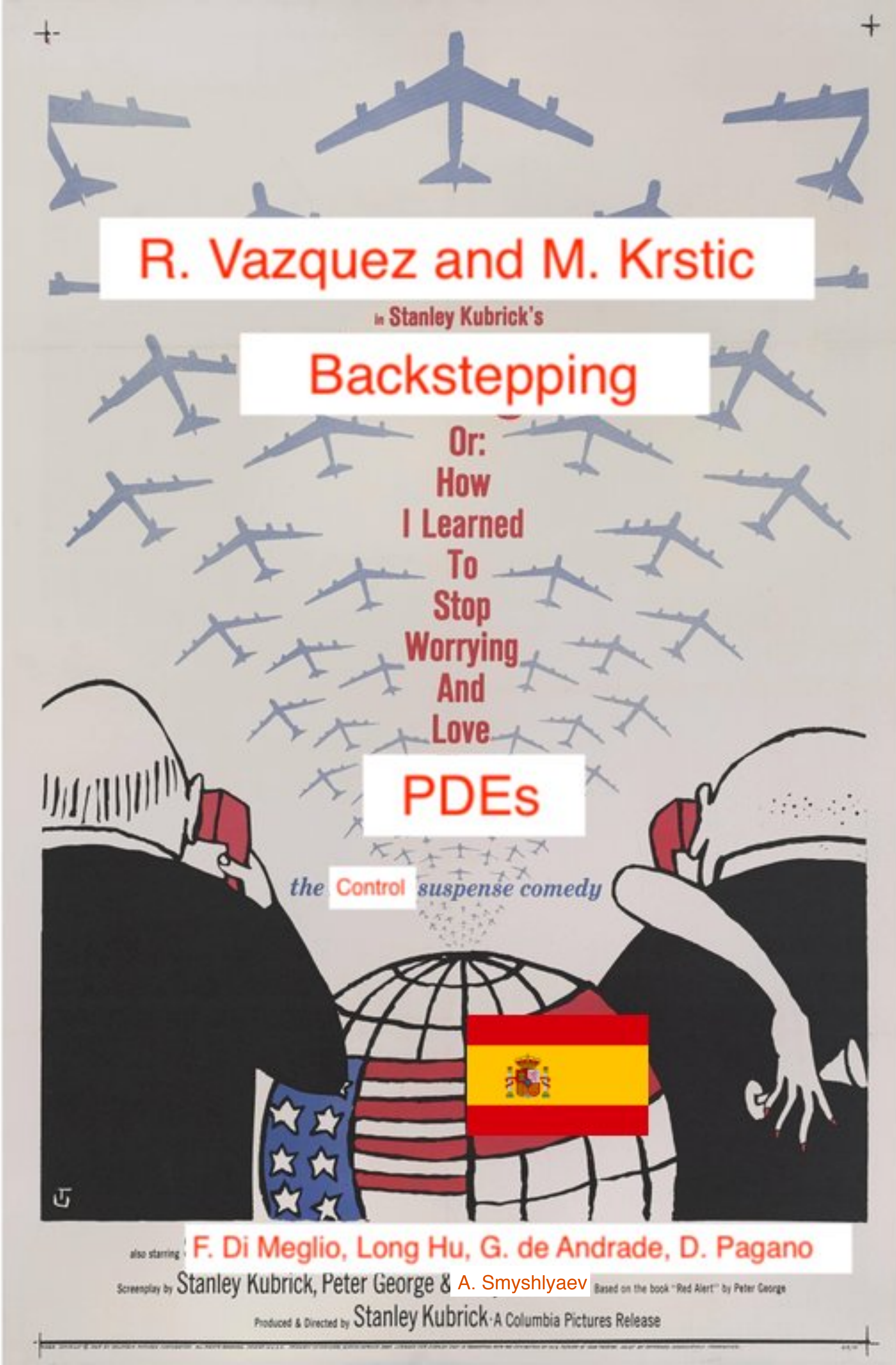
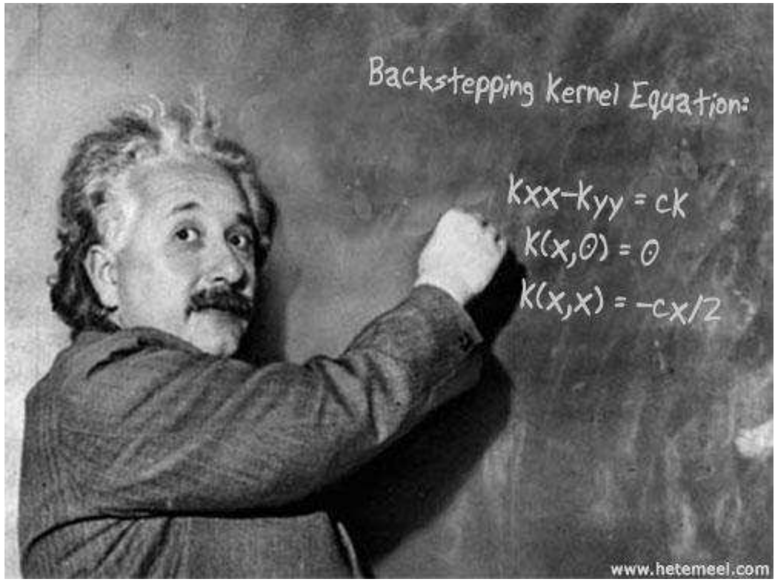

R. Vazquez and M. Krstic
in Stanley Kubrick's
Backstepping
Or:
How
I Learned
To
Stop
Worrying
And
Love
PDEs
the Control suspense comedy
also starring F. Di Meglio, Long Hu, G. de Andrade, D. Pagano
Screenplay by Stanley Kubrick, Peter George & A. Smyshlyaev
Based on the book "Red Alert" by Peter George
Produced & Directed by Stanley Kubrick · A Columbia Pictures Release

Disclaimer

- This presentation is exactly what a presentation should not be:
  - Too long
  - Full of equations
  - Some proofs nobody cares about
  - Jumping between topics
  - Clearly copy-pasted from past presentations with different formats
  - Potentially bad humor
- Please read Bob Bitmead's "Things I hate in other people's seminars" (`http://oodgeroo.ucsd.edu/~bob/Bob_zone_site/Literary_diversions_files/Things.pdf`)
- However ... when talking among close peers about topics you have been doing for more than a decade, perhaps some allowances are permitted...
- Solving kernel equations in your head is not really a pre-requisite ... but it helps!

The Age of Smyshlyaev & Krstic

San Diego, year 2002. Andrey Smyshlyaev starts his PhD.



Year 2004. Publication of **A. Smyshlyaev and M. Krstic, "Closed form boundary state feedbacks for a class of partial integro-differential equations," IEEE Transactions on Automatic Control, vol. 49, pp. 2185-2202, 2004.**

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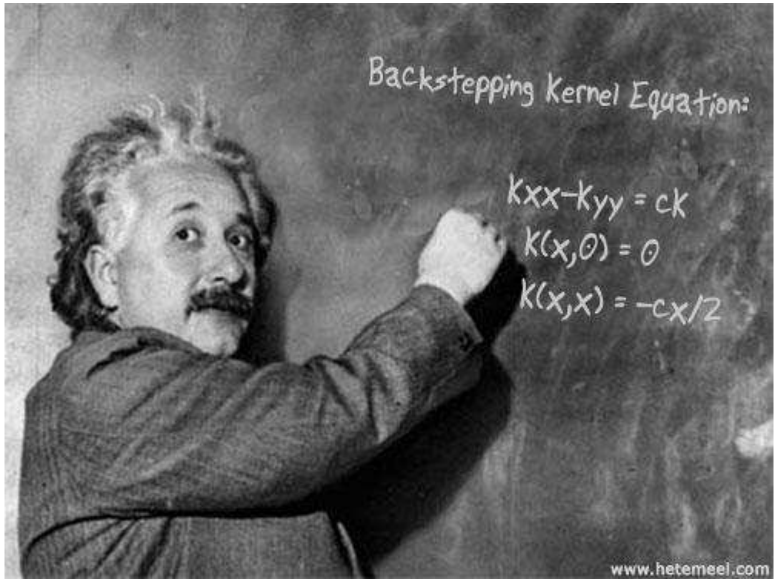

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This manuscript is a true template followed (sometimes too closely) by most backstepping papers.

A golden age starts for backstepping, and many problems are solved!

The Age of Smyshlyaev & Krstic

The main idea is to use a first-design, then discretize approach and follow the next steps:

1. Identify the undesirable terms in the PDE.

2. Choose a target system in which the undesirable terms are to be eliminated by state transformation and feedback, as in feedback linearization.

3. Find the state transformation typically as *identity minus a Volterra operator* (in x).
   Volterra operator = integral operator from 0 up to x (rather than from 0 to 1).
   A Volterra transformation is "triangular" or "spatially causal."

4. Obtain boundary feedback from the Volterra transformation. The transformation alone cannot eliminate the undesirable terms, but the transformantion brings them to the boundary, so control can cancel them.

5. Study the invertibility of the transformation (identity minus Volterra is always invertible).

The Age of Smyshlyaev & Krstic

Gain fcn of boundary controller = kernel of Volterra transformation.

Volterra kernel satisfies a *linear* PDE.

The Age of Smyshlyaev & Krstic

Gain fcn of boundary controller = kernel of Volterra transformation.

Volterra kernel satisfies a *linear* PDE.

Backstepping is not “one-size-fits-all.” Requires structure-specific effort by designer.

Reward: elegant controller, clear closed-loop behavior. ?

Outline

- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls
- Symmetric disk
- Rijke Tube
- My old friend Volterra
- Some open problems

Backstepping control of coupled hyperbolic 1-D systems

$$u_t(t,x) + \Sigma^+ u_x(t,x) = \Lambda^{++} u(t,x) + \Lambda^{+-} v(t,x)$$
$$v_t(t,x) - \Sigma^- v_x(t,x) = \Lambda^{-+} u(t,x) + \Lambda^{--} v(t,x)$$

with the following boundary conditions

$$u(t,0) = 0, \qquad v(t,1) = U(t)$$

where

$$u = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}^T, \qquad v = \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix}^T$$

$$\Sigma^+ = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix}, \qquad \Sigma^- = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_m \end{pmatrix}$$

with

$$-\mu_1 < \cdots < -\mu_m < 0 < \varepsilon_1 \leq \cdots \leq \varepsilon_n$$

Backstepping control of coupled hyperbolic 1-D systems

Backstepping transformation

$$\alpha(t,x) = u(t,x)$$
$$\beta(t,x) = v(t,x) - \int_0^x [L(x,\xi)u(\xi) + K(x,\xi)v(\xi)]\, d\xi$$

L and K defined on the triangular domain $\mathcal{T}$.

Target system

$$\alpha_t(t,x) + \Sigma^+ \alpha_x(t,x) = \Lambda^{++}\alpha(t,x) + \Lambda^{+-}\beta(t,x) + \int_0^x D^+(x,\xi)\alpha(\xi) d\xi + \int_0^x D^-(x,\xi)\beta(\xi) d\xi$$
$$\beta_t(t,x) - \Sigma^- \beta_x(t,x) = G(x)\beta(0)$$

with boundary conditions

$$\alpha(t,0) = \beta(t,1) = 0$$

Backstepping control of coupled hyperbolic 1-D systems

Structure of G is **lower-diagonal with diagonal of zeros**

$$G(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1}(x) & \cdots & g_{m,m-1}(x) & 0 \end{pmatrix}$$

It can be shown to make stable

$$\beta_t(t,x) - \Sigma^- \beta_x(t,x) = G(x)\beta(0)$$

From there follows target system stability.

Backstepping control of coupled hyperbolic 1-D systems

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$$G(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1}(x) & \cdots & g_{m,m-1}(x) & 0 \end{pmatrix}$$

It can be shown to make stable

$$\beta_t(t,x) - \Sigma^- \beta_x(t,x) = G(x)\beta(0)$$

From there follows target system stability.

G(x) is not chosen, but computed from the kernels.

Backstepping control of coupled hyperbolic 1-D systems

Kernel equations

$$
\begin{aligned}
0 =& \Sigma^- L_x(x,\xi) - L_\xi(x,\xi)\Sigma^+ - L(x,\xi)\Lambda^{++} - K(x,\xi)\Lambda^{-+} \\
0 =& \Sigma^- K_x(x,\xi) + K_\xi(x,\xi)\Sigma^- - K(x,\xi)\Lambda^{--} - L(x,\xi)\Lambda^{+-}
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
0 =& L(x,x)\Sigma^+ + \Sigma^- L(x,x) + \Lambda^{-+} \\
0 =& \Sigma^- K(x,x) - K(x,x)\Sigma^- + \Lambda^{--} \\
0 =& G(x) - K(x,0)\Sigma^-
\end{aligned}
$$

Too many boundary conditions?

Backstepping control of coupled hyperbolic 1-D systems

Kernel equations

$$
\begin{aligned}
0 =& \Sigma^- L_x(x,\xi) - L_\xi(x,\xi)\Sigma^+ - L(x,\xi)\Lambda^{++} - K(x,\xi)\Lambda^{-+} \\
0 =& \Sigma^- K_x(x,\xi) + K_\xi(x,\xi)\Sigma^- - K(x,\xi)\Lambda^{--} - L(x,\xi)\Lambda^{+-}
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
0 =& L(x,x)\Sigma^+ + \Sigma^- L(x,x) + \Lambda^{-+} \\
0 =& \Sigma^- K(x,x) - K(x,x)\Sigma^- + \Lambda^{--} \\
0 =& G(x) - K(x,0)\Sigma^-
\end{aligned}
$$

Again, too many boundary conditions?

No, in fact more boundary conditions are needed $\longrightarrow$ Nonuniqueness!

Backstepping control of coupled hyperbolic 1-D systems

Developing the equations:

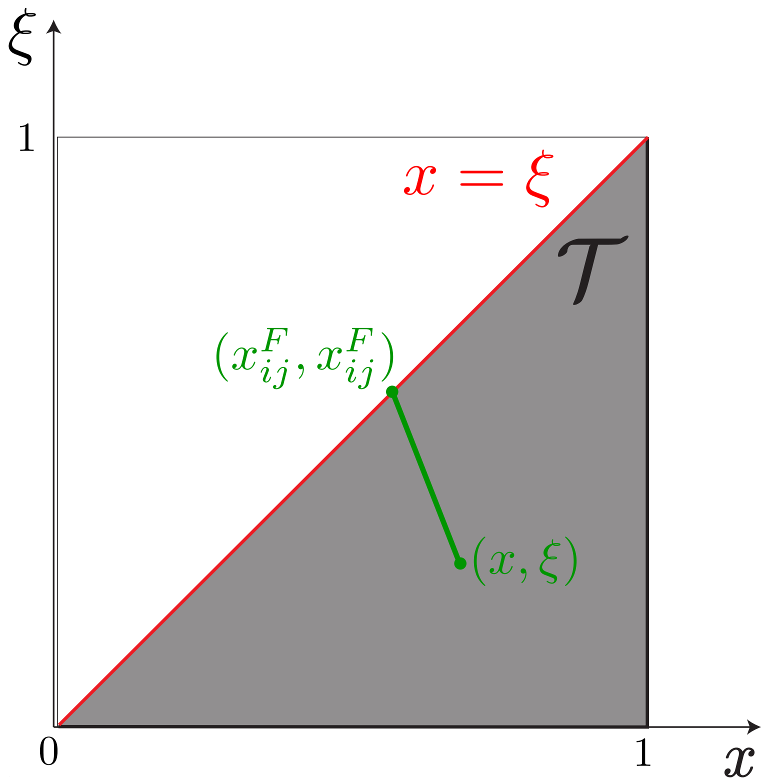
$$
\begin{aligned}
\mu_i \partial_x L_{ij}(x,\xi) - \varepsilon_j \partial_\xi L_{ij}(x,\xi) &= \sum_{k=1}^{n} \lambda_{kj}^{++} L_{ik}(x,\xi) + \sum_{p=1}^{m} \lambda_{pj}^{-+} K_{ip}(x,\xi) \\
\mu_i \partial_x K_{ij}(x,\xi) + \mu_j \partial_\xi K_{ij}(x,\xi) &= \sum_{p=1}^{m} \lambda_{pj}^{--} K_{ip}(x,\xi) + \sum_{k=1}^{n} \lambda_{kj}^{+-} L_{ik}(x,\xi)
\end{aligned}
$$

with boundary conditions:

$$
\begin{aligned}
\forall 1 \le i \le m, 1 \le j \le n, &\quad L_{ij}(x,x) = -\frac{\lambda_{ij}^{-+}}{\mu_i + \varepsilon_j} \\
\forall 1 \le i, j \le m, i \ne j, &\quad K_{ij}(x,x) = -\frac{\lambda_{ij}^{--}}{\mu_i - \mu_j} \\
\forall 1 \le i \le j \le m, &\quad K_{ij}(x,0) = 0 \\
\forall 1 \le j < i \le m, &\quad K_{ij}(1,\xi) = l_{ij} \\
\forall 1 \le j < i \le m, &\quad g_{ij}(x) = \mu_j K_{ij}(x,0)
\end{aligned}
$$

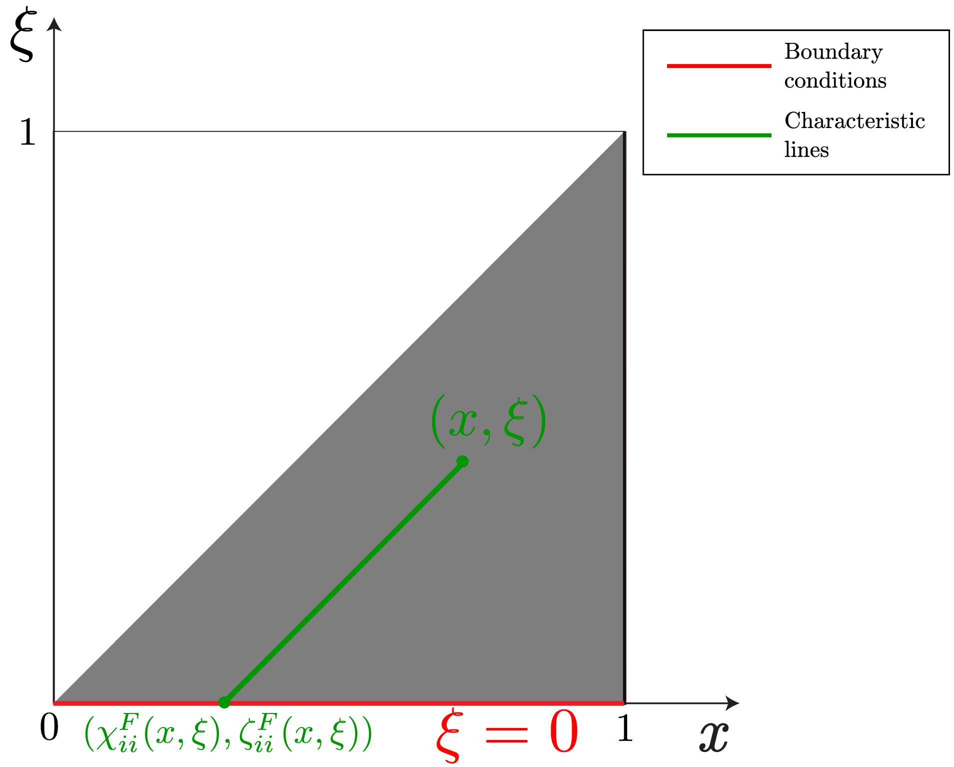
Well-posedness depends on the characteristics!

Characteristics for L_{ij}



$$
\begin{aligned}
\mu_i \partial_x L_{ij}(x,\xi) - \varepsilon_j \partial_\xi L_{ij}(x,\xi) &= \sum_{k=1}^{n} \lambda_{kj}^{++} L_{ik}(x,\xi) + \sum_{p=1}^{m} \lambda_{pj}^{-+} K_{ip}(x,\xi) \\
L_{ij}(x,x) &= -\frac{\lambda_{ij}^{-+}}{\mu_i + \varepsilon_j}
\end{aligned}
$$

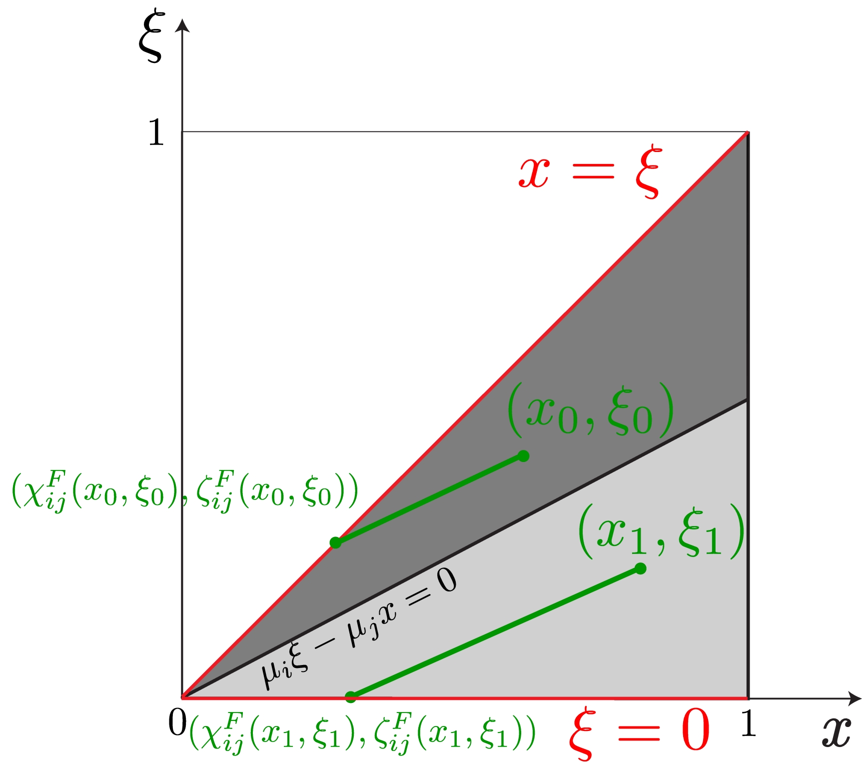
Characteristics for K_{ii}



$$\mu_i \partial_x K_{ii}(x,\xi) + \mu_i \partial_\xi K_{ii}(x,\xi) = \sum_{p=1}^{m} \lambda_{pi}^{--} K_{ip}(x,\xi) + \sum_{k=1}^{n} \lambda_{ki}^{+-} L_{ik}(x,\xi)$$

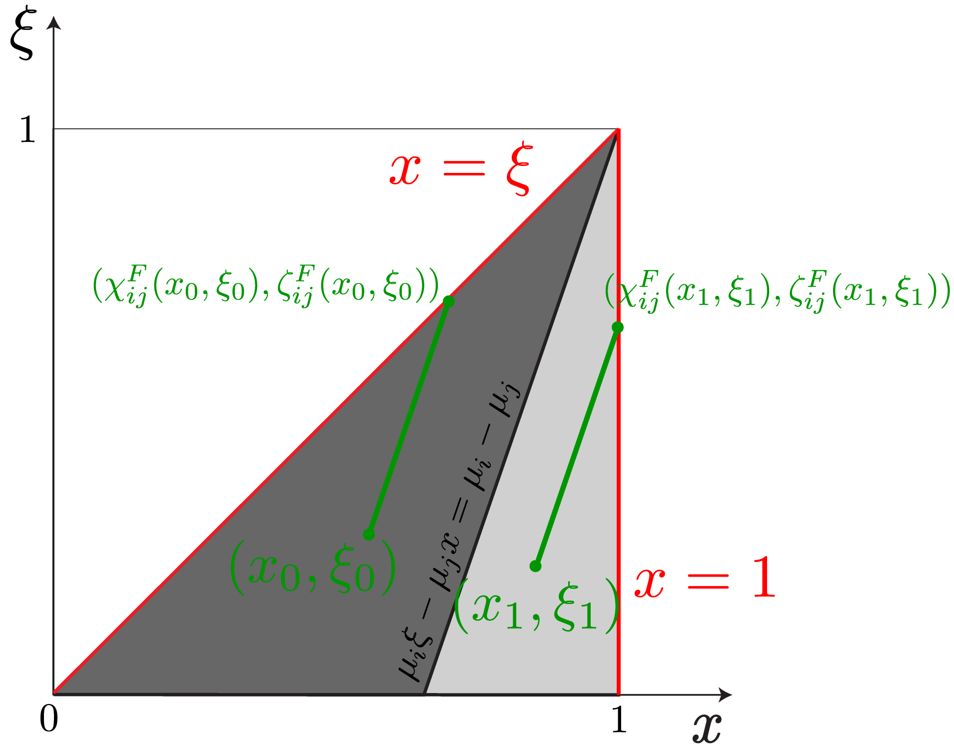
$$K_{ii}(x,0) = 0$$

Characteristics for $K_{ij}, i < j$



$$
\begin{aligned}
\mu_i \partial_x K_{ij}(x,\xi) + \mu_j \partial_\xi K_{ij}(x,\xi) &= \sum_{p=1}^{m} \lambda^{--}_{pj} K_{ip}(x,\xi) + \sum_{k=1}^{n} \lambda^{+-}_{kj} L_{ik}(x,\xi) \\
K_{ij}(x,x) &= -\frac{\lambda^{--}_{ij}}{\mu_i - \mu_j} \\
K_{ij}(x,0) &= 0
\end{aligned}
$$

Characteristics for $K_{ij}, i > j$



$$
\begin{aligned}
\mu_i \partial_x K_{ij}(x,\xi) + \mu_j \partial_\xi K_{ij}(x,\xi) &= \sum_{p=1}^{m} \lambda_{pj}^{--} K_{ip}(x,\xi) + \sum_{k=1}^{n} \lambda_{kj}^{+-} L_{ik}(x,\xi) \\
K_{ij}(x,x) &= -\frac{\lambda_{ij}^{--}}{\mu_i - \mu_j} \\
K_{ij}(1,\xi) &= l_{ij} \\
g_{ij}(x) &= \mu_j K_{ij}(x,0)
\end{aligned}
$$

Backstepping control of coupled hyperbolic 1-D systems

The presented approach produces piecewise continuous and differentiable kernels.

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There are potential lines of discontinuity, which complicate kernel calculation, but do not affect the stability result.

Backstepping control of coupled hyperbolic 1-D systems

The presented approach produces piecewise continuous and differentiable kernels.

There are potential lines of discontinuity, which complicate kernel calculation, but do not affect the stability result.

Next we see how we can produce a strikingly similar result for reaction-diffusion equations.

Outline

- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls
- Symmetric disk
- Rijke Tube
- My old friend Volterra
- Some open problems

Coupled parabolic systems

Consider

$$u_t(t,x) = \Sigma u_{xx}(t,x) + \Lambda(x)u(t,x)$$

$x \in [0,1]$, $t > 0$, $u \in \mathbb{R}^n$

$$\Sigma = \begin{bmatrix} \varepsilon_1 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon_n \end{bmatrix}, \quad \Lambda(x) = \begin{bmatrix} \lambda_{11}(x) & \lambda_{12}(x) & \dots & \lambda_{1n}(x) \\ \lambda_{21}(x) & \lambda_{22}(x) & \dots & \lambda_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1}(x) & \lambda_{n2}(x) & \dots & \lambda_{nn}(x) \end{bmatrix}$$

with $\varepsilon_i > 0$ ordered, i.e., $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_n > 0$, and boundary conditions

$$\begin{aligned} u(0,t) &= 0, \\ u(1,t) &= U(t) \end{aligned}$$

with $U \in \mathbb{R}^n$.

Backstepping approach

Consider the Backstepping Transformation :

$$w(t,x) = u(t,x) - \int_0^x K(x,\xi)u(t,\xi)d\xi$$

with $K(x,\xi)$ a $n \times n$ matrix of kernels, and w verifies the Target System :

$$w_t(t,x) = \Sigma w_{xx}(t,x) - Cw(t,x) - G(x)w_x(0,t),$$

with C and $G(x)$:

$$C = \begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_n \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ g_{21}(x) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{(n-1)1}(x) & g_{(n-1)2}(x) & \dots & 0 & 0 \\ g_{n1}(x) & g_{n2}(x) & \dots & g_{n(n-1)}(x) & 0 \end{bmatrix}$$

where $c_1, c_2, \dots, c_n > 0$. Control law is then

$$U(t) = \int_0^1 K(1,\xi)u(t,\xi)d\xi$$

The challenge is to prove that $K(x,\xi)$ exists and has good properties $\longrightarrow$ Kernel equations

Kernel equations

$$\Sigma K_{xx} - K_{\xi\xi}\Sigma = K\Lambda(\xi) + CK,$$

with b.c.

$$\begin{aligned} G(x) &= -K(x,0)\Sigma, \\ K(x,x)\Sigma &= \Sigma K(x,x), \\ C + \Lambda(x) &= -\Sigma K_x(x,x) - \Sigma \frac{d}{dx} K(x,x) - K_\xi(x,x)\Sigma. \end{aligned}$$

Kernel equations

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First b.c. with structure of G becomes:

$$K_{ij}(x,0) = 0, \qquad \forall j \geq i,$$

and

$$g_{ij}(x) = -K_{ij}(x,0)\varepsilon_j, \qquad \forall j < i,$$

Kernel equations

$$\Sigma K_{xx} - K_{\xi\xi}\Sigma = K\Lambda(\xi) + CK,$$

with b.c.

$$\begin{aligned} G(x) &= -K(x,0)\Sigma, \\ K(x,x)\Sigma &= \Sigma K(x,x), \\ C + \Lambda(x) &= -\Sigma K_x(x,x) - \Sigma \frac{d}{dx}K(x,x) - K_\xi(x,x)\Sigma. \end{aligned}$$

Second b.c. is:

$$K_{ij}(x,x) = 0, \qquad \forall j \neq i,$$

(no boundary condition for $K_{ii}(x,x)$)

Kernel equations

$$\Sigma K_{xx} - K_{\xi\xi}\Sigma = K\Lambda(\xi) + CK,$$

with b.c.

$$\begin{aligned} G(x) &= -K(x,0)\Sigma, \\ K(x,x)\Sigma &= \Sigma K(x,x), \\ C + \Lambda(x) &= -\Sigma K_x(x,x) - \Sigma \frac{d}{dx} K(x,x) - K_\xi(x,x)\Sigma. \end{aligned}$$

Third boundary condition:

$$0 = \lambda_{ij}(x) + \delta_{ij} c_i + K_{ij\xi}(x,x)\varepsilon_j + \varepsilon_i K_{ijx}(x,x) + \varepsilon_i \frac{d}{dx}\left(K_{ij}(x,x)\right),$$

Duplicating the kernel equations

Key idea ("duplication"): define

$$L(x,\xi) \quad = \quad \sqrt{\Sigma}K_x(x,\xi)+K_\xi(x,\xi)\sqrt{\Sigma} \longrightarrow L_{ij}(x,x)=\sqrt{\varepsilon_i}K_{ijx}(x,x)+\sqrt{\varepsilon_j}K_{ij\xi}(x,x)$$

Then we can rewrite the "duplicated" kernel equations as

$$\begin{aligned} \sqrt{\Sigma}K_x+K_\xi\sqrt{\Sigma} &= L \\ \sqrt{\Sigma}L_x-L_\xi\sqrt{\Sigma} &= K\Lambda(\xi)+CK \end{aligned}$$

Same structure as in the coupled hyperbolic result!

Third boundary condition becomes:

$$i=j\text{: } 0=\lambda_{ii}(x)+c_i+2\varepsilon_i(K_{iix}(x,x)+K_{ii\xi}(x,x)) \longrightarrow L_{ii}(x,x)=-\frac{\lambda_{ii}(x)+c_i}{2\sqrt{\varepsilon_i}}$$

$$i\neq j\text{: } 0=\lambda_{ij}(x)+(\varepsilon_i-\varepsilon_j)K_{ijx}(x,x) \longrightarrow L_{ij}(x,x)=-\frac{\lambda_{ij}(x)}{\sqrt{\varepsilon_i}+\sqrt{\varepsilon_j}}$$

Duplicating the kernel equations

The boundary conditions therefore are:

- If $i = j$

$$\begin{aligned} L_{ii}(x,x) &= -\frac{\lambda_{ii}(x) + c_i}{2\sqrt{\varepsilon_i}} \\ K_{ii}(x,0) &= 0 \end{aligned}$$

- If $i < j$

$$\begin{aligned} K_{ij}(x,x) &= K_{ij}(x,0) = 0 \\ L_{ij}(x,x) &= -\frac{\lambda_{ij}(x)}{\sqrt{\varepsilon_i} + \sqrt{\varepsilon_j}} \end{aligned}$$

- Finally if $i > j$

$$\begin{aligned} K_{ij}(x,x) &= 0 \\ K_{ij}(1,\xi) &= l_{ij}(\xi) \\ L_{ij}(x,x) &= -\frac{\lambda_{ij}(x)}{\sqrt{\varepsilon_i} + \sqrt{\varepsilon_j}} \end{aligned}$$

  and the additional condition $g_{ij}(x) = -K_{ij}(x,0)\varepsilon_j$

Same structure as in the coupled hyperbolic result!

Extension to reaction-advection-diffusion systems with spatially-varying coefficients

The method can be extended to

$$u_t = \partial_x (\Sigma(x) u_x) + \Phi(x) u_x + \Lambda(x) u$$

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Reaction-diffusion equation on an n-dimensional ball

Let the state $u = u(t,\vec{x})$, with $\vec{x} = [x_1, x_2, \ldots, x_n]^T$, verify

$$\frac{\partial u}{\partial t} = \varepsilon \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \right) + \lambda u = \varepsilon \triangle_n u + \lambda u,$$

for constant $\varepsilon > 0$, $\lambda(r,\vec{\theta})$, and for $t > 0$, in the n-ball $B^n(R)$ defined as

$$B^n(R) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| < R\},$$

with b.c. on the boundary of $B^n(R)$, the $(n-1)$-sphere $S^{n-1}(R)$:

$$S^{n-1}(R) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = R\}.$$

The b.c. is of Dirichlet type:

$$u(t,\vec{x})\Big|_{\vec{x} \in S^{n-1}(R)} = U(t,\vec{x})$$

where $U(t,\vec{x})$ is the actuation variable.

Ultraspherical coordinates

The n-ball domain is well described in n-dimensional spherical coordinates, also known as ultraspherical coordinates:

- one radial coordinate r, $r \in [0, R)$.
- $n-1$ angular coordinates: $\vec{\theta} = [\theta_1, \theta_2, \ldots, \theta_{n-1}]^T$, with $\theta_1 \in [0, 2\pi)$ and $\theta_i \in [0, \pi]$ for $2 \leq i \leq n-1$.

Definition:

$$
\begin{aligned}
x_1 &= r\cos\theta_1 \sin\theta_2 \sin\theta_3 \ldots \sin\theta_{n-1}, \\
x_2 &= r\sin\theta_1 \sin\theta_2 \sin\theta_3 \ldots \sin\theta_{n-1}, \\
x_3 &= r\cos\theta_2 \sin\theta_3 \ldots \sin\theta_{n-1}, \\
&\vdots \\
x_{n-1} &= r\cos\theta_{n-2} \sin\theta_{n-1}, \\
x_n &= r\cos\theta_{n-1}.
\end{aligned}
$$

Laplacian in ultraspherical coordinates

Writing the reaction diffusion equation in ultraspherical coordinates

$$\begin{aligned} u_t &= \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u\right) + \frac{1}{r^2}\triangle^*_{n-1}u + \lambda u, \\ u(t,R,\vec{\theta}\,) &= U(t,\vec{\theta}\,), \end{aligned}$$

where $\triangle^*_{n-1}$ is called the Laplace-Beltrami operator and represents the Laplacian over the $(n-1)$-sphere.

It is defined recursively as

$$\begin{aligned} \triangle^*_1 &= \frac{\partial^2}{\partial\theta_1^2}, \\ \triangle^*_n &= \frac{1}{\sin^{n-1}\theta_n}\frac{\partial}{\partial\theta_n}\left(\sin^{n-1}\theta_n\frac{\partial}{\partial\theta_n}\right) + \frac{\triangle^*_{n-1}}{\sin^2\theta_n}, \end{aligned}$$

Example:

$$\triangle^*_2 = \frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial}{\partial\theta_2}\right) + \frac{1}{\sin^2\theta_2}\frac{\partial^2}{\partial\theta_1^2}.$$

Designing a boundary feedback law

- Exploit periodicity in $\vec{\theta}$ by using Spherical Harmonics
- Apply the backstepping method to each harmonic coefficient
- Solve the backstepping kernel equations to find a feedback law for each harmonic
- Re-assemble the feedback law in Spherical Harmonics back to physical space

Spherical Harmonics

Develop u and U in term of Spherical Harmonics coefficients u_l^m and U_l^m:

$$u(t,r,\vec{\theta}) = \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=N(l,n)-1} u_l^m(r,t) Y_{lm}^n(\vec{\theta}), \quad U(t,\vec{\theta}) = \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=N(l,n)-1} U_l^m(t) Y_{lm}^n(\vec{\theta}),$$

$N(l,n)$: number of (linearly independent) n-dimensional spherical harmonics of degree l

$$N(l,n) = \frac{2l+n-2}{l} \begin{pmatrix} l+n-3 \\ l-1 \end{pmatrix}, \quad l > 0; \qquad N(0,n) = 1$$

$Y_{lm}^n(\vec{\theta})$: m-th order n-dimensional spherical harmonic of degree l

Coefficients are defined as:

$$u_l^m(r,t) = \int_0^{\pi} \ldots \int_0^{\pi} \int_0^{2\pi} u(t,r,\vec{\theta}) \bar{Y}_{lm}^n(\vec{\theta}) \sin^{n-2}\theta_{n-1} \sin^{n-3}\theta_{n-2} \ldots \sin\theta_2 d\vec{\theta},$$

$$U_l^m(t) = \int_0^{\pi} \ldots \int_0^{\pi} \int_0^{2\pi} U(t,\vec{\theta}) \bar{Y}_{lm}^n(\vec{\theta}) \sin^{n-2}\theta_{n-1} \sin^{n-3}\theta_{n-2} \ldots \sin\theta_2 d\vec{\theta},$$

($d\vec{\theta} = d\theta_{n-1} d\theta_{n-2} \ldots d\theta_2 d\theta_1$, $\bar{Y}_{lm}^n$ is the complex conjugate of Y_{lm}^n)

Spherical Harmonics

The n-dimensional spherical harmonics are **eigenfunctions** for the Laplacian $\triangle^*_{n-1}$:

$$\triangle^*_{n-1} Y^n_{lm} = -l(l+n-2) Y^n_{lm}.$$

Thus, each harmonic coefficient $u^m_l(t,r)$ for $l \in \mathbb{N}$ and $0 \leq m \leq N(l,n)$, verifies

$$\partial_t u^m_l = \frac{\varepsilon}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r u^m_l \right) - l(l+n-2) \frac{\varepsilon}{r^2} u^m_l + \lambda u^m_l,$$

evolving in $r \in [0,R]$, $t > 0$, with boundary conditions

$$u^m_l(t,R) \quad = \quad U^m_l(t),$$

The PDEs for the harmonics are not coupled: we can independently design each U^m_l and later assemble all of the them to find an expression for U.

Backstepping control of Spherical Harmonics coefficients

To design $U_l^m(t)$ seek transformation of

$$\partial_t u_l^m = \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}u_l^m + \lambda u_l^m$$

into the (stable) target system

$$\partial_t w_l^m = \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r w_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}w_l^m$$

with boundary conditions

$$w_l^m(t,R) = 0$$

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with boundary conditions

$$w_l^m(t,R) = 0$$

The transformation is

$$w_l^m(t,r) = u_l^m(t,r) - \int_0^r K_{lm}^n(r,\rho) u_l^m(t,\rho) d\rho$$

with kernels K_{lm}^n to be found.

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with kernels K_{lm}^n to be found.

Substituting at $r = R$ we find $U_l^m(t)$ as

$$U_l^m(t)(t) = \int_0^R K_{lm}^n(R,\rho)u_l^m(t,\rho)d\rho$$

Kernel equation

The control kernels $K_{lm}^n(r,\rho)$ are found, for a given $n \geq 2$ and each l, m, from

$$\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r K_{lm}^n\right) - \partial_\rho\left(\rho^{n-1}\partial_\rho\left(\frac{K_{lm}^n}{\rho^{n-1}}\right)\right) - l(l+n-2)\left(\frac{1}{r^2}-\frac{1}{\rho^2}\right)K_{lm}^n = \frac{\lambda}{\varepsilon}K_{lm}^n.$$

with BC

$$\begin{aligned}
\lambda + 2\varepsilon\frac{d}{dr}\left(K_{lm}^n(r,r)\right) &= 0 \\
K_{lm}^n(r,0) &= 0 \\
(n-2)\partial_\rho K_{lm}^n(r,\rho)|_{\rho=0} &= 0
\end{aligned}$$

Kernel equation

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$$\begin{aligned} \lambda + 2\varepsilon\frac{d}{dr}\left(K_{lm}^n(r,r)\right) &= 0 \\ K_{lm}^n(r,0) &= 0 \\ (n-2)\partial_\rho K_{lm}^n(r,\rho)|_{\rho=0} &= 0 \end{aligned}$$

The first BC integrates (using $K_{lm}^n(0,0)=0$) to

$$K_{lm}^n(r,r) = -\int_0^r \frac{\lambda}{2\varepsilon}d\rho = -\frac{\lambda r}{2\varepsilon}$$

Explicit Kernel equation solution and feedback law

It is found that

$$K_{lm}^{n}(r,\rho) = -\rho\left(\frac{\rho}{r}\right)^{l+n-2}\frac{\lambda}{\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}}$$

Thus the feedback law for each spherical harmonic is

$$U_l^m(t) = \int_0^R K_{lm}^{n}(R,\rho)u_l^m(t,\rho)d\rho = \int_0^R -\rho\left(\frac{\rho}{R}\right)^{l+n-2}\frac{\lambda}{\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}}u_l^m(t,\rho)d\rho$$

Explicit feedback law

Using some spherical harmonics machinery one obtains an explicit feedback law

$$
\begin{aligned}
U(t,\theta) &= -\frac{\lambda}{\varepsilon}\int_0^R \rho \frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}} \\
&\quad \times \left[\int_0^{\pi}\!\!\cdots\!\int_0^{\pi}\int_0^{2\pi} P(R,\rho,\vec{\theta},\vec{\phi})u(t,\rho,\vec{\phi})\rho^{n-1}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\ldots\sin\phi_2 d\vec{\phi}\right] d
\end{aligned}
$$

where $P(R,\rho,\vec{\theta},\vec{\phi})$ is the Poisson kernel for the n-ball.

Back in rectangular coordinates

$$
U(t,\vec{x}) = -\frac{1}{\mathrm{Area}(S^{n-1})}\sqrt{\frac{\lambda}{\varepsilon}}\int_{B^n(R)} \mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\|\vec{\xi}\|^2)}\right]\frac{\sqrt{R^2-\|\vec{\xi}\|^2}}{\|\vec{x}-\vec{\xi}\|^n}u(t,\vec{\xi})d\vec{\xi},
$$

where the integral is extended to the complete n-ball $B^n(R)$ and $\vec{x}\in S^{n-1}(R)$.

Outline

- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls
- Symmetric disk
- Rijke Tube
- My old friend Volterra
- Some open problems

Extension to spatially-varying λ

Consider now the same problem but with spatially-varying coefficient λ:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \varepsilon \triangle_n u + \lambda(\vec{x}) u, \\ u(t,\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} &= U(t,\vec{x}) \end{aligned}$$

the question is: can backstepping still be applied?

Extension to spatially-varying λ

Consider now the same problem but with spatially-varying coefficient $\lambda(\vec{x})$:

$$
\begin{aligned}
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u(t,\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} &= U(t,\vec{x})
\end{aligned}
$$

the question is: can backstepping still be applied?

Consider two cases:

- Sphere
- Disk (harder!)

Under a simplifying assumption, we solve the problem and get a taste of the challenges

Revolution symmetry condition

Revolution Symmetry Condition : if the initial conditions are symmetric (do not depend on the angle or angles in 3-D), and U is chosen constant (do not depend on the position in the boundary) nothing depends on the angle.

Typical engineering simplification. Equations becomes 1-D in radius, with singularities.

Disk: $u_t = \frac{\varepsilon}{r}(ru_r)_r + \lambda(r)u$

Sphere: $u_t = \frac{\varepsilon}{r^2}\left(r^2 u_r\right)_r + \lambda(r)u$

We apply the method as before but only one kernel (corresponding to the constant Fourier mode or Spherical Harmonic) is needed.

3-D case—revolution symmetry

Kernel equation is:

$$
\begin{aligned}
K_{rr} + 2\frac{K_r}{r} - K_{\rho\rho} + 2\frac{K_\rho}{\rho} - 2\frac{K}{\rho^2} &= \frac{\lambda(r)}{\varepsilon}K \\
K(r,0) &= K_\rho(r,0) = 0, \\
K(r,r) &= -\frac{\lambda r}{2\varepsilon},
\end{aligned}
$$

3-D case—revolution symmetry

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Define $K(r,\rho) = \frac{\rho}{r}\bar{K}(r,\rho)$. Then:

$$\begin{aligned} \bar{K}_{rr} - \bar{K}_{\rho\rho} &= \frac{\lambda(r)}{\varepsilon}\bar{K} \\ \bar{K}(r,0) &= 0, \\ \bar{K}(r,r) &= -\frac{\lambda r}{2\varepsilon}, \end{aligned}$$

which is the 1-D backstepping equation! Can be proved solvable by successive approximations (classical backstepping papers).

3-D case—revolution symmetry

For instance if λ is constant we directly get:

$$K(r,\rho) = \frac{\rho}{r}\bar{K}(r,\rho) = \frac{\rho^2}{r}\frac{c}{\varepsilon}\frac{I_1\left[\sqrt{\frac{c}{\varepsilon}\left(r^2-\rho^2\right)}\right]}{\sqrt{\frac{c}{\varepsilon}\left(r^2-\rho^2\right)}}$$

2-D case—revolution symmetry

Interestingly, the 2-D case is harder than the 3-D case. Kernel equations are

$$
\begin{aligned}
K_{rr} + \frac{K_r}{r} - K_{\rho\rho} + \frac{K_\rho}{\rho} - \frac{K}{\rho^2} &= \frac{\lambda(\rho)}{\varepsilon} K, \\
K(r,0) &= 0, \\
K(r,r) &= -\int_0^r \frac{\lambda(\rho)}{2\varepsilon} d\rho
\end{aligned}
$$

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Interestingly, the 2-D case is harder than the 3-D case. Kernel equations are

$$\begin{aligned} K_{rr} + \frac{K_r}{r} - K_{\rho\rho} + \frac{K_\rho}{\rho} - \frac{K}{\rho^2} &= \frac{\lambda(\rho)}{\varepsilon} K, \\ K(r,0) &= 0, \\ K(r,r) &= -\int_0^r \frac{\lambda(\rho)}{2\varepsilon} d\rho \end{aligned}$$

Define $G = \sqrt{\frac{r}{\rho}} K$. Then, for G we have:

$$\begin{aligned} G_{rr} - G_{\rho\rho} + \frac{G}{4r^2} - \frac{G}{4\rho^2} &= \frac{\lambda(\rho)}{\varepsilon} G \\ G(r,0) &= 0, \\ G(r,r) &= -\int_0^r \frac{\lambda(\rho)}{2\varepsilon} d\rho. \end{aligned}$$

and we can try to prove existence & uniqueness of a solution by using the classical successive approximation method.

2-D case—revolution symmetry

Define new variables $\alpha = r + \rho$, $\beta = r - \rho$. The G equations become

$$\begin{aligned}
4G_{\alpha\beta} + \frac{G}{(\alpha+\beta)^2} - \frac{G}{(\alpha-\beta)^2} &= \frac{\lambda\left(\frac{\alpha-\beta}{2}\right)}{\varepsilon} G \\
G(\beta,\beta) &= 0, \\
G(\alpha,0) &= -\int_0^{\alpha/2} \frac{\lambda(\rho)}{2\varepsilon} d\rho.
\end{aligned}$$

2-D case—revolution symmetry

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This can be transformed into the (singular) integral equation

$$\begin{aligned} G(\alpha,\beta) &= -\int_{\beta/2}^{\alpha/2} \frac{\lambda(\rho)}{2\varepsilon} d\rho + \int_{\beta}^{\alpha} \int_0^{\beta} \frac{\lambda\left(\frac{\eta-\sigma}{2}\right)}{4\varepsilon} G(\eta,\sigma)\, d\sigma\, d\eta \\ &\quad + \int_{\beta}^{\alpha} \int_0^{\beta} \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} G(\eta,\sigma)\, d\sigma\, d\eta \end{aligned}$$

2-D case—revolution symmetry

Try the successive approximations scheme, by defining

$$G_0(\alpha,\beta) \;=\; -\int_{\beta/2}^{\alpha/2} \frac{\lambda(\rho)}{2\varepsilon} d\rho$$

and for $k > 0$,

$$G_k(\alpha,\beta) \;=\; \int_{\beta}^{\alpha}\int_{0}^{\beta} \frac{\lambda\left(\frac{\eta-\sigma}{2}\right)}{4\varepsilon} G_{k-1}(\eta,\sigma)\,d\sigma d\eta + \int_{\beta}^{\alpha}\int_{0}^{\beta} \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} G_{k-1}(\eta,\sigma)\,d\sigma d\eta$$

then, the solution to the integral equation would be

$$G = \sum_{k=0}^{\infty} G_k(\alpha,\beta)$$

if the series converges.

2-D case—revolution symmetry

Call $\bar{\lambda} = \max_{(\alpha,\beta)\in\mathcal{T}'} \left| \frac{\lambda\left(\frac{\alpha-\beta}{2}\right)}{4\varepsilon} \right|$.

Then one clearly obtains $|G_0(\alpha,\beta)| \leq \bar{\lambda}(\alpha-\beta)$.

However when trying to substitute in G_1 even the first integral is not so easy to perform.

2-D case—revolution symmetry

Call $\bar{\lambda} = \max_{(\alpha,\beta)\in\mathcal{T}'} \left| \frac{\lambda\left(\frac{\alpha-\beta}{2}\right)}{4\varepsilon} \right|$.

Then one clearly obtains $|G_0(\alpha,\beta)| \le \bar{\lambda}(\alpha-\beta)$.

However when trying to substitute in G_1 even the first integral is not so easy to perform. We use an alternative approach based on the following Lemma:

Define, for $n \ge 0, k \ge 0$,

$$F_{nk}(\alpha,\beta) = \frac{\bar{\lambda}^{n+1}\alpha^n\beta^n}{n!(n+1)!}(\alpha-\beta)\frac{\log^k\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}{k!}.$$

and $F_{nk} = 0$ if $n < 0$ or $k < 0$. Then F_{nk} is well-defined and nonnegative in the integration domain for all n, k, $F_{nk}(\beta,\beta) = 0$ for all n and k, $F_{nk}(\alpha,0) = 0$ if $n \ge 1$ or $k \ge 1$ and $F_{00}(\alpha,0) = \alpha$, and we have the following identity valid for $n \ge 1$ or $k \ge 1$.

$$F_{nk} = \int_\beta^\alpha \int_0^\beta \bar{\lambda} F_{(n-1)k}(\eta,\sigma)\,d\sigma d\eta + 4\int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2}\left(F_{n(k-1)}(\eta,\sigma) - F_{n(k-2)}(\eta,\sigma)\right) d\sigma d$$

2-D case—revolution symmetry

We use the lemma to try to find estimates for the terms in the successive approximation series:

$$|G_0| \leq F_{00}$$

next

$$|G_1| \leq \int_\beta^\alpha \int_0^\beta \bar{\lambda} F_{00}(\eta,\sigma)\, d\sigma d\eta + \int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} F_{00}(\eta,\sigma)\, d\sigma d\eta = F_{10} + \frac{F_{01}}{4}$$

where we have used the formulas of the lemma. The next term is

$$\begin{aligned} |G_2| &\leq \int_\beta^\alpha \int_0^\beta \bar{\lambda} \left(F_{10} + \frac{F_{01}}{4} \right) d\sigma d\eta + \int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} \left(F_{10} + \frac{F_{01}}{4} \right) d\sigma d\eta \\ &= F_{20} + \frac{F_{11}}{4} + \frac{F_{01}+F_{02}}{16} \end{aligned}$$

If we keep going we find

$$|G_3| \leq F_{30} + \frac{F_{21}}{4} + \frac{F_{11}+F_{12}}{16} + \frac{2F_{01}+2F_{02}+F_{03}}{64}$$

2-D case—revolution symmetry

The key to find these numbers is the following. Call:

$$
\begin{aligned}
I_1[F] &= \int_\beta^\alpha \int_0^\beta \bar{\lambda} F(\eta,\sigma)\, d\sigma\, d\eta \\
I_2[F] &= \int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} F(\eta,\sigma)\, d\sigma\, d\eta
\end{aligned}
$$

For instance, to find a bound on G_4 we find the following:

$$
\begin{aligned}
I_1[F_{30}] &= F_{40} \\
I_2[F_{30}] + \frac{I_1[F_{21}]}{4} &= \frac{F_{31}}{4} \\
\frac{I_2[F_{21}]}{4} + \frac{I_1[F_{11}+F_{12}]}{16} &= \frac{F_{21}+F_{22}}{16} \\
\frac{I_2[F_{11}+F_{12}]}{16} + \frac{I_1[2F_{01}+2F_{02}+F_{03}]}{64} &= \frac{2F_{11}+2F_{12}+F_{13}}{64} \\
\frac{I_2[2F_{01}+2F_{02}+F_{03}]}{64} &= \frac{5F_{01}+5F_{02}+3F_{03}+F_{04}}{256}
\end{aligned}
$$

Thus,

$$
|G_4| \le F_{40} + \frac{F_{31}}{4} + \frac{F_{21}+F_{22}}{16} + \frac{2F_{11}+2F_{12}+F_{13}}{64} + \frac{5F_{01}+5F_{02}+3F_{03}+F_{04}}{256}
$$

2-D case—revolution symmetry

Based on this structure, we propose the following recursive formula for $n > 0$:

$$|G_n| \leq F_{n0} + \sum_{i=0}^{n-1} \sum_{j=1}^{j=n-i} \frac{C_{(n-i)j}}{4^{n-i}} F_{ij}$$

where C_{ij} verifies $C_{ij} = C_{(i-1)(j-1)} + C_{i(j+1)}$, taking $C_{11} = 1$, $C_{i0} = 0$, and $C_{ij} = 0$ if $j > i$, for all i. This set of numbers, known as the "Catalan's Triangle", verifies many interesting properties.

In particular it can be shown

$$\begin{aligned} C_{ii} &= 1. \\ C_{ij} &= \sum_{k=j-1}^{i-1} C_{(i-1)k}. \end{aligned}$$

which allows us to write the recursive formula

2-D case—revolution symmetry

Let us show in a table the first few numbers.

C_{ij}	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$
$i=1$	1									
$i=2$	1	1								
$i=3$	2	2	1							
$i=4$	5	5	3	1						
$i=5$	14	14	9	4	1					
$i=6$	42	42	28	14	5	1				
$i=7$	132	132	90	48	20	6	1			
$i=8$	429	429	297	165	75	27	7	1		
$i=9$	1430	1430	1001	572	275	110	35	8	1	
$i=10$	4862	4862	3432	2002	1001	429	154	44	9	1

Catalan’s Triangle

2-D case—revolution symmetry

Now, since the solution verifies

$$|G| \leq \sum_{n=0}^{\infty} |G_n(\alpha,\beta)|$$

and we found

$$|G_n| \leq F_{n0} + \sum_{i=0}^{n-1} \sum_{j=1}^{j=n-i} \frac{C_{(n-i)j}}{4^{n-i}} F_{ij}$$

We get

$$|G| \leq \sum_{n=0}^{\infty} F_{n0} + \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \sum_{j=1}^{j=n-i} \frac{C_{(n-i)j}}{4^{n-i}} F_{ij}$$

and we only need to prove convergence of this series.

2-D case—revolution symmetry

First term of the series:

$$\sum_{n=0}^{\infty} F_{n0} = \frac{\bar{\lambda}^{n+1}\alpha^n\beta^n}{n!(n+1)!}(\alpha-\beta) = \frac{\sqrt{\bar{\lambda}}}{2}(\alpha-\beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}}$$

For the next term, we use the fact that

$$\sum_{n=1}^{\infty}\sum_{i=0}^{n-1} H(n,i) = \sum_{i=0}^{\infty}\sum_{l=1}^{\infty} H(l+i,i)$$

Therefore

$$\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\sum_{j=1}^{j=n-i}\frac{C_{(n-i)j}}{4^{n-i}}F_{ij} = \sum_{i=0}^{\infty}\sum_{l=1}^{\infty}\sum_{j=1}^{j=l}\frac{C_{lj}}{4^l}F_{ij} = \sum_{i=0}^{\infty}\sum_{j=1}^{j=\infty}\left(\sum_{l=j}^{\infty}\frac{C_{lj}}{4^l}\right)F_{ij}$$

It turns out that the parenthesis can be calculated and gives an exact sum for each j.

2-D case—revolution symmetry

To find the sum, consider first the generating function of the Catalan numbers C_{l1}:

$$f_1(x) = \frac{2}{1+\sqrt{1-4x}}$$

Remember that a generating function of a sequence of number is a function such that the coefficients of its power series is exactly those of the sequence of numbers.

Thus,

$$f_1(x) = C_{11} + C_{21}x + C_{31}x^2 + \ldots = \sum_{l=1}^{\infty} C_{l1}x^{l-1}$$

Therefore if we evaluate the function at $x = 1/4$ we find that

$$f_1(\frac{1}{4}) = \sum_{l=1}^{\infty} C_{l1}\frac{1}{4^{l-1}}$$

thus we find

$$\sum_{l=1}^{\infty} \frac{C_{l1}}{4^l} = \frac{1}{4}\sum_{l=1}^{\infty} \frac{C_{lj}}{4^{l-1}} = \frac{f_1(\frac{1}{4})}{4} = \frac{1}{2}$$

2-D case—revolution symmetry

Following the previous argument, it is clear that

$$\sum_{l=j}^{\infty} \frac{C_{lj}}{4^l} = \frac{1}{4} \sum_{l=j}^{\infty} \frac{C_{lj}}{4^{l-1}} = \frac{f_j(\frac{1}{4})}{4}$$

where we define the generating function f_j as

$$f_j(x) = \sum_{l=j}^{\infty} C_{lj} x^{l-1}$$

Now since $C_{l2} = C_{l1}$ but obviously $C_{12} = 0$, it is clear that $f_2 = f_1 - C_{11} = f_1 - 1$. Thus $f_2(1/4) = 1$ and we find

$$\sum_{l=2}^{\infty} \frac{C_{l2}}{4^l} = \frac{f_2(\frac{1}{4})}{4} = \frac{1}{4}$$

2-D case—revolution symmetry

To find successive generating functions we use the properties of the Catalan's Triangle and make the following claim:

$$f_n(x) = f_{n-1}(x) - x f_{n-2}(x)$$

Based on this fact, we can now prove that

$$\sum_{l=j}^{\infty} \frac{C_{lj}}{4^l} = \frac{1}{2^j}$$

Thus we obtain

$$\begin{aligned} |G| &\leq \frac{\sqrt{\bar{\lambda}}}{2}(\alpha - \beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}} + \sum_{i=0}^{\infty}\sum_{j=1}^{j=\infty}\frac{F_{ij}}{2^j} \\ &= \frac{\sqrt{\bar{\lambda}}}{2}(\alpha - \beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}} + \sum_{i=0}^{\infty}\sum_{j=1}^{j=\infty}\frac{\bar{\lambda}^{i+1}\alpha^i\beta^i}{i!(i+1)!}(\alpha - \beta)\frac{\log^j\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}{2^j j!} \end{aligned}$$

2-D case—revolution symmetry

Summing the series

$$|G| \;\le\; \frac{\sqrt{\bar\lambda}}{2}(\alpha-\beta)\frac{I_1\left[2\sqrt{\bar\lambda\alpha\beta}\right]}{2\sqrt{\alpha\beta}}\left(\sum_{j=0}^{j=\infty}\frac{\log^j\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}{2^j j!}\right),$$

therefore

$$|G| \;\le\; \frac{\sqrt{\bar\lambda}}{2}(\alpha-\beta)\frac{I_1\left[2\sqrt{\bar\lambda\alpha\beta}\right]}{2\sqrt{\alpha\beta}}e^{\log\left(\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\right)} = \frac{\sqrt{\bar\lambda}}{2}\sqrt{\alpha^2-\beta^2}\frac{I_1\left[2\sqrt{\bar\lambda\alpha\beta}\right]}{2\sqrt{\alpha\beta}}$$

In physical variables r and ρ:

$$|G| \;\le\; \sqrt{\bar\lambda}\sqrt{r\rho}\frac{I_1\left[2\sqrt{\bar\lambda(r^2-\rho^2)}\right]}{2\sqrt{r^2-\rho^2}}$$

2-D case—revolution symmetry

Finally, going back to the original K, we find

$$|K(r,\rho)| \leq \rho\sqrt{\bar{\lambda}}\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}(r^2-\rho^2)}\right]}{2\sqrt{r^2-\rho^2}}$$

Thus, we have shown that the successive approximation series converges, with the solution K verifying the above bound. Uniqueness can be proved easily from the successive approximation series.

Unfortunately, this approach does not seem to be extensible for other Fourier coefficients.

Outline

- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls
- Symmetric disk
- Rijke Tube
- My old friend Volterra
- Some open problems

Backstepping-based linear boundary observer for estimation of thermoacoustic instabilities in a Rijke tube

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  - Mathematical model
  - Characteristic coordinates model description
- Backstepping-based observer design
  - Target system
  - Backstepping transformation
  - Well-posedness of the kernel equations and invertibility of the transformation
- Experimental results
- Suggested literature

- de Andrade, G. A., Vazquez, R., and Pagano, D. J. (2016). Boundary feedback control of unstable thermoacoustic oscillations in the rijke tube. In Proceedings of the 2nd IFAC workshop on control of systems governed by partial differential equations (pp. 48–53).

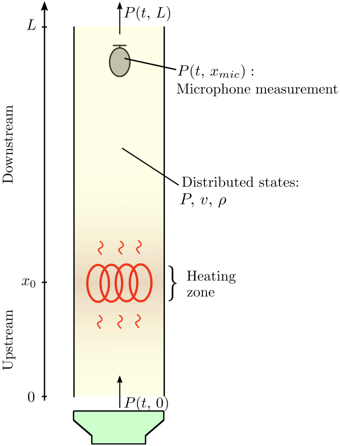
- de Andrade, G. A., Vazquez, R., and Pagano, D. J. (2017). Boundary control of a Rijke tube using irrational transfer functions with experimental validation. In Proceedings of the 20th IFAC world congress (pp. 4528–4533)

- de Andrade, G. A., Vazquez, R., and Pagano, D. J. (2018). Backstepping stabilization of a linearized ODE–PDE Rijke tube model. Automatica (pp. 98–109)

- de Andrade, G. A., Vazquez, R., and Pagano, D. J. (2018). Backstepping-based linear boundary observer for estimation of thermoacoustic instabilities in a Rijke tube. Accepted in 2018 CDC.

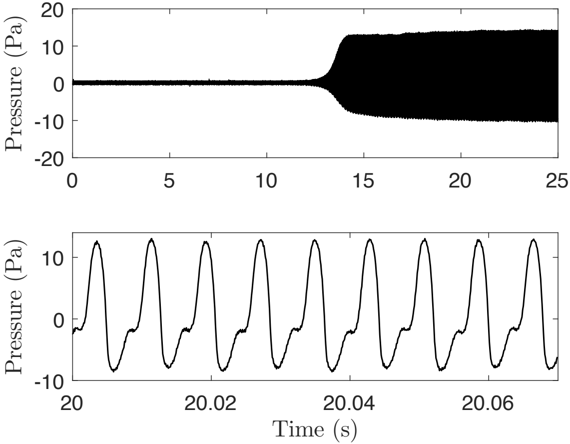
The Rijke Tube Experiment

- A vertical tube opened in both ends.
- A heat source is inserted in the lower half of the tube.
- A speaker under the tube is used as actuator, while a microphone at the top provides the pressure measurement.
- Under the right conditions, the tube begins to hum loudly (thermoacoustic instability).



The Rijke Tube Experiment

Microphone signal at the onset of instability showing growth,and then saturation of the limit cycle. A zoomed-in picture shows the periodic, but nonsymetric, limit-cycle behavior.



The Rijke Tube Experiment

- Thermoacoustic instabilities are often encountered in steam and gas turbines, industrial burners, and jet and ramjet engines.
- These instabilities are undesirable and notorious difficult to model and study.
- The absence of combustion process in the Rijke tube makes the modeling and analysis more tractable.
- The Rijke tube experiment provides an accessible platform to explore and study thermoacoustic instabilities.

Nonlinear mathematical model

- The thermoacoustic oscillations can be captured using an one-dimensional model of compressible gas dynamics (Euler equations)

$$\partial_t \rho(t,x) + v(t,x)\partial_x \rho(t,x) + \rho(t,x)\partial_x v(t,x) = 0, \tag{1}$$

$$\partial_t v(t,x) + \partial_x v(t,x) + \frac{1}{\rho(t,x)}\partial_x P(t,x) = 0, \tag{2}$$

$$\partial_t P(t,x) + \gamma P(t,x)\partial_x v(t,x) + v(t,x)\partial_x P(t,x) = \overline{\gamma}\frac{1}{A}\delta(x-x_0)Q(t), \tag{3}$$

- Heat release dynamics:

$$\tau \dot{Q}(t) = -Q(t) + l_w(T_w - T_{gas})(\kappa + \kappa_v \sqrt{|v(t,x_0)|}), \tag{4}$$

- Boundary conditions:

$$P(t,0) = P_a + U(t), \tag{5}$$

$$P(t,L) = P_a + f(v(t,L)), \tag{6}$$

Linearized mathematical model

- Assume constant steady-state solution, $(\rho,\, v\, P) = (\overline{\rho},\, \overline{v},\, \overline{P}), \forall t \in [0,\, +\infty)$, $\forall x \in [0,\, L]$, and subsonic conditions for the gas flow, i.e., $\overline{v} \approx 0$. Then,

$$\partial_t \tilde{v}(t,\, x) + \frac{1}{\overline{\rho}} \partial_x \tilde{P}(t,\, x) = 0, \tag{7}$$

$$\partial_t \tilde{P}(t,\, x) + \gamma \overline{P} \partial_x \tilde{v}(t,\, x) = \frac{\overline{\gamma}}{A} \delta(x - x_0) \tilde{Q}(t), \tag{8}$$

  and the linearized expression of the heat release dynamics

$$\tau \dot{\tilde{Q}}(t) = -\tilde{Q}(t) + f'(\overline{v})(T_w - \overline{T}_{gas}) \tilde{v}(t,\, x_0), \tag{9}$$

- Boundary conditions:

$$\tilde{P}(t,\, 0) = U(t), \tag{10}$$

$$\tilde{P}(t,\, L) = Z_L \tilde{v}(t,\, L), \tag{11}$$

Representation in characteristic coordinates

- Using the characteristic coordinates, the system (7)-(11) can be rewritten as

$$\partial_t R_1 + \lambda \partial_x R_1 = c_1 \delta(x - x_0) \tilde{Q}(t), \tag{12}$$

$$\partial_t R_2 - \lambda \partial_x R_2 = c_1 \delta(x - x_0) \tilde{Q}(t), \tag{13}$$

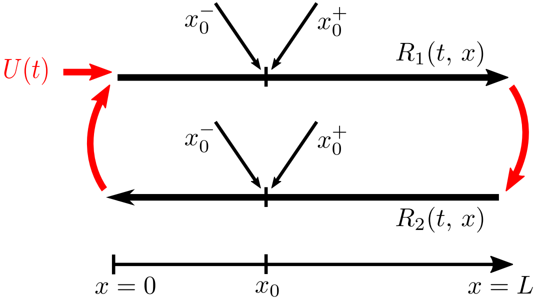
$$R_1(t, 0) = -R_2(t, 0) + 2U(t), \tag{14}$$

$$R_2(t, L) = \alpha R_1(t, L), \tag{15}$$

$$\tau \dot{\tilde{Q}}(t) = -\tilde{Q}(t) + c_2 (R_1(t, x_0) - R_2(t, x_0)), \tag{16}$$

with λ, α, c_1, $c_2 > 0$.

Representation in characteristic coordinates



Schematic view of the jumping point at the solution of the PDE system (12)-(16).

- The following relations are satisfied:

$$R_1(t, x_0^+) = R_1(t, x_0^-) + c_1\tilde{Q}(t),$$
$$R_2(t, x_0^-) = R_2(t, x_0^+) + c_1\tilde{Q}(t).$$

Representation in characteristic coordinates

- Now, we introduce the following state variables

$$\begin{aligned} R_{11}(t,x) &\triangleq R_1(t,x), && \text{if } x \in [0,\, x_0] \\ R_{12}(t,x) &\triangleq R_2(t,x), && \text{if } x \in [0,\, x_0] \\ R_{21}(t,x) &\triangleq R_1(t,x), && \text{if } x \in [x_0,\, L] \\ R_{22}(t,x) &\triangleq R_2(t,x), && \text{if } x \in [x_0,\, L] \end{aligned}$$

and the rescaled spatial variable, so that everything evolves on the same domain:

$$z = \begin{cases} \frac{x}{x_0} & \text{if } x \in [0,\, x_0] \\ \frac{L-x}{L-x_0} & \text{if } x \in [x_0,\, L] \end{cases}$$

- Then, the system (12)-(16) is equivalent to

$$\partial_t R_{11}(t,\, z) + \lambda_1 \partial_z R_{11}(t,\, z) = 0, \tag{17}$$

$$\partial_t R_{12}(t,\, z) - \lambda_1 \partial_z R_{12}(t,\, z) = 0, \tag{18}$$

$$\partial_t R_{21}(t,\, z) - \lambda_2 \partial_z R_{21}(t,\, z) = 0, \tag{19}$$

$$\partial_t R_{22}(t,\, z) + \lambda_2 \partial_z R_{22}(t,\, z) = 0, \tag{20}$$

- The boundary conditions of (17)-(20) are given by

$$R_{11}(t,0) = -R_{12}(t,\, 0) + 2U(t), \tag{21}$$

$$R_{12}(t,1) = R_{22}(t,\, 1) + c_1 \tilde{Q}(t), \tag{22}$$

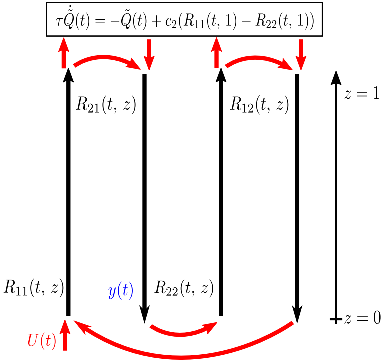
$$R_{21}(t,1) = R_{11}(t,\, 1) + c_1 \tilde{Q}(t), \tag{23}$$

$$R_{22}(t,0) = \alpha R_{21}(t,\, 0), \tag{24}$$

$$\tau \dot{\tilde{Q}}(t) = -\tilde{Q}(t) + c_2 (R_{11}(t,\, 1) - R_{22}(t,\, 1)). \tag{25}$$

Representation in characteristic coordinates

- The boundary conditions represent two effects: reflection of the acoustic waves; and the feedback coupling between R_{21} and R_{22}, and between R_{11} and R_{12}.
- Under the right conditions the system becomes unstable due to this feedback between the states.



Backstepping-based observer design

- We design the observe as a copy of the plant (17)-(24) plus output injection terms:

$$\partial_t \hat{R}_{11}(t,\, z) + \lambda_1 \partial_z \hat{R}_{11}(t,\, z) = -p_{11}(z)\tilde{Y}(t), \tag{26}$$

$$\partial_t \hat{R}_{12}(t,\, z) - \lambda_1 \partial_z \hat{R}_{12}(t,\, z) = -p_{12}(z)\tilde{Y}(t), \tag{27}$$

$$\partial_t \hat{R}_{21}(t,\, z) - \lambda_2 \partial_z \hat{R}_{21}(t,\, z) = -p_{21}(z)\tilde{Y}(t), \tag{28}$$

$$\partial_t \hat{R}_{22}(t,\, z) + \lambda_2 \partial_z \hat{R}_{22}(t,\, z) = -p_{22}(z)\tilde{Y}(t), \tag{29}$$

$$\tau_{hr}\hat{Q}'(t) = -\hat{Q}(t) + c_2(\hat{R}_{11}(t,\, 1) - \hat{R}_{22}(t,\, 1)) - p_Q\tilde{Y}(t), \tag{30}$$

with $\tilde{Y}(t) = R_{21}(t,\, 0) - \hat{R}_{21}(t,\, 0)$.

- The boundary conditions of (26)-(30) are given by

$$\hat{R}_{11}(t, 0) = -\hat{R}_{12}(t,\, 0) + 2U(t), \tag{31}$$

$$\hat{R}_{12}(t, 1) = \hat{R}_{22}(t,\, 1) + c_1\hat{Q}(t), \tag{32}$$

$$\hat{R}_{21}(t, 1) = \hat{R}_{11}(t,\, 1) + c_1\hat{Q}(t), \tag{33}$$

$$\hat{R}_{22}(t, 0) = \alpha R_{21}(t,\, 0), \tag{34}$$

- p_{11}, p_{12}, p_{21}, p_{22}, and p_Q are gains to be found.

Target system

Define the error estimation $\tilde{R}_{ij} = R_{ij} - \hat{R}_{ij}$, $i,\, j = 1,\, 2$, whose dynamics is given by

$$\partial_t \tilde{R}_{11}(t,\, z) + \lambda_1 \partial_z \tilde{R}_{11}(t,\, z) = p_{11}(z)\tilde{Y}(t), \tag{35}$$

$$\partial_t \tilde{R}_{12}(t,\, z) - \lambda_1 \partial_z \tilde{R}_{12}(t,\, z) = p_{12}(z)\tilde{Y}(t), \tag{36}$$

$$\partial_t \tilde{R}_{21}(t,\, z) - \lambda_2 \partial_z \tilde{R}_{21}(t,\, z) = p_{21}(z)\tilde{Y}(t), \tag{37}$$

$$\partial_t \tilde{R}_{22}(t,\, z) + \lambda_2 \partial_z \tilde{R}_{22}(t,\, z) = p_{22}(z)\tilde{Y}(t), \tag{38}$$

$$\tau_{hr}\tilde{Q}'(t) = -\tilde{Q}(t) + c_2(\tilde{R}_{11}(t,\, 1) - \tilde{R}_{22}(t,\, 1)) + p_Q\tilde{Y}(t), \tag{39}$$

and boundary conditions

$$\tilde{R}_{11}(t, 0) = -\tilde{R}_{12}(t,\, 0), \tag{40}$$

$$\tilde{R}_{12}(t, 1) = \tilde{R}_{22}(t,\, 1) + c_1\tilde{Q}(t), \tag{41}$$

$$\tilde{R}_{21}(t, 1) = \tilde{R}_{11}(t,\, 1) + c_1\tilde{Q}(t), \tag{42}$$

$$\tilde{R}_{22}(t, 0) = \alpha\tilde{R}_{21}(t,\, 0) - p_0\tilde{Y}. \tag{43}$$

Target system

- To design the observer output injection gains, we map (35)-(43) to the following appropriate target system:

$$\partial_t \check{R}_{11}(t,\, z) + \lambda_1 \partial_z \check{R}_{11}(t,\, z) = 0, \tag{44}$$

$$\partial_t \check{R}_{12}(t,\, z) - \lambda_1 \partial_z \check{R}_{12}(t,\, z) = 0, \tag{45}$$

$$\partial_t \check{R}_{21}(t,\, z) - \lambda_2 \partial_z \check{R}_{21}(t,\, z) = 0, \tag{46}$$

$$\partial_t \check{R}_{22}(t,\, z) + \lambda_2 \partial_z \check{R}_{22}(t,\, z) = 0, \tag{47}$$

$$\tau_{hr} \check{Q}'(t) = -(1 + c_1 c_2) \check{Q}(t) - c_2 \check{R}_{22}(t,\, 1), \tag{48}$$

with boundary conditions

$$\check{R}_{11}(t, 0) = -\check{R}_{12}(t,\, 0), \tag{49}$$

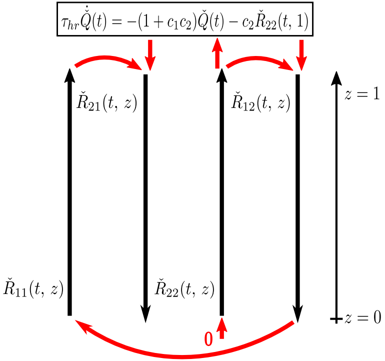
$$\check{R}_{12}(t, 1) = \check{R}_{22}(t,\, 1) + c_1 \check{Q}(t), \tag{50}$$

$$\check{R}_{21}(t, 1) = \check{R}_{11}(t,\, 1) + c_1 \check{Q}(t), \tag{51}$$

$$\check{R}_{22}(t, 0) = 0. \tag{52}$$

Target system

- The mechanism of the proof of stability of the target system is based on this scheme.
- R_{22} is identically zero for all $t \geq \lambda_2^{-1}$.
- By the cascade structure of the target system, it follows that $\check{Q} \to 0$ as $t \to \infty$.
- Finally, by computing the explicit solution of $\check{R}_{11}$, $\check{R}_{12}$ and $\check{R}_{21}$, we get that the target system is exponentially stable.



Backstepping transformation

- To map system (35)-(43) into (44)-(52), we consider the following backstepping transformation:

$$\tilde{R}_{11}(t,\ z) = \check{R}_{11}(t,\ z) - \int_0^1 P_{11}(z,\xi)\check{R}_{21}(t,\xi)d\xi, \tag{53}$$

$$\tilde{R}_{12}(t,\ z) = \check{R}_{12}(t,\ z) - \int_0^1 P_{12}(z,\xi)\check{R}_{21}(t,\xi)d\xi, \tag{54}$$

$$\tilde{R}_{21}(t,\ z) = \check{R}_{21}(t,\ z) - \int_0^z P_{21}(z,\xi)\check{R}_{21}(t,\xi)d\xi, \tag{55}$$

$$\tilde{Q}(t) = \check{Q}(t) - \int_0^1 P_Q(\xi)\check{R}_{21}(t,\xi)d\xi, \tag{56}$$

- Note that P_{21} is the kernel of a Volterra-type integral transformation, whereas P_{11} and P_{12} are the kernels of a ~~Fredholm~~-type integral transformation. P_Q is a finite dimensional kernel.

Backstepping transformation

- Differentiating (53)-(56) with respect to space and time, plugging the target system equation and integrating by parts, we obtain that (35)-(39) is mapped into (44)-(48) if and only if the kernels satisfy the following equations:

$$\lambda_2 \partial_\xi P_{11}(z,\xi) - \lambda_1 \partial_z P_{11}(z,\xi) = 0, \tag{57}$$

$$\lambda_2 \partial_\xi P_{12}(z,\xi) + \lambda_1 \partial_z P_{12}(z,\xi) = 0, \tag{58}$$

$$\partial_\xi P_{21}(z,\xi) + \partial_z P_{21}(z,\xi) = 0, \tag{59}$$

$$\tau_{hr} \lambda_2 P'_Q(\xi) = P_Q(\xi) - c_2 P_{11}(1,\xi), \tag{60}$$

and

$$P_{11}(z,1) = 0, \tag{61}$$

$$P_{12}(z,1) = 0, \tag{62}$$

$$P_Q(1) = -\frac{c_2}{\tau_{hr}\lambda_2}, \tag{63}$$

$$P_{11}(0,\xi) = -P_{12}(0,\xi), \tag{64}$$

$$P_{12}(1,\xi) = c_1 P_Q(\xi), \tag{65}$$

$$P_{21}(1,\xi) = P_{11}(1,\xi) + c_1 P_Q(\xi). \tag{66}$$

Backstepping transformation

- The observer gains are given by

$$p_{11}(z) = \lambda_2 P_{11}(z,0), \tag{67}$$

$$p_{12}(z) = \lambda_2 P_{12}(z,0), \tag{68}$$

$$p_{21}(z) = \lambda_2 P_{21}(z,0), \tag{69}$$

$$p_Q = \tau_{hr} \lambda_2 P_Q(0). \tag{70}$$

Well-posedness of the kernel equations and invertibility of the transformation

- The existence and uniqueness of the solution of the kernel equations were shown in

  - de Andrade, G. A., Vazquez, R., and Pagano, D. J. (2018). Backstepping stabilization of a linearized ODE–PDE Rijke tube model. Automatica (pp. 98–109)

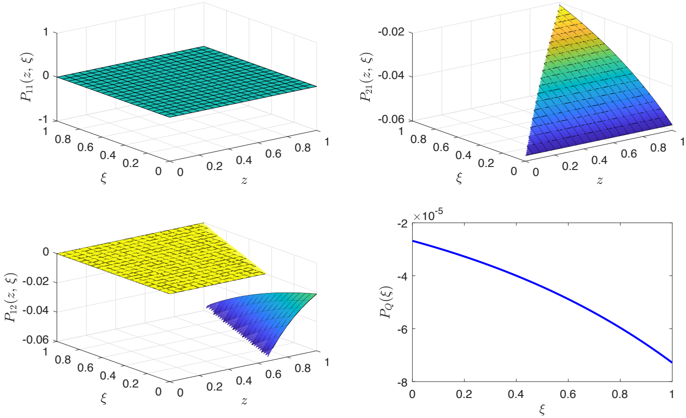
- Since these equations have a simple structure, a closed solution can be obtained by using the method of characteristics.

- In particular, the solution is piecewise-differentiable, where the number of pieces of the solution depends on the position of the heat release element.

- Finally, the transformation (53)-(56) is invertible, ensuring that the target system and the observer error system have equivalent stability properties.

Numerical solution of the kernel equations

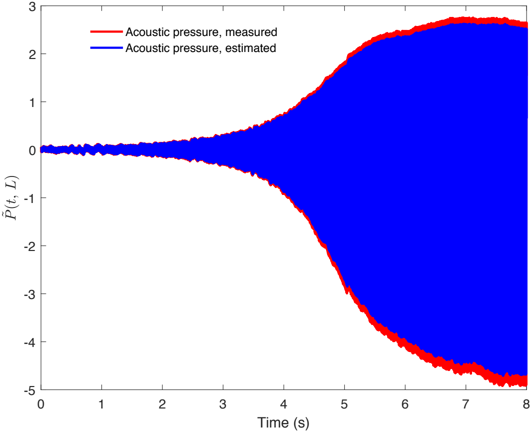
- Numerical solution of the kernel equations for the case $\lambda_1 < \lambda_2$, i.e., $x_0 > \frac{1}{2}L$ (the heater element is near from the measured boundary).



Experimental results

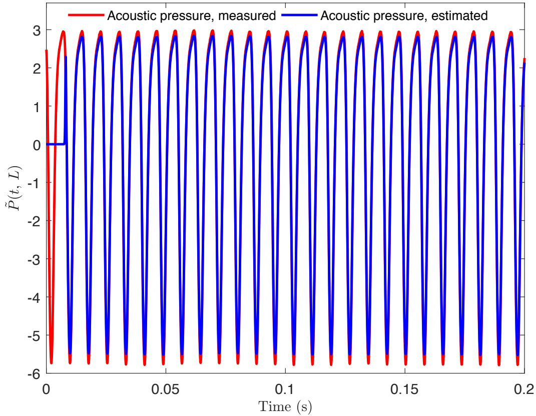
Real view of the Rijke tube experiment.

Experimental results



Time response of the measured and estimated acoustic pressure at the onset of instability.

Experimental results



Detailed view of the measured and estimated acoustic pressure.

Final remarks

- The design, requires measurements from one boundary condition and the observer gains can be computed analytically
- The resulting kernels are piecewise differentiable, with the number of pieces depending on the heat release position
- As future works, we will combine the observer design proposed in this paper with the backstepping controller that we have developed to produce real-life closed-loop experiments.
- The closed-loop experiments must be done in a real time framework because of the fast dynamics of the system and large amount of computations required to obtain the control law.

Outline

- Coupled hyperbolic systems
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Class of PDEs

Plant:

$$
\begin{aligned}
u_t &= u_{xx} + \lambda(x)u + F[u] + uH[u], \\
u_x(t,0) &= qu(t,0), \quad u(t,1) = U(t).
\end{aligned}
$$

$F[u]$, $H[u]$ are Volterra series, which are **functionals** (i.e., functions that depends on another function) defined as

$$
F[u](x,t) = \sum_{n=1}^{\infty} F_n[u](x,t) = \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} f_n(x,\xi_1,\ldots,\xi_n) \left(\prod_{j=1}^{n} u(\xi_j,t) \right) d\xi_1 \ldots d\xi_n.
$$

f_n is called the n-th (Volterra) kernel of F.

Using the Volterra Series definition in the plant:

$$
\begin{aligned}
u_t = \; & u_{xx} + \lambda(x)u + \sum_{n=1}^{\infty} \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} f_n(x,\xi_1,\ldots,\xi_n) \left(\prod_{j=1}^{n} u(\xi_j,t) \right) d\xi_1 \ldots d\xi_n \\
& + u(t,x) \sum_{n=1}^{\infty} \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} h_n(x,\xi_1,\ldots,\xi_n) \left(\prod_{j=1}^{n} u(\xi_j,t) \right) d\xi_1 \ldots d\xi_n.
\end{aligned}
$$

Target System:

$$\begin{aligned} w_t &= w_{xx}, \\ w_x(t,0) &= qw(t,0), \quad w(t,1) = 0. \end{aligned}$$

Exponentially stable plant!

Change of Variables:

$$w(x,t) = u(x,t) - K[u](x,t),$$

which expanded is

$$w(x,t) = u(x,t) - \sum_{n=1}^{\infty} \int_0^x \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} k_n(x,\xi_1,\ldots,\xi_n) \left(\prod_{j=1}^{n} u(\xi_j,t) \right) d\xi_1 \ldots d\xi_n.$$

Control: From the transformation at $x = 1$, since $u(t,1) = U(t)$ and $w(t,1) = 0$,

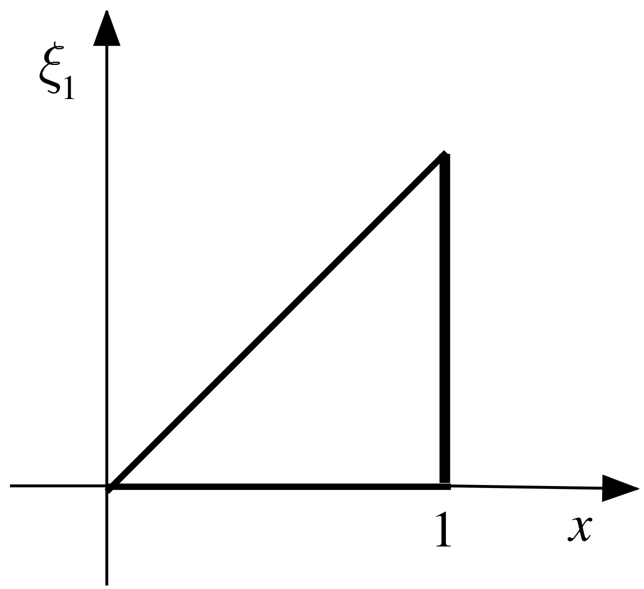
$$U(t) = \sum_{n=1}^{\infty} \int_0^1 \int_0^{\xi_1} \cdots \int_0^{\xi_{n-1}} k_n(1,\xi_1,\ldots,\xi_n) \left(\prod_{j=1}^{n} u(\xi_j,t) \right) d\xi_1 \ldots d\xi_n,$$

i.e., the control kernels are

$$\kappa_n = k_n(1,\xi_1,\ldots,\xi_n).$$

Kernel equations: First-order kernel k_1

$$
\begin{aligned}
\partial_{xx} k_1 &= \partial_{\xi_1 \xi_1} k_1 + \lambda(\xi_1) k_1 - f_1(x, \xi_1) + \int_{\xi_1}^{x} k_1(x,s) f_1(s, \xi_1) ds, \\
k_1(x,x) &= -\frac{1}{2} \int_0^x \lambda(s) ds, \\
k_{1\xi_1}(x,0) &= q k_1(x,0).
\end{aligned}
$$



Domain: $T_1 = \{(x, \xi_1) : 0 \leq \xi_1 \leq x \leq 1\}$. Control kernel $k_1(1, \xi_1)$ (along bold line).

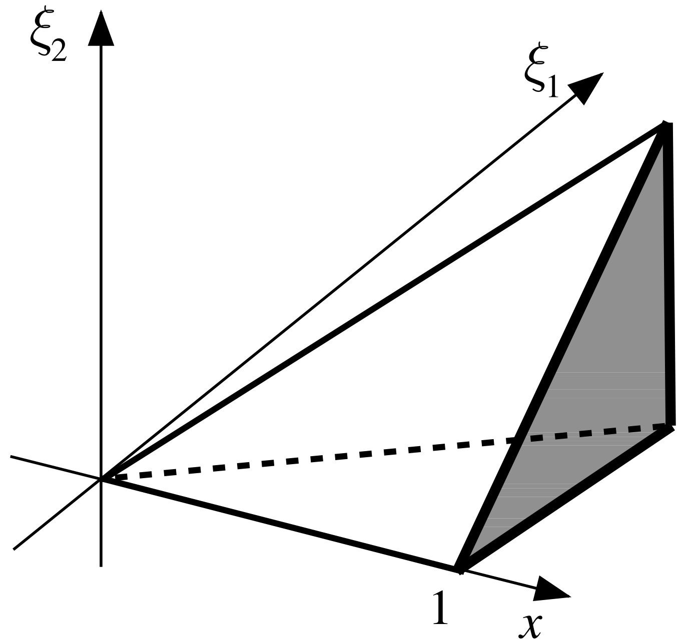
Autonomous equation in k_1. Same kernel equation for backstepping control of linear parabolic PDEs [Smyshlyaev & Krstic, 2004].

Kernel equations: Second-order kernel k_2

$$
\begin{aligned}
\partial_{xx}k_2 &= \partial_{\xi_1\xi_1}k_2 + \partial_{\xi_2\xi_2}k_2 + (\lambda(\xi_1)+\lambda(\xi_2))\,k_2 - f_2(x,\xi_1,\xi_2) \\
&\quad + \int_{\xi_1}^{x} k_1(x,s) f_2(s,\xi_1,\xi_2)ds + \int_{\xi_2}^{\xi_1} k_2(x,\xi_1,s) f_1(s,\xi_2)ds \\
&\quad + \int_{\xi_1}^{x} k_2(x,s,\xi_1) f_1(s,\xi_2)ds + \int_{\xi_1}^{x} k_2(x,s,\xi_2) f_1(s,\xi_1)ds \\
&\quad + k_1(x,\xi_1) h_1(\xi_1,\xi_2), \\
k_2(x,x,\xi_2) &= -\frac{1}{2}\int_{\xi_2}^{x} h_1(s,\xi_2)ds, \\
k_{2x}(x,x,\xi_2) &= -\frac{1}{4}(3h_1(\xi_2,\xi_2) + h_1(x,\xi_2)) + \frac{1}{2}\int_{\xi_2}^{x}\phi_2(s,\xi_2), \\
k_{2\xi_2}(x,\xi_1,0) &= qk_2(x,\xi_1,0), \\
\partial_{\xi_1}k_2(x,\xi_1,\xi_2)\Big|_{\xi_2=\xi_1} &= \partial_{\xi_2}k_2(x,\xi_1,\xi_2)\Big|_{\xi_2=\xi_1},
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_2(x,\xi_2) &= \int_{\xi_2}^{x}\Big(h_{1\xi_2\xi_2}(s,\xi_2) + [\lambda(x)+\lambda(\xi_2)]\,h_1(s,\xi_2)\Big)\,ds - 2h_{1\xi_2}(\xi_2,\xi_2) - h_{1\xi_1}(\xi_2,\xi_2) \\
&\quad + 2f_2(x,x,\xi_2) + \int_{\xi_2}^{x}\int_{s}^{x} h_1(\xi,s) f_1(s,\xi_2)d\xi ds + h_1(x,\xi_2)\int_0^x \lambda(s)ds.
\end{aligned}
$$



Domain $T_2 = \{(x,\xi_1,\xi_2) : 0 \leq \xi_2 \leq \xi_1 \leq x \leq 1\}$. Control kernel $k_2(1,\xi_1,\xi_2)$ (shaded).

In general, in the equation for k_n:

- The exact number of right-hand side terms is $3 \cdot 2^n - (n+3)$.
- But the kernel domain volume is $\frac{1}{(n+1)!}$.

Outline

- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls
- Symmetric disk
- Rijke Tube
- My old friend Volterra
- Some open problems

Some open problems

- Underactuated coupled hyperbolic and parabolic systems.
- Robustness properties of backstepping controllers.
- Non-strict-feedback terms (terms that are not “spatially-causal”).
- Reaction-diffusion equation in the n-ball with non-constant diffusion.

Design on the disk with $\lambda(r,\theta)$

$$u_t = \frac{\varepsilon}{r}(ru_r)_r + \frac{\varepsilon}{r^2}u_{\theta\theta} + \lambda(r,\theta)u,$$

It is not possible to use spherical harmonics (they are no longer eigenfunctions that decouple the problem).

Pose a physical-space transformation:

$$w = u - \int_0^r \int_{-\pi}^{\pi} K(r,\rho,\theta,\psi)u(\rho,\psi)d\psi d\rho,$$

to transform the u equation into the target system

$$w_t = \frac{\varepsilon}{r}(rw_r)_r + \frac{\varepsilon}{r^2}w_{\theta\theta},$$

Design on the disk with $\lambda(r,\theta)$

The kernel verifies the ultrahyperbolic equation

$$K_{rr}+\frac{K_r}{r}-K_{\rho\rho}+\frac{K_\rho}{\rho}-\frac{K}{\rho^2}-\frac{K_{\theta\theta}}{r^2}+\frac{K_{\psi\psi}}{\rho^2}=\frac{\lambda(\rho,\psi)}{\varepsilon}K$$

with BC

$$\begin{aligned}
K(r,\rho,0,\psi) &= K(r,\rho,\pi,\psi)\\
K(r,\rho,\theta,0) &= K(r,\rho,\theta,\pi)\\
K(r,0,\theta,\psi) &= 0,\\
\int_{-\pi}^{\pi}K(r,r,\theta,\psi)u(r,\psi)d\psi &= -\int_0^r\frac{\lambda(\rho,\theta)}{2\varepsilon}d\rho u(r,\theta),
\end{aligned}$$

this last boundary condition can be verified if

$$\lim_{\rho\to r}K(r,\rho,\theta,\psi) = -\delta(\theta-\psi)\int_0^r\frac{\lambda(\rho,\theta)}{2\varepsilon}d\rho.$$

Design on the disk with $\lambda(r,\theta)$

The kernel verifies the ultrahyperbolic equation

$$K_{rr}+\frac{K_r}{r}-K_{\rho\rho}+\frac{K_\rho}{\rho}-\frac{K}{\rho^2}-\frac{K_{\theta\theta}}{r^2}+\frac{K_{\psi\psi}}{\rho^2}=\frac{\lambda(\rho,\psi)}{\varepsilon}K$$

with BC

$$\begin{aligned} K(r,\rho,0,\psi) &= K(r,\rho,\pi,\psi) \\ K(r,\rho,\theta,0) &= K(r,\rho,\theta,\pi) \\ K(r,0,\theta,\psi) &= 0, \\ \int_{-\pi}^{\pi} K(r,r,\theta,\psi)u(r,\psi)d\psi &= -\int_0^r \frac{\lambda(\rho,\theta)}{2\varepsilon}d\rho u(r,\theta), \end{aligned}$$

this last boundary condition can be verified if

$$\lim_{\rho\to r} K(r,\rho,\theta,\psi) = -\delta(\theta-\psi)\int_0^r \frac{\lambda(\rho,\theta)}{2\varepsilon}d\rho.$$

We don't know how to solve it, only know there is a solution for constant λ!

$$K(r,\rho,\theta,\psi)=-\rho\frac{\lambda}{2\pi\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}}\frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cos(\theta-\psi)}$$

Gracias!

US has more Spanish speakers than Spain
New York Post - Jun 29, 2015
The **United States** now has **more Spanish** speakers **than Spain** ... there are now an estimated 52.6 million **people** in the **US** who can **speak** the ...
US now has **more Spanish** speakers **than Spain** – only Mexico has **more**
Highly Cited - The Guardian - Jun 29, 2015

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