq M_0 \tag{11}$$

and zero otherwise. In this way, one has $E[m_0] = \bar{m}_0$ and

$$\mathrm{Var}[m_0] = \frac{\delta_m^2}{3}$$

(independently of k), as for the previously chosen uniform distribution.

Note that for $k \to \infty$, one has

$$\frac{G(k,1) - k}{\sqrt{k}} \to N(0,1)$$

which implies

$$m_0 \to N\left(\bar{m}_0, \frac{\delta_m^2}{3}\right)$$

(i.e., for large k the gamma distribution resembles a Gaussian distribution). However, the maximum value of k is limited by the fact that M_0 should be greater than zero. Therefore, the value of k must be chosen taking into account Eq. (10).

In Fig. 1, the shape of the probability density function of m_0 is plotted for different values of k and compared with the uniform distribution.

IV. Analysis of Mass Mean and Variance

To compute the mean and variance of the mass, the GPC method is used (see [7]), in which the process is represented as a Fourier-type series, with time-dependent coefficients, and orthogonal polynomials in terms of random variables are used as basis functions. The orthogonal polynomials used in GPC are chosen from the Askey scheme (a way of organizing certain orthogonal polynomials into a hierarchy, see [16]). If one chooses a family of polynomials that are orthogonal, the convergence of the series is exponential. The orthogonality property implies that, when taking expectation with respect to the random variable for two polynomials of the family ϕ_i and ϕ_j, then $E[\phi_i\phi_j] = \delta_{ij}E[\phi_i^2]$, where δ_{ij} is the Kronecker delta. For the uniform distribution Δ, the adequate orthogonal polynomials are the Legendre polynomials $L_n(\Delta)$, whereas, for the gamma distribution $G(k,1)$, one must use the generalized Laguerre polynomials $\phi_n^{k-1}(G)$.

To apply the GPC method, one must first write the initial mass distribution m_0 in terms of the orthogonal polynomials. For the uniform distribution, one can write $m_0 = \bar{m}_0 L_0(\Delta) + \delta_m L_1(\Delta)$, whereas, for the gamma distribution, it follows that

$$m_0 = \bar{m}_0\phi_0^{k-1}(G) - \frac{\delta_m}{\sqrt{3k}}\phi_1^{k-1}(G)$$

In the following, the uniform distribution case is considered (the gamma distribution is handled analogously). It is assumed that $m(t;m_0)$ can be written as

$$m(t;m_0) = \sum_{i=0}^{P} h_i(t)L_i(\Delta) \tag{12}$$

where the coefficients h_i are to be found using the mass equation (3), and P is the order of the approximation, which is to be taken sufficiently large. Substituting Eq. (12) in Eq. (3), the following equation is obtained

$$\sum_{i=0}^{P}\dot{h}_i(t)L_i(\Delta) = -A - B\sum_{i=0}^{P}\sum_{j=0}^{P}h_i(t)h_j(t)L_i(\Delta)L_j(\Delta) \tag{13}$$

Now, multiplying Eq. (13) by $L_l(\Delta)$ for $l = 0, \ldots, P$, taking expectation with respect to Δ, and using the orthogonality property of the L_l polynomials, one obtains $P+1$ equations

$$\begin{aligned}\dot{h}_l(t)E[L_l^2(\Delta)] &= -A\delta_{0l}\\ &- B\sum_{i=0}^{P}\sum_{j=0}^{P}h_i(t)h_j(t)E[L_i(\Delta)L_j(\Delta)L_l(\Delta)], \qquad l = 0,\ldots,P\end{aligned} \tag{14}$$

and calling

$$C_{ijl} = \frac{E[L_iL_jL_l]}{E[L_l^2]}$$

(which is a number that can be exactly computed because the involved expectations are just integrals of polynomials), it follows that

$$\dot{h}_l = -A\delta_{0l} - B\sum_{i=0}^{P}\sum_{j=0}^{P}h_ih_jC_{ijl}, \qquad l = 0,\ldots,P \tag{15}$$

which is a system of $P+1$ nonlinear coupled ordinary differential equations. The same result is reached for the gamma distribution case, with the corresponding C_{ijl} coefficients. The initial conditions of Eq. (15) depend on the initial mass distribution. For the uniform distribution case, they are

$$h_0(0) = \bar{m}_0, \qquad h_1(0) = \delta_m, \qquad h_l(0) = 0, \quad \text{for } l = 2,\ldots,P \tag{16}$$

whereas, for the gamma distribution, they are given by

$$h_0(0) = \bar{m}_0, \qquad h_1(0) = -\frac{\delta_m}{\sqrt{3k}}, \qquad h_l(0) = 0, \quad \text{for } l = 2,\ldots,P \tag{17}$$

The advantage of the GPC method is that a small or moderate value of P is enough to get good results, thus resulting in a method that is not very intensive computationally.

Once the coefficients h_i are found, it is possible to compute from Eq. (12) approximate values for quantities of interest such as mean and variance. For the uniform distribution, taking into account Eq. (12) and $L_0(\Delta) = 1$, it follows that

$$\begin{aligned}E[m(t;m_0)] &= \sum_{i=0}^{P}h_i(t)E[L_i(\Delta)] = \sum_{i=0}^{P}h_i(t)E[L_i(\Delta)L_0(\Delta)]\\ &= h_0(t)E[L_0^2(\Delta)] = h_0(t)\end{aligned} \tag{18}$$

To compute the variance

$$\begin{aligned}\mathrm{Var}[m(t;m_0)] &= E[m^2(t;m_0)] - E[m(t;m_0)]^2\\ &= \sum_{i=0}^{P}\sum_{j=0}^{P}h_i(t)h_j(t)E[L_i(\Delta)L_j(\Delta)] - h_0^2 = \sum_{i=1}^{P}h_i^2(t)E[L_i^2(\Delta)]\end{aligned} \tag{19}$$

For the gamma distribution, similar results hold:

$$E[m(t;m_0)] = h_0(t) \tag{20}$$

$$\mathrm{Var}[m(t;m_0)] = \sum_{i=1}^{P} h_i^2(t)E[(\phi_i^{k-1}(G))^2] \tag{21}$$

V. Analysis of the Evolution of the Mass Distribution Function

In this section, because the GPC method cannot be used to obtain distribution functions (see [11]), two original approximate methods to obtain the distribution function of the mass (which evolves in time) are developed.

Recall first that, given a random variable x with distribution function $f_x(x)$, if one defines another random variable y using a transformation g such that $y = g(x)$, then it is known that the distribution function $f_y(y)$ of y is given by (see [15])

$$f_y(y) = \frac{f_x(g^{-1}(y))}{|g'(g^{-1}(y))|} \tag{22}$$

with expression (22) valid only if the function $g(x)$ is invertible in the domain of x.

Denoting $m = m(t;m_0) = \varphi_t(m_0)$ as the solution of the differential equation (3) with initial condition (4), it follows from standard uniqueness results in differential equations (see [17]) that the function relating m and m_0 (for a given time t) is always monotonous. Indeed, if it were not monotonous, there would be values of mass (for a given time t) that could be reached from two different initial conditions, which would contradict uniqueness. Because it is monotonous, it is therefore invertible. Thus, it is possible to write

$$f_m(m,t) = \frac{f_{m_0}(\varphi_t^{-1}(m))}{|\varphi_t'(\varphi_t^{-1}(m))|} \tag{23}$$

where f_{m_0} is the distribution of the initial mass and $f_m(m,t)$ is the distribution of the mass at time t.

A. Approximate Method 1

The objective is to numerically approximate Eq. (23). For that, take n consecutive points from the domain of m_0, denoted as m_0^i, $i = 1, \ldots, n$, so that $m_0^1 < m_0^2 < \ldots < m_0^n$. Now, fix a time $\tau > 0$; solving the mass equation (3) for each i with m_0^i as the initial condition, one can compute the value of mass at time τ, $m^i(\tau) = \varphi_\tau(m_0^i)$. The numerator of Eq. (23) is computed for each i as $f_{m_0}(m_0^i)$. To compute the denominator of Eq. (23), the theory of differential equations is used. Noting that

$$\varphi_t'(m_0) = \frac{\partial m}{\partial m_0}(t)$$

is the value of the derivative of the solution m with respect to m_0 (also known as the sensitivity function with respect to the initial condition), a differential equation can be written for $\varphi_t'(m_0)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t'(m_0) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial m}{\partial m_0}\right) = -2Bm\frac{\partial m}{\partial m_0} = -2Bm\varphi_t'(m_0) \tag{24}$$

with initial condition [obtained from Eq. (4)]

$$\varphi_0'(m_0) = 1 \tag{25}$$

This is the so-called variational equation, which is linear, and its solution is given by

$$\varphi_t'(m_0) = \exp\left(-2B\int_0^t m(t;m_0)\,\mathrm{d}t\right) \tag{26}$$

Numerically solving Eq. (26) to find $\varphi_t'(m_0^i)$ at time $t = \tau$, the denominator of Eq. (23) is computed for each i.

Thus, for a fixed time τ, one finds the value of $f_m(m,\tau)$ at the n points $m^i = \varphi_\tau(m_0^i)$, $i = 1, \ldots, n$, as

$$f_m(m^i,\tau) = \frac{f_{m_0}(m_0^i)}{\varphi_\tau'(m_0^i)} \tag{27}$$

B. Approximate Method 2

Now, another method that avoids having to solve the differential equation for the sensitivity function (24) is formulated. As in the previous method, take n consecutive points from the domain of m_0, $m_0^1 < m_0^2 < \ldots < m_0^n$, fix a time $\tau > 0$, and solve the mass equation (3) to compute the value of mass at time $t = \tau$, $m^i(\tau) = \varphi_\tau(m_0^i)$. To find the value of $f_m(m,\tau)$ at these points, the intermediate value theorem for integrals is used:

$$Pr(m^i \le m \le m^{i+1}) = \int_{m^i}^{m^{i+1}} f_m(\mu,\tau)\,\mathrm{d}\mu = (m^{i+1} - m^i)f_m(\xi^i,\tau) \tag{28}$$

where Pr is the probability measure and $\xi^i \in [m^i, m^{i+1}]$, for $i = 1, \ldots, n-1$.

Given the uniqueness of the solution, intervals in the initial condition are univocally mapped into intervals in the solution (as illustrated in Fig. 2), thus the probability of the mass m being in the interval (m^i, m^{i+1}) is the same as the probability of the initial mass m_0 being in the interval (m_0^i, m_0^{i+1}), that is, $Pr(m^i \le m \le m^{i+1}) = Pr(m_0^i \le m_0 \le m_0^{i+1})$. These probabilities can be computed (numerically or analytically) from the distribution function of m_0. Thus, one has

$$f_m(\xi^i,\tau) = \frac{Pr(m_0^i \le m_0 \le m_0^{i+1})}{m^{i+1} - m^i}, \qquad i = 1, \ldots, n-1 \tag{29}$$

Taking

$$\begin{aligned} f_m(m^1,\tau) &= f_m(\xi^1,\tau) \\ f_m(m^i,\tau) &= \frac{f_m(\xi^{i-1},\tau) + f_m(\xi^i,\tau)}{2}, \qquad i = 2, \ldots, n-1 \\ f_m(m^n,\tau) &= f_m(\xi^{n-1},\tau) \end{aligned} \tag{30}$$

an approximation of f_m is obtained at n points.

VI. Analysis of the Distribution Function of the Flight Time and the Fuel Consumption

In this section, the distribution functions of the flight time t_f (in the case of a given fuel load) and of the fuel consumption m_F (in the case of a given range) are analyzed using the approximate methods developed in Sec. V.

A. Distribution Function of the Flight Time

The flight time t_f is defined explicitly by Eq. (6), where it can be seen that it is a function of the initial mass and hence a random variable itself. Calling $t_f = \varphi(m_0)$, one has that

$$m(\varphi(m_0);m_0) = m_0 - m_F \tag{31}$$

The distribution function of t_f is given, similarly to Eq. (23), by

$$f_{t_f}(t_f) = \frac{f_{m_0}(\varphi^{-1}(t_f))}{|\varphi'(\varphi^{-1}(t_f))|} \tag{32}$$

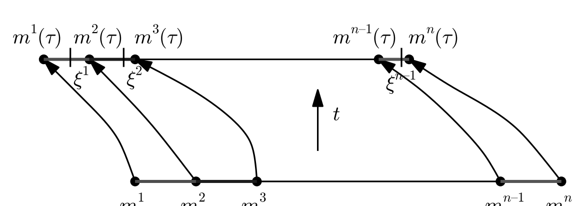

Fig. 2 Evolution of the initial mass intervals in time.

if φ is invertible. To see that this is the case, take the derivative with respect to m_0 in Eq. (31),

$$\frac{\partial m}{\partial t}(t_f; m_0)\varphi'(m_0) + \frac{\partial m}{\partial m_0}(t_f; m_0) = 1 \tag{33}$$

Note that

$$\frac{\partial m}{\partial t}(t_f; m_0) = \dot{m}(t_f)$$

thus using the mass equation (3), it is found that

$$\frac{\partial m}{\partial t}(t_f; m_0) = -(A + Bm^2(t_f)) = -(A + B(m_0 - m_F)^2) < 0 \tag{34}$$

Thus, from Eqs. (33) and (34), one has

$$\varphi'(m_0) = -\frac{1 - (\partial m/\partial m_0)(t_f; m_0)}{A + B(m_0 - m_F)^2} \tag{35}$$

On the other hand, $\partial m/\partial m_0$ satisfies the differential equation (24), hence, one has from Eq. (26) that $(\partial m/\partial m_0)(t) < 1$for $t > 0$ and, in particular, $(\partial m/\partial m_0)(t_f; m_0) < 1$Thus, $\varphi'(m_0) < 0$ and it follows that $t_f = \varphi(m_0)$ is monotonically decreasing with m_0 and hence invertible. Therefore Eq. (32) is a valid equation to compute $f_{t_f}(t_f)$.

1. Approximate Method 1

Take n consecutive points from the domain of m_0, denoted as m_0^i, $i = 1, \ldots, n$, so that $m_0^1 < m_0^2 < \ldots < m_0^n$. Each of these points determines a value t_f^i by solving the mass equation (3) with initial condition m_0^i and stopping when $m = m_0^i - m_F$. Then, combining Eqs. (32) and (35),

$$f_{t_f}(t_f^i) = \frac{f_{m_0}(\varphi^{-1}(t_f^i))}{|\varphi'(\varphi^{-1}(t_f^i))|} = f_{m_0}(m_0^i)\frac{A + B(m_0^i - m_F)^2}{1 - (\partial m/\partial m_0)(t_f^i; m_0^i)} \tag{36}$$

where $(\partial m/\partial m_0)(t_f^i; m_0^i)$ is obtained by computing Eq. (26) for $t = t_f^i$ and $m_0 = m_0^i$. Thus, the value of f_{t_f} at n points is obtained.

2. Approximate Method 2

Take n consecutive points from the domain of m_0, as before, each of which determines a value t_f^i. It has to be noted that, because it was found before that $\varphi'(m_0) < 0$, increasing values of m_0 produce decreasing values of t_f and thus $t_f^{i+1} < t_f^i$. As it was done for the distribution of the mass, the intermediate value theorem for integrals can be applied to find

$$Pr(t_f^{i+1} \le t_f \le t_f^i) = \int_{t_f^{i+1}}^{t_f^i} f_{t_f}(\mu)\,\mathrm{d}\mu = (t_f^i - t_f^{i+1})f_{t_f}(\xi^i) \tag{37}$$

where $\xi^i \in [t_f^{i+1}, t_f^i]$, for $i = 1, \ldots, n-1$.

Reasoning as in Sec. V.B, it can be seen that intervals in the initial condition m_0 are univocally mapped into intervals of t_f. However, noting that increasing values of m_0 produce decreasing values of t_f, one has that the interval (m_0^i, m_0^{i+1}) is mapped into the interval (t_f^{i+1}, t_f^i). Thus, it is deduced that $Pr(t_f^{i+1} \le t_f \le t_f^i) = Pr(m_0^i \le m_0 \le m_0^{i+1})$, hence

$$f_{t_f}(\xi^i) = \frac{Pr(m_0^i \le m_0 \le m_0^{i+1})}{t_f^i - t_f^{i+1}}, \qquad i = 1, \ldots, n-1 \tag{38}$$

Taking

$$\begin{aligned} f_{t_f}(t_f^1) &= f_{t_f}(\xi^1) \\ f_{t_f}(t_f^i) &= \frac{f_{t_f}(\xi^{i-1}) + f_{t_f}(\xi^i)}{2}, \qquad i = 2, \ldots, n-1 \\ f_{t_f}(t_f^n) &= f_{t_f}(\xi^{n-1}) \end{aligned} \tag{39}$$

an approximation of f_{t_f} is obtained at n points.

B. Distribution Function of the Fuel Consumption

The fuel consumption m_F is defined explicitly by Eq. (9) as a function of the initial mass; thus m_F is a random variable itself. Calling this function $m_F = \psi(m_0)$, the distribution function of m_F is given, similar to Eq. (23), by

$$f_{m_F}(m_F) = \frac{f_{m_0}(\psi^{-1}(m_F))}{|\psi'(\psi^{-1}(m_F))|} \tag{40}$$

if ψ is invertible. To prove that this is the case, notice from Eq. (9) that

$$\psi'(m_0) = \frac{\partial m_F}{\partial m_0}(m_0) = 1 - \frac{\partial m}{\partial m_0}(x_f; m_0) \tag{41}$$

Similar to Eq. (24), the variable $\partial m/\partial m_0$ satisfies now a differential equation with respect to distance

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial m}{\partial m_0}\right) = -\frac{2Bm}{V}\frac{\partial m}{\partial m_0} \tag{42}$$

with initial condition [from Eq. (4)]

$$\frac{\partial m}{\partial m_0}(0) = 1 \tag{43}$$

whose solution is given by

$$\frac{\partial m}{\partial m_0}(x; m_0) = \exp\left(-\frac{2B}{V}\int_0^x m(x; m_0)\,\mathrm{d}x\right) \tag{44}$$

Thus, from Eq. (41), one has $\psi'(m_0) > 0$ for $x_f > 0$ which implies invertibility of $\psi(m_0)$. Hence, Eq. (40) is a valid equation to compute $f_{m_F}(m_F)$.

1. Approximate Method 1

Take n consecutive points from the domain of m_0, denoted as m_0^i, $i = 1, \ldots, n$, so that $m_0^1 < m_0^2 < \ldots < m_0^n$. Each of these points determines a value $m_F^i = m_0^i - m(x_f; m_0^i)$ by solving the mass equation (7) with initial condition m_0^i and stopping when $x = x_f$. Then, using Eq. (41),

$$f_{m_F}(m_F^i) = \frac{f_{m_0}(\psi^{-1}(m_F^i))}{|\psi'(\psi^{-1}(m_F^i))|} = \frac{f_{m_0}(m_0^i)}{1 - (\partial m/\partial m_0)(x_f; m_0^i)} \tag{45}$$

where $\frac{\partial m}{\partial m_0}(x_f; m_0^i)$ is obtained by computing Eq. (44) for $x = x_f$ and $m_0 = m_0^i$. Thus, the value of f_{m_F} at n points is obtained.

2. Approximate Method 2

Take n consecutive points from the domain of m_0, as before, each of which determines a value m_F^i. Because it was found before that $\psi'(m_0) > 0$, increasing values of m_0 produce increasing values of m_F. As it was done for the distribution of the mass, the intermediate value theorem for integrals can be applied to find

$$Pr(m_F^i \le m_F \le m_F^{i+1}) = \int_{m_F^i}^{m_F^{i+1}} f_{t_f}(\mu)\,\mathrm{d}\mu = (m_F^{i+1} - m_F^i)f_{m_F}(\xi^i) \tag{46}$$

where $\xi^i \in [m_F^i, m_F^{i+1}]$, for $i = 1, \ldots, n-1$.

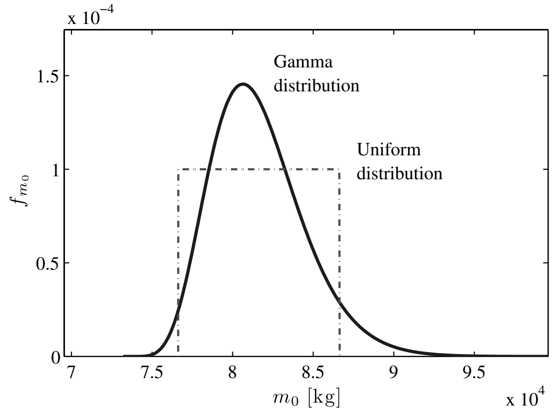

Fig. 3 Probability density functions of the initial mass ($\bar{m}_0 =$ 81,633 kg, $\delta_m = 5000$ kg, and $k = 8.5$).

Reasoning as in Sec. V.B, it can be seen that intervals in the initial condition m_0 are univocally mapped into intervals of m_F. Thus, it is deduced that $Pr(m_F^i \leq m_F \leq m_F^{i+1}) = Pr(m_0^i \leq m_0 \leq m_0^{i+1})$, hence

$$f_{m_F}(\xi^i) = \frac{Pr(m_0^i \leq m_0 \leq m_0^{i+1})}{m_F^{i+1} - m_F^i}, \qquad i = 1, \ldots, n-1 \tag{47}$$

Taking

$$\begin{aligned} f_{m_F}(m_F^1) &= f_{m_F}(\xi^1) \\ f_{m_F}(m_F^i) &= \frac{f_{m_F}(\xi^{i-1}) + f_{m_F}(\xi^i)}{2}, \qquad i = 2, \ldots, n-1 \\ f_{m_F}(m_F^n) &= f_{m_F}(\xi^{n-1}) \end{aligned} \tag{48}$$

an approximation of f_{m_F} is obtained at n points.

VII. Results

Now, the methods presented in previous sections are applied to the two initial mass distributions defined in Sec. III. The numerical resolution of the different problems is performed using the MATLAB environment. The numerical results are compared with the exact results of the problem, so that their accuracy can be assessed; these exact results are presented in the Appendix.

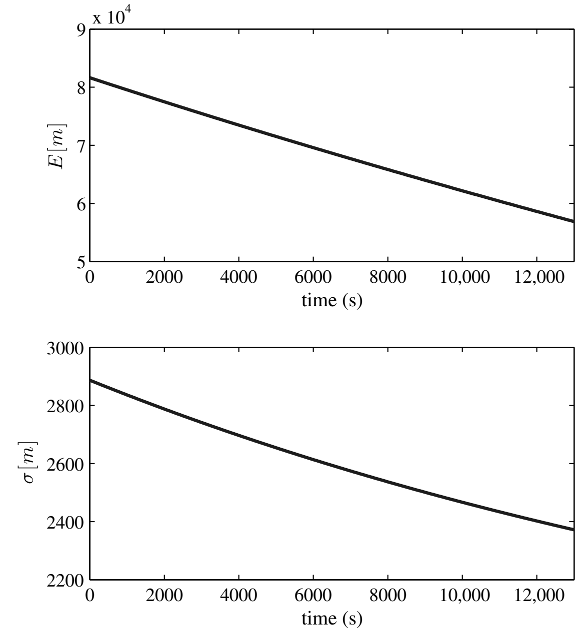

Fig. 5 Evolution of mass mean and typical deviation for the uniform distribution case.

For the numerical application, the following values are used: $C_{d_0} = 0.015$, $C_{d_2} = 0.042$, $\rho = 0.5\rho_0$, $\rho_0 = 1.225$ kg/m^3, $V = 200$ m/s, $c = 5 \cdot 10^{-5}$ s/m, $S = 150$ m^2, $g = 9.8$ m/s^2, $m_F = 25,000$ kg in the case of a given fuel load, and $x_f = 2500$ km in the case of a given range. For the initial mass distributions, the nominal values chosen for mean and width are $\bar{m}_0 = 81,633$ kg and $\delta_m = 5000$ kg, which yields a typical deviation

$$\sigma[m_0] = \sqrt{\mathrm{Var}[m_0]} = \frac{\delta_m}{\sqrt{3}} = 2887 \text{ kg}$$

and for the gamma distribution, the nominal value $k = 8.5$ is considered. A parametric study as a function of δ_m and k is also presented. For the nominal values, the two initial mass distributions are shown in Fig. 3. For the uniform distribution, the values of m_0 with nonzero probability are in the interval $[\bar{m}_0 - \delta_m, \bar{m}_0 + \delta_m] =$

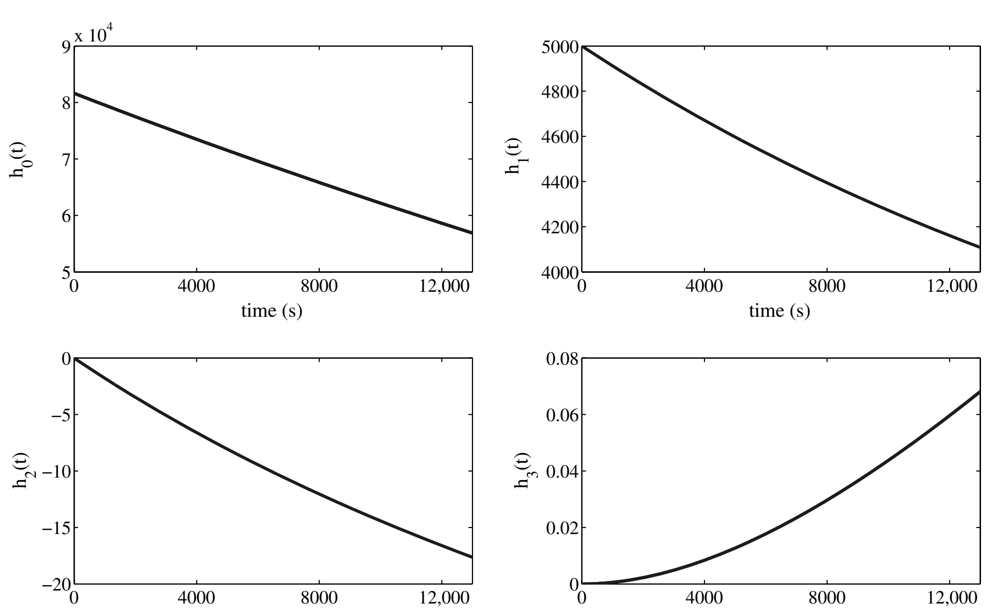

Fig. 4 GPC coefficients for the uniform distribution case.

Table 1 Values of mass mean and typical deviation at selected times for the uniform distribution case

Time, s	$E[m(t; m_0)]$, kg	$\sigma[m(t; m_0)]$, kg
2×10^3	77,485	2787
4×10^3	73,477	2696
6×10^3	69,596	2613
8×10^3	65,831	2536
10^4	62,175	2467
1.2×10^4	58,616	2402

$[76, 633, 86, 633]$ in kilograms, and for the gamma distribution, they are in $[M_0, \infty) = [73, 217, \infty)$ in kilograms.

For the GPC method, the number of terms used in the expansions is $P = 3$, which turns out to be enough to obtain a good representation of m. In the computation of the distribution functions, the number of discretization points considered is $n = 1000$, which has proven to be good enough. All the integrations have been performed using the MATLAB environment.

In Sec. VII.A, the GPC method is applied to obtain the evolution of mass mean and variance. The distribution function of the mass and its evolution in time are analyzed in Sec. VII.B. The distribution function of the flight time in the case of a given fuel load is studied in Sec. VII.C and that of the fuel consumption in the case of a given range is studied in Sec. VII.D.

A. Mass Mean and Variance

1. *Uniform Distribution of the Initial Mass*

To find the mean and variance using GPC, the value $P = 3$ is chosen in the GPC expansion of the mass [Eq. (12)], which, as already mentioned, is enough to obtain a good representation of m. The coefficients of the GPC expansion are shown in Fig. 4. Note the fast decrease of their order of magnitude (six orders of magnitude from h_0 to h_3).

The evolution of mean $E[m(t; m_0)]$ and typical deviation $\sigma[m(t; m_0)] = \sqrt{\mathrm{Var}[m(t; m_0)]}$ is shown in Fig. 5. Selected values of mean and typical deviation are given in Table 1. The difference between the GPC solution and the analytical solution [Eqs. (A1) and (A2)] of mean and variance is negligible; the absolute error is less than 10^{-4} for the mean and less than $2 \cdot 10^{-3}$ for the typical deviation. Thus a low-order GPC expansion, which is very fast to compute, is enough to capture well the mean and variance evolution.

Although the fact that the mean mass decreases with time is to be expected (because fuel mass in consumed), it is remarkable that the standard deviation of the mass also decreases with time. Thus, the dispersion of the distribution function and, therefore, the uncertainty decreases with time. This result can be explained by noting that the larger the aircraft mass, the larger its rate of decrease (which is given at each instant by $A + Bm^2$). Thus, if one computes the solution $m(t)$ given by Eq. (5) for $m_0 = \bar{m}_0 \pm \delta_m$, say, $m_+(t) = m(t; \bar{m}_0 + \delta_m)$ and $m_-(t) = m(t; \bar{m}_0 - \delta_m)$, the distance $\Delta m(t) = m_+(t) - m_-(t)$ decreases with time: For example, at $t = 0$, one has $\Delta m = 2\delta_m = 10,000$ kg and at $t = 1.2 \times 10^4$ s, $\Delta m = 8321$ kg.

2. *Gamma Distribution of the Initial Mass*

As in the uniform distribution case, to find the mean and variance using GPC, choosing $P = 3$ in the GPC expansion of the mass [Eq. (12)] is good enough. The coefficients of the GPC expansion are shown in Fig. 6. Note again the fast decrease of their order of magnitude (seven orders of magnitude from h_0 to h_3).

The evolution of mean and typical deviation is shown in Fig. 7. Selected values of mean and typical deviation are given in Table 2. As before, the difference between the GPC solution and the analytical solution [Eqs. (A5) and (A6)] of mean and variance is negligible; the absolute error is less than $4 \cdot 10^{-5}$ for the mean and less than $2 \cdot 10^{-3}$ for the typical deviation. Again, both the mean and the standard deviation decrease with time.

Note that the plots and values are very similar to the ones obtained with the uniform distribution. Thus, the results show that the evolution of mean and standard deviation is very weakly affected by the specific distribution function chosen for the initial mass (at least for the two cases studied).

B. Distribution Function of the Mass

1. *Uniform Distribution of the Initial Mass*

The mass distribution is represented at several time instants in Fig. 8. Both approximate methods developed in Sec. V to approximate Eq. (23) show excellent agreement with the exact analytical results [Eq. (A12)] and are indistinguishable from them. The results in Fig. 8 show that, as time increases (and m decreases), the width of the distribution function decreases, whereas the probability density increases. Thus, uncertainty decreases with time (as it was seen in Fig. 5). Note also that the uniform shape is approximately maintained.

2. *Gamma Distribution of the Initial Mass*

In this case, the mass distribution is represented at several time instants in Fig. 9. Again, both numerical methods developed in Sec. V

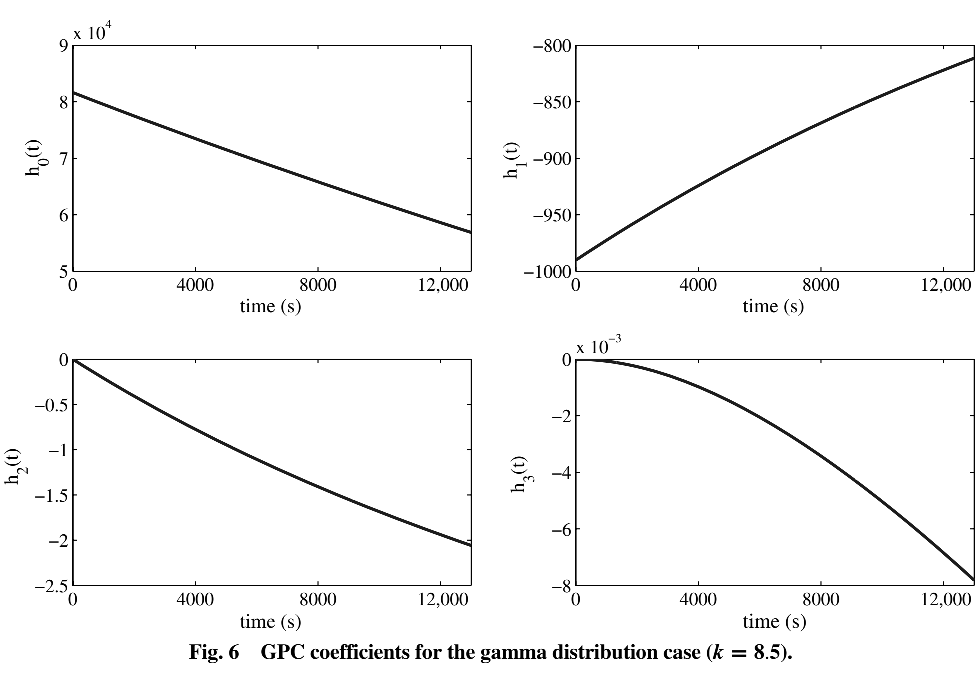

Fig. 6 GPC coefficients for the gamma distribution case ($k = 8.5$).

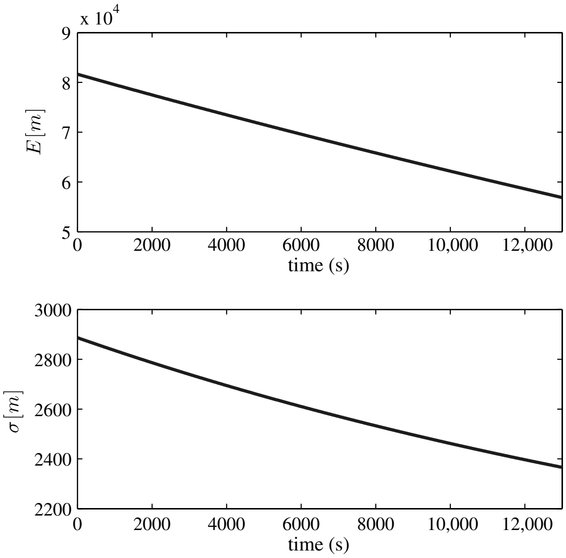

Fig. 7 Evolution of mass mean and typical deviation for the gamma distribution case ($k = 8.5$).

to approximate Eq. (23) show excellent agreement with the exact analytical results [Eq. (A14)] and are indistinguishable from them. As in Figs. 8 and 9 show that uncertainty decreases as time increases. Also, the shape of the distribution function is approximately of gamma type at all times.

C. Distribution Function of the Flight Time

1. Uniform Distribution of the Initial Mass

The distribution function of the flight time is represented in Fig. 10. Note that it looks approximately uniform, similar to the initial mass distribution. As in the computation of the mass distribution function, both approximate methods developed in Sec. V to approximate Eq. (32) show excellent agreement with the exact analytical result [Eq. (A19)]. The values of t_f with nonzero probability are those in the interval $[T_1, T_2] = [12,625, 13,664]$ in seconds, for values of m_0 with nonzero probability in $[\bar{m}_0 - \delta_m, \bar{m}_0 + \delta_m] = [76,633, 86,633]$ in kilograms, where, as shown in the Appendix,

$$T_1 = \frac{1}{\sqrt{AB}} \arctan\left(\frac{\sqrt{AB} m_F}{A + B(\bar{m}_0 + \delta_m)(\bar{m}_0 + \delta_m - m_F)}\right) \tag{49}$$

$$T_2 = \frac{1}{\sqrt{AB}} \arctan\left(\frac{\sqrt{AB} m_F}{A + B(\bar{m}_0 - \delta_m)(\bar{m}_0 - \delta_m - m_F)}\right) \tag{50}$$

The mean and the typical deviation of the flight time are obtained using the distribution function, computed numerically from

$$E[t_f] = \int_0^\infty t_f f_{t_f}(t_f)\, \mathrm{d}t_f \tag{51}$$

Table 2 Values of mass mean and typical deviation at selected times for the gamma distribution case ($k = 8.5$)

Time, s	$E[m(t; m_0)]$, kg	$\sigma[m(t; m_0)]$, kg
2×10^3	77,485	2786
4×10^3	73,477	2695
6×10^3	69,596	2610
8×10^3	65,831	2533
10^4	62,175	2462
1.2×10^4	58,616	2397

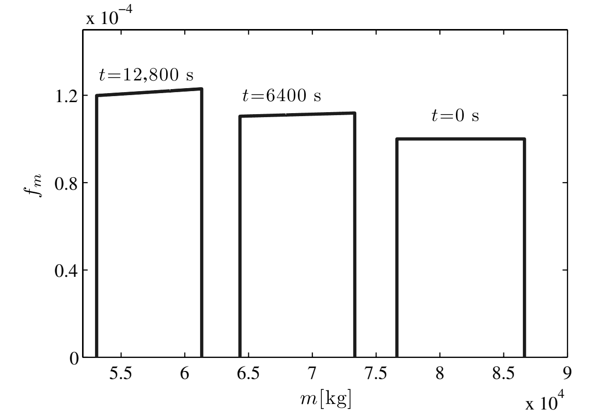

Fig. 8 Mass distribution at several time instants for the uniform distribution case.

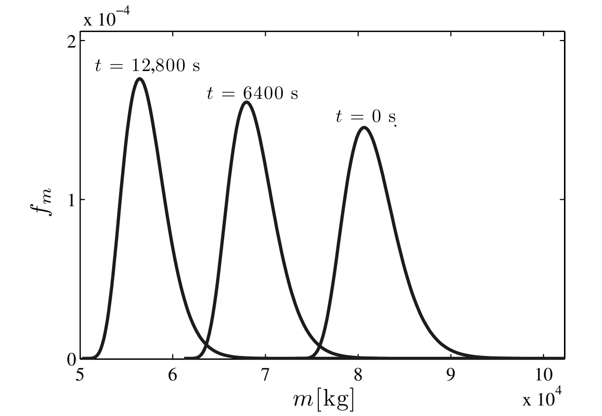

Fig. 9 Mass distribution at several time instants for the gamma distribution case ($k = 8.5$).

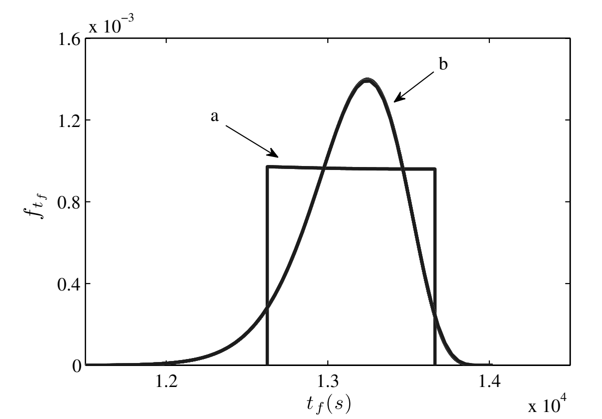

Fig. 10 Distribution functions of the flight time: a) uniform distribution case and b) gamma distribution case ($k = 8.5$).

$$(\sigma[t_f])^2 = \int_0^\infty t_f^2 f_{t_f}(t_f)\, \mathrm{d}t_f - (E[t_f])^2 \tag{52}$$

The results are given in Table 3.

Now the effect of δ_m on the results is analyzed. Values of $\sigma[t_f]$ for different values of δ_m (obtained using the exact solution) are given in Fig. 11, in which it is seen that there is a proportionality between the two parameters. The values of $E[t_f]$ are not significantly affected by changing δ_m.

2. Gamma Distribution of the Initial Mass

The distribution function in this case is represented in Fig. 10. Note that this distribution function is somewhat different from a gamma distribution, because the values of t_f with nonzero probability are

Table 3 Computed values of $E[t_f]$ and $\sigma[t_f]$ for the uniform distribution case

	Exact	Method 1	Method 2
$E[t_f]$, s	13,144	13,144	13,146
$\sigma[t_f]$, s	299.8	296.9	290.1

those in the finite interval $(0, T] = (0, 14019]$ in seconds, for values of m_0 with nonzero probability in $[M_0, \infty) = [73, 217, \infty)$ in kilograms, where, as shown in the Appendix,

$$T = \frac{1}{\sqrt{AB}} \arctan\left(\frac{\sqrt{AB} m_F}{A + BM_0(M_0 - m_F)}\right) \tag{53}$$

Moreover, because t_f decreases when m_0 increases, the bell of the distribution is sort of inverted (with respect to the bell of the initial mass distribution).

As in the computation of the mass distribution function, both numerical methods developed in Sec. V to approximate Eq. (32) show excellent agreement with the exact analytical result [Eq. (A21)].

Again, the mean and the typical deviation are computed numerically from Eqs. (51) and (52) using the distribution function. The results are given in Table 4.

Note that these results of mean and standard deviation are very close to the ones obtained before for the uniform distribution (especially for the exact distribution functions), showing again that the initial mass distribution chosen affects the results very weakly.

Now the effect of k on the results is analyzed. Values of $\sigma[t_f]$ for different values of k and δ_m (obtained using the exact solution) are given in Fig. 12, in which it is seen that there is no significant effect from changing k and, as in the uniform distribution case, there is a proportionality between the values of $\sigma[t_f]$ and δ_m. The values of $E[t_f]$ are not significantly affected by changing k or δ_m.

D. Distribution Function of the Fuel Consumption

1. Uniform Distribution of the Initial Mass

The distribution function of the fuel consumption is represented in Fig. 13. Note that it looks approximately uniform, similar to the initial mass distribution, although smaller values of m_F show a slightly higher probability. As before, both approximate methods developed in Sec. V to approximate Eq. (40) show excellent agreement with the exact analytical results [Eq. (A25)].

The values of m_F with nonzero probability are those in the interval $[M_1, M_2] = [23, 043, 24, 775]$ in kilograms, for values of m_0 with nonzero probability in $[\bar{m}_0 - \delta_m, \bar{m}_0 + \delta_m] = [76, 633, 86, 633]$ in kilograms, where, as shown in the Appendix,

$$M_1 = \frac{((\bar{m}_0 - \delta_m)^2 + \frac{A}{B}) \tan(\frac{1}{V}\sqrt{AB} x_f)}{\sqrt{\frac{A}{B}} + (\bar{m}_0 - \delta_m) \tan(\frac{1}{V}\sqrt{AB} x_f)} \tag{54}$$

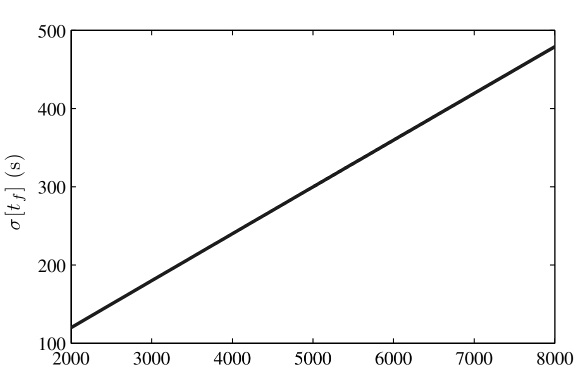

Fig. 11 Typical deviation of the flight time vs δ_m in the uniform distribution case.

Table 4 Computed values of $E[t_f]$ and $\sigma[t_f]$ for the gamma distribution case ($k = 8.5$)

	Exact	Method 1	Method 2
$E[t_f]$, s	13,144	13,144	13,143
$\sigma[t_f]$, s	299.3	301.6	303.9

$$M_2 = \frac{((\bar{m}_0 + \delta_m)^2 + \frac{A}{B}) \tan(\frac{1}{V}\sqrt{AB} x_f)}{\sqrt{\frac{A}{B}} + (\bar{m}_0 + \delta_m) \tan(\frac{1}{V}\sqrt{AB} x_f)} \tag{55}$$

As for the flight time, the mean and the typical deviation of the fuel consumption are obtained using the distribution function, computed numerically from

$$E[m_F] = \int_0^\infty m_F f_{m_F}(m_F)\, \mathrm{d}m_F \tag{56}$$

$$(\sigma[m_F])^2 = \int_0^\infty m_F^2 f_{m_F}(m_F)\, \mathrm{d}m_F - (E[m_F])^2 \tag{57}$$

The results are given in Table 5.

Now the effect of δ_m on the results is analyzed. Values of $\sigma[m_F]$ for different values of δ_m (obtained using the exact solution) are given in Fig. 14, in which one can see that there is a proportionality between both parameters. The values of $E[m_F]$ are not significantly affected by changing δ_m.

2. Gamma Distribution of the Initial Mass

The distribution function in this case is represented in Fig. 13 for $k = 8.5$. Note that its shape is approximately of gamma type, as the

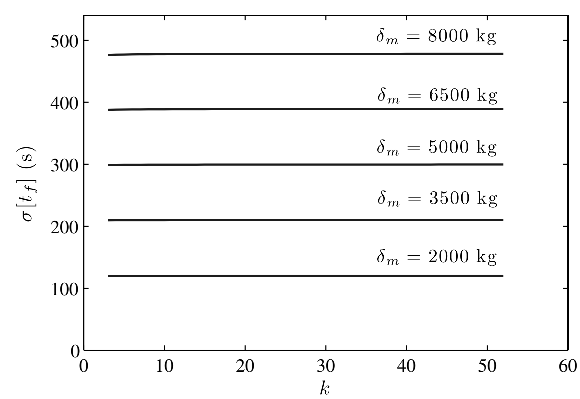

Fig. 12 Typical deviation of the flight time vs k, for different values of δ_m, in the gamma distribution case.

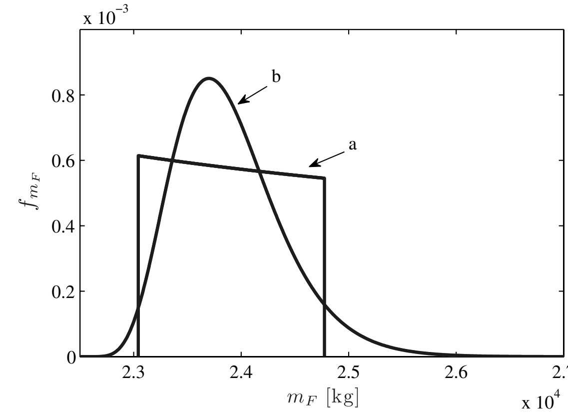

Fig. 13 Distribution functions of the fuel consumption: a) uniform distribution case and b) gamma distribution case ($k = 8.5$).

Table 5 Computed values of $E[m_F]$ and $\sigma[m_F]$ for the uniform distribution case

	Exact	Method 1	Method 2
$E[m_F]$, kg	23,892	23,892	23,892
$\sigma[m_F]$, kg	499.96	499.81	500.04

initial mass distribution. The values of m_F with nonzero probability are those in the interval $[M, \infty) = [22,499, \infty)$ in kilograms, for values of m_0 with nonzero probability in $[M_0, \infty) = [73,217, \infty)$ in kilograms, where, as shown in the Appendix,

$$M = \frac{(M_0^2 + \frac{A}{B}) \tan(\frac{1}{V} \sqrt{AB} x_f)}{\sqrt{\frac{A}{B}} + M_0 \tan(\frac{1}{V} \sqrt{AB} x_f)} \tag{58}$$

As before, the approximate methods developed in Sec. V to approximate Eq. (40) show excellent agreement with the exact analytical result [Eq. (A27)].

Again, the mean and the typical deviation are computed numerically from Eqs. (56) and (57) using the distribution function. The results are given in Table 6. Note that these results of mean and standard deviation are very close to the ones obtained before for the uniform distribution, showing again that the initial mass distribution chosen affects very weakly the results.

Now the effect of k on the results is analyzed. Values of $\sigma[m_F]$ for different values of k and δ_m (obtained using the exact solution) are given in Fig. 15, in which one can see that the influence of k in $\sigma[m_F]$ is negligible. Also, as in the uniform distribution case, there is a proportionality between the values of $\sigma[m_F]$ and δ_m. The values of $E[m_F]$ are not significantly affected by changing k or δ_m.

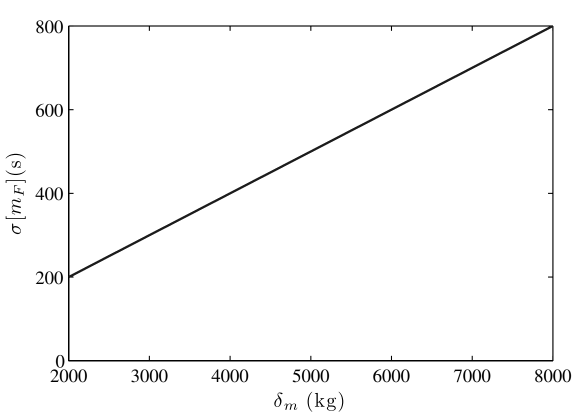

Fig. 14 Typical deviation of fuel consumption vs δ_m in the uniform distribution case.

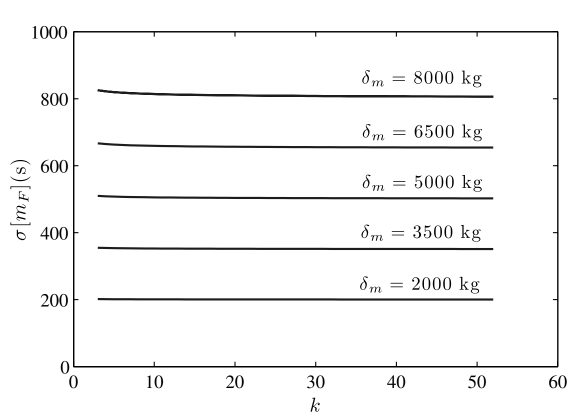

Fig. 15 Typical deviation of the fuel consumption vs k, for different values of δ_m, in the gamma distribution case.

Table 6 Computed values of $E[m_F]$ and $\sigma[m_F]$ for the gamma distribution case ($k = 8.5$)

	Exact	Method 1	Method 2
$E[m_F]$, kg	23,891	23,891	23,894
$\sigma[m_F]$, kg	506.46	506.37	506.46

VIII. Conclusions

The problem of propagation of initial mass uncertainty in cruise flight has been studied, using a nonlinear model that has known analytical solution. To study the evolution of mean and variance of the aircraft mass, the generalized polynomial chaos (GPC) method has been used, in which an expansion with just four terms has proven to be accurate enough. The study of the evolution of the mass distribution function has also been considered and two approximate methods have been developed. These two methods are applicable to problems in which there is just one random variable and for the analysis of distribution functions of functions of the random variable which are invertible. Using these methods, the distribution functions of the flight time in the case of a given fuel load, and of the fuel consumption in the case of a given range, have been also studied. The results obtained with these methods have been compared with the exact analytical results, showing an excellent agreement in all cases; thus, the accuracy of the methods has been assessed and, therefore, they are proposed as accurate and computationally efficient candidates to study uncertainty propagation.

The results presented in this work show that both mass mean and standard deviation decrease with time, with the distribution function getting narrower and more concentrated around the mean; thus, an important conclusion of this analysis is that uncertainty (represented by the dispersion of the distribution function) decreases with time. On the other hand, the shape of the distribution function of the mass is fundamentally unchanged from its initial shape. The results also show that the values of both mean and standard deviation are very weakly affected by the specific distribution function chosen for the initial mass (at least in the uniform and gamma cases).

The distribution functions of other flight properties different from mass (flight time and fuel consumption) have been analyzed as well, and their main statistical properties have been computed. Again, it has been shown that the results are affected very weakly by the choice of the initial mass distribution. The influence of the parameters of the initial mass distributions (δ_m and k) has been studied: The mean is not significantly affected by changing δ_m or k, and the typical deviation varies almost linearly with δ_m and is not affected by k. In these cases, the mean and variance have been obtained directly using the known distribution functions (and not the GPC method, as in the case of the mass distribution).

The approximate methods developed in this paper can be applied to other flight phases defined by more complicated flight conditions, and they can be extended to consider other sources of uncertainty, not only in the initial conditions but, for example, persistently affecting the system, such as wind. The analysis of these problems is left for future work.

Appendix: Exact Results

A1. Mean and Typical Deviation of the Mass

In this Appendix, the different analytic expressions used for comparison purposes throughout the paper are presented, and their derivation is briefly explained. To simplify the notation, the following parameters are defined: $c_1(t) = \tan\big(\sqrt{AB}t\big) \geq 0$,

$$c_2 = \sqrt{\frac{A}{B}} > 0$$

and

$$c_3 = \tan\left(\frac{1}{V}\sqrt{AB}x_f\right) > 0$$

.

A. Uniform Distribution of the Initial Mass

The analytical value of the mean is computed directly from Eq. (5), obtaining

$$\begin{aligned}E[m(t;m_0)] &= \frac{1}{2\delta_m}\int_{\bar{m}_0-\delta_m}^{\bar{m}_0+\delta_m} m(t;m_0)\,\mathrm{d}m_0\\ &= \frac{c_2}{2\delta_m}\int_{\bar{m}_0-\delta_m}^{\bar{m}_0+\delta_m}\frac{m_0-c_1c_2}{c_2+c_1m_0}\,\mathrm{d}m_0\\ &= \frac{c_2}{c_1(t)}\left(1-\frac{c_2}{c_1(t)}\frac{c_1^2(t)+1}{2\delta_m}\log\left[\frac{c_2+(\bar{m}_0+\delta_m)c_1(t)}{c_2+(\bar{m}_0-\delta_m)c_1(t)}\right]\right)\end{aligned} \tag{A1}$$

Similarly, the computation of the variance of $m(t)$ gives

$$\begin{aligned}\mathrm{Var}[m(t;m_0)] &= E[m^2(t;m_0)] - (E[m(t;m_0)])^2\\ &= \frac{1}{2\delta_m}\int_{\bar{m}_0-\delta_m}^{\bar{m}_0+\delta_m} m^2(t;m_0)\,\mathrm{d}m_0 - (E[m(t;m_0)])^2\\ &= \frac{c_2^2}{2\delta_m}\int_{\bar{m}_0-\delta_m}^{\bar{m}_0+\delta_m}\frac{(m_0-c_1c_2)^2}{(c_2+c_1m_0)^2}\,\mathrm{d}m_0 - (E[m(t;m_0)])^2\\ &= \frac{c_2^4}{c_1^4(t)}\frac{(c_1^2(t)+1)^2}{2\delta_m}\left[\frac{2\delta_m}{(\bar{m}_0+\frac{c_2}{c_1(t)})^2-\delta_m^2}\right.\\ &\quad\left.-\frac{1}{2\delta_m}\left(\log\left[\frac{c_2+(\bar{m}_0+\delta_m)c_1(t)}{c_2+(\bar{m}_0-\delta_m)c_1(t)}\right]\right)^2\right]\end{aligned} \tag{A2}$$

Expressions (A1) and (A2) are both indeterminate for $t=0$ (which implies $c_1=0$). For numerical purposes, it is convenient to develop both expressions as a second-order Taylor series for small t (i.e., small values of c_1) as follows:

$$E[m(t;m_0)] \approx \bar{m}_0 - \frac{c_1}{c_2}\left(\bar{m}_0^2 + c_2^2 + \frac{\delta_m^2}{3}\right) + \frac{c_1^2}{c_2^2}\bar{m}_0(c_2^2+\delta_m^2+\bar{m}_0^2) \tag{A3}$$

$$\begin{aligned}\mathrm{Var}[m(t;m_0)] &\approx \frac{\delta_m^2}{3} - \frac{4c_1}{3c_2}(\bar{m}_0\delta_m^2)\\ &\quad + \frac{2c_1^2}{45c_2^2}\delta_m^2(15c_2^2+11\delta_m^2+75\bar{m}_0^2)\end{aligned} \tag{A4}$$

B. Gamma Distribution of the Initial Mass

For the gamma distribution, the exact value of the mean obtained from Eq. (5) is

$$\begin{aligned}E[m(t;m_0)] &= c_2\int_{M_0}^{\infty}\frac{m_0-c_1c_2}{c_2+c_1m_0}(m_0-M_0)^{k-1}\frac{e^{-(m_0-M_0)\frac{\sqrt{3k}}{\delta_m}}}{(\frac{\delta_m}{\sqrt{3k}})^k\Gamma(k)}\,\mathrm{d}m_0\\ &= \frac{c_2}{c_1(t)} - \frac{c_2^2(c_1^2(t)+1)}{c_1^2(t)\frac{\delta_m}{\sqrt{3k}}}e^{\frac{\sqrt{3k}}{\delta_m}(M_0+\frac{c_2}{c_1(t)})}\\ &\quad\times\left(\frac{M_0+\frac{c_2}{c_1(t)}}{\frac{\delta_m}{\sqrt{3k}}}\right)^{k-1}\Gamma\left(1-k,\frac{M_0+\frac{c_2}{c_1(t)}}{\frac{\delta_m}{\sqrt{3k}}}\right)\end{aligned} \tag{A5}$$

where M_0 is defined by Eq. (10) and $\Gamma(s,x)$ is the upper incomplete Euler gamma function defined as

$$\Gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}\,\mathrm{d}t$$

(see [18]).

The variance of $m(t)$ is as follows:

$$\begin{aligned}\mathrm{Var}[m(t;m_0)] &= E[m^2(t;m_0)] - (E[m(t;m_0)])^2\\ &= c_2^2\int_{M_0}^{\infty}\frac{(m_0-c_1c_2)^2}{(c_2+c_1m_0)^2}(m_0-M_0)^{k-1}\frac{e^{-(m_0-M_0)\frac{\sqrt{3k}}{\delta_m}}}{(\frac{\delta_m}{\sqrt{3k}})^k\Gamma(k)}\,\mathrm{d}m_0\\ &\quad-(E[m(t;m_0)])^2 = \left[\left(1-k-\frac{M_0+\frac{c_2}{c_1(t)}}{\frac{\delta_m}{\sqrt{3k}}}\right)e^{\frac{\sqrt{3k}}{\delta_m}(M_0+\frac{c_2}{c_1(t)})}\right.\\ &\quad\times\left(\frac{M_0+\frac{c_2}{c_1(t)}}{\frac{\delta_m}{\sqrt{3k}}}\right)^k\Gamma\left(1-k,\frac{M_0+\frac{c_2}{c_1(t)}}{\frac{\delta_m}{\sqrt{3k}}}\right)+\frac{M_0+\frac{c_2}{c_1(t)}}{\frac{\delta_m}{\sqrt{3k}}}\\ &\quad\left.-e^{\frac{\sqrt{3k}}{\delta_m}(M_0+\frac{c_2}{c_1(t)})}\left(\frac{M_0+\frac{c_2}{c_1(t)}}{\frac{\delta_m}{\sqrt{3k}}}\right)^{2k}\left(\Gamma\left(1-k,\frac{M_0+\frac{c_2}{c_1(t)}}{\frac{\delta_m}{\sqrt{3k}}}\right)\right)^2\right]\\ &\quad\times\frac{c_2^4}{c_1^2(t)}\left(\frac{c_1^2(t)+1}{c_1(t)M_0+c_2}\right)^2\end{aligned} \tag{A6}$$

Expressions (A5) and (A6) are both indeterminate for $t=0$ (which implies $c_1=0$). For numerical purposes, it is convenient to approximate both expressions up to order 3 in $1/c_1$ using the asymptotic series

$$\Gamma(s,x) = x^{s-1}e^{-x}\left(1+(s-1)\frac{1}{x}+(s-1)(s-2)\frac{1}{x^2}+\ \ldots\right)$$

valid for $x\to\infty$ (see [18]). It follows that

$$\begin{aligned}E[m(t;m_0)] &\approx \frac{c_2M_0-c_2^2c_1}{c_1M_0+c_2}\\ &\quad+k\frac{c_2^2(c_1^2+1)}{c_1M_0+c_2}\left[\left(\frac{\frac{\delta_m}{\sqrt{3k}}}{c_1M_0+c_2}\right)-(k+1)c_1\left(\frac{\frac{\delta_m}{\sqrt{3k}}}{c_1M_0+c_2}\right)^2\right.\\ &\quad+(k+1)(k+2)c_1^2\left(\frac{\frac{\delta_m}{\sqrt{3k}}}{c_1M_0+c_2}\right)^3\\ &\quad-(k+1)(k+2)(k+3)c_1^3\left(\frac{\frac{\delta_m}{\sqrt{3k}}}{c_1M_0+c_2}\right)^4\\ &\quad\left.+(k+1)(k+2)(k+3)(k+4)c_1^4\left(\frac{\frac{\delta_m}{\sqrt{3k}}}{c_1M_0+c_2}\right)^5\right]\end{aligned} \tag{A7}$$

$$\begin{aligned}&\mathrm{Var}[m(t;m_0)]\\ &\approx\frac{\delta_m^2}{3}\frac{(c_1^2+1)^2c_2^4}{(c_1M_0+c_2)^4}\left[1-4(k+1)\left(\frac{c_1\frac{\delta_m}{\sqrt{3k}}}{c_1M_0+c_2}\right)\right.\\ &\quad+2(k+1)(5k+9)\left(\frac{c_1\frac{\delta_m}{\sqrt{3k}}}{c_1M_0+c_2}\right)^2\\ &\quad-4(1+k)(2+k)(12+5k)\left(\frac{c_1\frac{\delta_m}{\sqrt{3k}}}{c_1M_0+c_2}\right)^3\\ &\quad\left.+(1+k)(2+k)(300+7k(29+5k))\left(\frac{c_1\frac{\delta_m}{\sqrt{3k}}}{c_1M_0+c_2}\right)^4\right]\end{aligned} \tag{A8}$$

A2. Distribution Function of the Mass

To compute the distribution function of the mass, note that $\partial m/\partial m_0$ can be exactly computed from Eq. (5) as

$$\frac{\partial m}{\partial m_0}(t;m_0) = \frac{A}{B}\frac{1+\tan^2(\sqrt{AB}t)}{(\sqrt{\frac{A}{B}}+m_0\tan(\sqrt{AB}t))^2} = c_2^2\frac{1+c_1^2(t)}{(c_2+m_0c_1(t))^2} \tag{A9}$$

Also from Eq. (5), m_0 can be written in terms of $m(t)$ as follows:

$$m_0 = c_2 \frac{m(t) + c_2 c_1(t)}{c_2 - c_1(t) m(t)} \tag{A10}$$

Thus $\partial m / \partial m_0$ in terms of m is written as

$$\frac{\partial m}{\partial m_0}(t; m) = \frac{(c_2 - c_1(t)m)^2}{c_2^2(1 + c_1^2(t))} \tag{A11}$$

A. Uniform Distribution of the Initial Mass

Since

$$f_{m_0} = \frac{1}{2\delta_m}$$

using Eqs. (23) and (A11), the exact distribution function of the mass as a function of time is

$$f_m(m, t) = \frac{c_2^2(1 + c_1^2(t))}{2\delta_m (c_2 - c_1(t)m)^2} \tag{A12}$$

if

$$m \in \left[\frac{c_2(\bar{m}_0 - \delta_m) - c_2^2 c_1(t)}{c_2 + c_1(t)(\bar{m}_0 - \delta_m)}, \frac{c_2(\bar{m}_0 + \delta_m) - c_2^2 c_1(t)}{c_2 + c_1(t)(\bar{m}_0 + \delta_m)} \right]$$

and zero otherwise. The limit points in the interval have been found from Eq. (5) evaluated at the limit points in the initial mass distribution ($\bar{m}_0 - \delta_m$ and $\bar{m}_0 + \delta_m$).

B. Gamma Distribution of the Initial Mass

In this case, the distribution function f_{m_0} given by Eq. (11) has to be written in terms of $m(t)$ using Eq. (A10) as follows:

$$f_{m_0}(m, t) = \left(\frac{c_2 m + c_2^2 c_1(t)}{c_2 - c_1(t)m} - M_0 \right)^{k-1} \frac{e^{-\frac{\sqrt{3k}}{\delta_m}\left(\frac{c_2 m + c_2^2 c_1(t)}{c_2 - c_1(t)m} - M_0\right)}}{\left(\frac{\delta_m}{\sqrt{3k}}\right)^k \Gamma(k)} \tag{A13}$$

Then, using Eqs. (23), (A11), and (A13), the exact distribution function of the mass as a function of time is

$$f_m(m, t) = \left(\frac{c_2 m + c_2^2 c_1(t)}{c_2 - c_1(t)m} - M_0 \right)^{k-1} \times \frac{e^{-\frac{\sqrt{3k}}{\delta_m}\left(\frac{c_2 m + c_2^2 c_1(t)}{c_2 - c_1(t)m} - M_0\right)}}{\left(\frac{\delta_m}{\sqrt{3k}}\right)^k \Gamma(k)} \frac{c_2^2(1 + c_1^2(t))}{(c_2 - c_1(t)m)^2} \tag{A14}$$

if

$$m \geq c_2 \frac{M_0 - c_2 c_1(t)}{c_2 + M_0 c_1(t)}$$

and zero otherwise. The lower limit is found evaluating Eq. (5) at $m_0 = M_0$.

A3. Distribution Function of the Flight Time

To compute Eq. (32), using Eq. (6), the values of φ and its inverse can be explicitly obtained as

$$t_f = \varphi(m_0) = \frac{1}{\sqrt{AB}} \arctan\left(\frac{\sqrt{AB} m_F}{A + B m_0 (m_0 - m_F)} \right) \tag{A15}$$

$$m_0 = \varphi^{-1}(t_f) = \frac{m_F}{2} + \sqrt{\frac{m_F^2}{4} - \frac{A}{B} + \frac{\sqrt{\frac{A}{B}} m_F}{\tan(\sqrt{AB} t_f)}} = \frac{m_F}{2} + \Phi(t_f) \tag{A16}$$

where

$$\Phi(t_f) = \sqrt{\frac{m_F^2}{4} - \frac{A}{B} + \frac{\sqrt{\frac{A}{B}} m_F}{\tan(\sqrt{AB} t_f)}}$$

is defined to simplify the expressions. Also $\varphi'(m_0)$ is given by

$$\varphi'(m_0) = \frac{-B m_F (2m_0 - m_F)}{(A + B m_0 (m_0 - m_F))^2 + AB m_F^2} \tag{A17}$$

Hence,

$$|\varphi'(\varphi^{-1}(t_f))| = \frac{2\sin^2(\sqrt{AB} t_f)}{A m_F} \Phi(t_f) \tag{A18}$$

These results are now used to derive an explicit expression for f_{t_f}, for the two initial mass distributions under consideration.

A. Uniform Distribution of the Initial Mass

From Eq. (32), using

$$f_{m_0} = \frac{1}{2\delta_m}$$

and Eq. (A18), the resulting expression for the exact distribution function of the flight time is

$$f_{t_f}(t_f) = \frac{A m_F}{4\delta_m \sin^2(\sqrt{AB} t_f) \Phi(t_f)} \tag{A19}$$

if $t_f \in [T_1, T_2]$, and zero otherwise, where the endpoints of this interval are found evaluating Eq. (A15) at the endpoints of the uniform distribution of m_0 (namely, $\bar{m}_0 - \delta_m$ and $\bar{m}_0 + \delta_m$) and are given by

$$T_1 = \frac{1}{\sqrt{AB}} \arctan\left(\frac{\sqrt{AB} m_F}{A + B(\bar{m}_0 + \delta_m)(\bar{m}_0 + \delta_m - m_F)} \right)$$

and

$$T_2 = \frac{1}{\sqrt{AB}} \arctan\left(\frac{\sqrt{AB} m_F}{A + B(\bar{m}_0 - \delta_m)(\bar{m}_0 - \delta_m - m_F)} \right)$$

B. Gamma Distribution of the Initial Mass

To find f_{t_f} now, the distribution function f_{m_0} for the gamma case [Eq. (11)] has to be written in terms of t_f using Eq. (A16) as follows:

$$f_{m_0}(t_f) = \left(\frac{m_F}{2} + \Phi(t_f) - M_0 \right)^{k-1} \frac{e^{-\frac{\sqrt{3k}}{\delta_m}\left(\frac{m_F}{2} + \Phi(t_f) - M_0\right)}}{\left(\frac{\delta_m}{\sqrt{3k}}\right)^k \Gamma(k)} \tag{A20}$$

Then, from Eq. (32), using Eqs. (A24) and (A20), the resulting expression for the exact distribution function of the flight time is

$$f_{t_f}(t_f) = \left(\frac{m_F}{2} + \Phi(t_f) - M_0 \right)^{k-1} \frac{A m_F}{2\sin^2(\sqrt{AB} t_f)} \frac{e^{-\frac{\sqrt{3k}}{\delta_m}\left(\frac{m_F}{2} + \Phi(t_f) - M_0\right)}}{\left(\frac{\delta_m}{\sqrt{3k}}\right)^k \Gamma(k) \Phi(t_f)} \tag{A21}$$

for

$$t_f \in \left(0, \frac{1}{\sqrt{AB}} \arctan\left(\frac{\sqrt{AB} m_F}{A + BM_0(M_0 - m_F)}\right)\right]$$

and zero otherwise. The upper limit value is found evaluating Eq. (A15) at $m_0 = M_0$.

A4. Distribution Function of the Fuel Consumption

To compute Eq. (40), the inverse of $\psi(m_0)$ is necessary. For that, one has to solve for m_0 in Eq. (9), finding

$$m_0 = \psi^{-1}(m_F) = \frac{m_F}{2} + \sqrt{\frac{m_F^2}{4} + \frac{m_F c_2}{c_3} - c_2^2} = \frac{m_F}{2} + \Psi(m_F) \tag{A22}$$

where

$$\Psi(m_F) = \sqrt{\frac{m_F^2}{4} + \frac{m_F c_2}{c_3} - c_2^2}$$

has been defined.

Also, taking the derivative with respect to m_0 in Eq. (9), the value of $\psi'(m_0)$ is found as

$$\psi'(m_0) = c_3 \frac{2c_2 m_0 + (m_0^2 - c_2^2)c_3}{(c_2 + m_0 c_3)^2} \tag{A23}$$

For Eq. (40), it is necessary to explicitly compute $|\psi'(\psi^{-1}(m_F))|$ using Eqs. (A22) and (A23), finding

$$|\psi'(\psi^{-1}(m_F))| = c_3 \frac{(\frac{m_F}{2} + \Psi(m_F))^2 c_3 + 2(\frac{m_F}{2} + \Psi(m_F))c_2 - c_3 c_2^2}{[c_2 + c_3(\frac{m_F}{2} + \Psi(m_F))]^2} \tag{A24}$$

This result is now used to derive an explicit expression for f_{m_F}, for the two initial mass distributions under consideration.

A. Uniform Distribution of the Initial Mass

From Eq. (40), using

$$f_{m_0} = \frac{1}{2\delta_m}$$

and Eq. (A24), the resulting expression for the exact distribution function of the flight time is

$$f_{m_F}(m_F) = \frac{1}{2\delta_m c_3} \frac{[c_2 + c_3(\frac{m_F}{2} + \Psi(m_F))]^2}{(\frac{m_F}{2} + \Psi(m_F))^2 c_3 + 2(\frac{m_F}{2} + \Psi(m_F))c_2 - c_3 c_2^2} \tag{A25}$$

if

$$m_F \in \left[\frac{((\bar{m}_0 - \delta_m)^2 + c_2^2)c_3}{c_2 + (\bar{m}_0 - \delta_m)c_3}, \frac{((\bar{m}_0 + \delta_m)^2 + c_2^2)c_3}{c_2 + (\bar{m}_0 + \delta_m)c_3}\right]$$

and zero otherwise. The endpoints of this interval are found by evaluating Eq. (9) at the endpoints of the uniform distribution of m_0 ($\bar{m}_0 - \delta_m$ and $\bar{m}_0 + \delta_m$).

B. Gamma Distribution of the Initial Mass

To find f_{m_F} now, the distribution function f_{m_0} for the gamma case [Eq. (11)] has to be written in terms of m_F using Eq (A22) as follows:

$$f_{m_0}(m_F) = \left(\frac{m_F}{2} + \Psi(m_F) - M_0\right)^{k-1} \frac{e^{-(\frac{m_F}{2} + \Psi(m_F) - M_0)\frac{\sqrt{3k}}{\delta_m}}}{(\frac{\delta_m}{\sqrt{3k}})^k \Gamma(k)} \tag{A26}$$

Then, from Eq. (40), using Eqs. (A24) and (A26), the resulting expression for the exact distribution function of the fuel consumption is

$$f_{m_F}(m_F) = \frac{e^{-(\frac{m_F}{2} + \Psi(m_F) - M_0)\frac{\sqrt{3k}}{\delta_m}}}{(\frac{\delta_m}{\sqrt{3k}})^k \Gamma(k) c_3} \frac{(\frac{m_F}{2} + \Psi(m_F) - M_0)^{k-1}[c_2 + c_3(\frac{m_F}{2} + \Psi(m_F))]^2}{(\frac{m_F}{2} + \Psi(m_F))^2 c_3 + 2(\frac{m_F}{2} + \Psi(m_F))c_2 - c_3 c_2^2} \tag{A27}$$

for

$$m_F \geq \frac{(M_0^2 + c_2^2)c_3}{c_2 + M_0 c_3}$$

and zero otherwise. The lower limit is found evaluating Eq. (9) at $m_0 = M_0$.

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