SIAM J. CONTROL OPTIM.
Vol. 57, No. 2, pp. 963–998

BOUNDARY EXPONENTIAL STABILIZATION OF 1-DIMENSIONAL INHOMOGENEOUS QUASI-LINEAR HYPERBOLIC SYSTEMS*

LONG HU†, RAFAEL VAZQUEZ‡, FLORENT DI MEGLIO§, AND MIROSLAV KRSTIC¶

Abstract. This paper deals with the problem of boundary stabilization of first-order $n \times n$ inhomogeneous quasi-linear hyperbolic systems. A backstepping method is developed. The main result supplements the previous works on how to design multiboundary feedback controllers to achieve exponential stability with arbitrary decay rate of the original nonlinear system in the spatial H^2 sense.

Key words. rapid exponential stability, $n \times n$ quasi-linear hyperbolic systems, backstepping transformation, Lyapunov functions method

AMS subject classifications. 93B52, 35L60

DOI. 10.1137/15M1012712

1. Introduction and main result.

Consider the following 1-dimensional $n \times n$ quasi-linear hyperbolic system with source terms

$$\frac{\partial u}{\partial t} + A(x,u)\frac{\partial u}{\partial x} = F(x,u),\ x \in [0,1],\ t \in [0,+\infty), \tag{1.1}$$

where $u = (u_1, \dots, u_n)^T$ is an unknown vector function of (t,x), $A(x,u)$ is an $n \times n$ matrix with C^2 entries $a_{ij}(x,u)\,(i,j = 1,\dots,n)$, $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector valued function with C^2 components $f_i(x,u)\,(i = 1,\dots,n)$ with respect to u and

$$F(x,0) \equiv 0. \tag{1.2}$$

Denote

$$\frac{\partial F}{\partial u}(x,0) := (f_{ij}(x))_{n\times n}, \tag{1.3}$$

where we assume that $f_{ij} \in C^2([0,1])$.

By the definition of hyperbolicity, we assume that $A(x,0)$ is a diagonal matrix with distinct and nonzero eigenvalues $A(x,0) = \mathrm{diag}(\Lambda_1(x),\dots,\Lambda_n(x))$, which are, without loss of generality, ordered as follows:

$$\Lambda_1(x) < \Lambda_2(x) < \cdots < \Lambda_m(x) < 0 < \Lambda_{m+1}(x) < \cdots < \Lambda_n(x)\ \forall x \in [0,1]. \tag{1.4}$$

*Received by the editors March 17, 2015; accepted for publication (in revised form) June 4, 2018; published electronically March 19, 2019.
http://www.siam.org/journals/sicon/57-2/M101271.html
Funding: The first author's work was supported by the National Natural Science Foundation of China (11601284), the China Scholarship Council for Ph.D. study at UPMC (201306100081), the ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7), the Young Scholars Program of Shandong University (2016WLJH52), and the National Postdoctoral Program for Innovative Talents (BX201600096).
†School of Mathematics, Shandong University, Jinan, Shandong 250100, China, Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France, and School of Mathematical Sciences, Fudan University, Shanghai 200433, China (hu@ann.jussieu.fr, hul10@fudan.edu.cn).
‡Department of Aerospace Engineering, Universidad de Sevilla, Camino de los Descubrimiento s.n., 41092 Sevilla, Spain (rvazquez1@us.es).
§MINES ParisTech, PSL Research University, CAS - Centre automatique et systèmes, 60 bd St Michel, 75006 Paris, France (florent.di_meglio@mines-paristech.fr).
¶Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA 92093-0411 (krstic@ucsd.edu).

Here and in what follows, $\mathrm{diag}(\Lambda_1(x), \dots, \Lambda_n(x))$ denotes the diagonal matrix whose ith element on the diagonal is $\Lambda_i(x)$.

The boundary conditions are given as

$$x = 0 : u_s = G_s(u_1, \dots, u_m), \quad s = m+1, \dots, n, \tag{1.5}$$

$$x = 1 : u_r = h_r(t), \quad r = 1, \dots, m, \tag{1.6}$$

where G_s are C^2 functions, and we assume that they vanish at the origin, i.e.,

$$G_s(0, \dots, 0) \equiv 0, s = m+1, \dots, n, \tag{1.7}$$

while $H = (h_1, \dots, h_m)^T$ is a vector boundary function of H^2.

Let us first point out the well-posedness result for this hyperbolic system (1.1) with (1.5)–(1.6) in the sense of the following lemma (the detailed proof can be found in [6] for the conservation laws (i.e., $F \equiv 0$), in [2, Appendix B] for the corresponding general inhomogeneous case, and in [13] for the isothermal Euler equations; actually these references deal with the closed-loop system (i.e., $H \equiv 0$), but the proofs given there can be adapted).

LEMMA 1.1. *For any given $0 < T < +\infty$, there exist $\delta_0 > 0$ such that for every $\phi \in H^2((0,1);\mathbb{R}^n)$ and $H \in H^2((0,T);\mathbb{R}^m)$ satisfying*

$$\|\phi\|_{H^2((0,1);\mathbb{R}^n)} + \|H\|_{H^2((0,T);\mathbb{R}^m)} \le \delta_0, \tag{1.8}$$

and the C^1 compatibility conditions at the points $(t,x) = (0,0)$ and $(0,1)$, the mixed initial boundary value problem (1.1), (1.5)–(1.6) *and the initial conditions*

$$t = 0 : u(0,x) := \phi(x) = (\phi_1(x), \dots, \phi_n(x)), \tag{1.9}$$

admits a unique solution $u = u(t,x)$ in the space $C^0([0,T]; H^2((0,1);\mathbb{R}^n))$.

Remark 1.1. In what follows, for simplicity, for the norm $H^p((0,1);\mathbb{R}^n)$ $(p \in \mathbb{N}^+)$, when no confusion is possible, we use $H^p(0,1)$ for short.

Remark 1.2. The C^1 compatibility conditions at the point $(t,x) = (0,0)$ are given by

(1.10)

$$\phi_s(0) = G_s(\phi_1(0), \dots, \phi_m(0)), \qquad s = m+1, \dots, n,$$

$$f_s(0,\phi(0)) - \sum_{j=1}^{n} a_{sj}(0,\phi(0))\phi'_j(0)$$

(1.11)

$$= \sum_{r=1}^{m} \frac{\partial G_s}{\partial u_r}(\phi_1(0), \dots, \phi_m(0)), \cdot \left(f_r(0,\phi(0)) - \sum_{j=1}^{n} a_{rj}(0,\phi(0))\phi'_j(0)\right), \quad s = m+1, \dots, n.$$

The C^1 compatibility conditions at the point $(t,x) = (0,1)$ are similar.

Our concern, in this paper, is to design a feedback control law for $H(t)$ in order to ensure that the closed-loop system is locally exponentially stable in the H^2 norm. In other words, we are interested in the following stabilization problem for the system (1.1) and (1.5)–(1.6):

Problem (ES). For any given $\lambda > 0$, suppose that C^1 compatibility conditions are satisfied at the point $(t, x) = (0, 0)$. Does there exist a linear feedback control $\mathcal{B} : H^2((0,1);\mathbb{R}^n) \to \mathbb{R}^m$, verifying the C^1 compatibility conditions at the point $(t, x) = (0, 1)$, such that for some $\varepsilon > 0$, the mixed initial boundary value problem (1.1), (1.5)–(1.6) and the initial conditions

$$t = 0 : u(0, x) := \phi(x) = (\phi_1(x), \ldots, \phi_n(x)) \tag{1.12}$$

with $H(t) = \mathcal{B}(u(t,\cdot))$ admits a unique $C^0([0,\infty); (H^2(0,1))^n)$ solution $u = u(t, x)$, which satisfies

$$\|u(t,\cdot)\|_{H^2(0,1)} \le Ce^{-\lambda t}\|\phi\|_{H^2(0,1)} \quad \forall t > 0 \tag{1.13}$$

for some $C > 0$, provided that $\|\phi\|_{H^2(0,1)} \le \varepsilon$?

The boundary stabilization problem for linear and nonlinear hyperbolic systems has been widely studied in the last three decades or so. During this time, three parallel mathematical approaches have emerged. The first one is the so-called characteristic method, i.e., computing corresponding bounds by using explicit evolution of the solution along the characteristic curves. With this method, Problem (ES) has been previously investigated by Greenberg and Li (see [12]) for 2×2 systems and Li and Qin (see [18, 20]) for a generalization to $n \times n$ homogeneous systems in the framework of the C^1 norm. Also, this method was developed by Li and Rao [19] to study the exact boundary controllability for general inhomogeneous quasi-linear hyperbolic systems.

The second method is the "control Lyapunov functions method," which is a useful tool to analyze the asymptotic behavior of dynamical systems. This method was first used by Coron and coworkers to design dissipative boundary conditions for nonlinear homogeneous hyperbolic systems in the context of both the C^1 and H^2 norms [5, 6, 7]. More recently, it has been shown in [8] that the exponential stability strongly depends on the considered norm, i.e., a previously known sufficient condition for exponential stability with respect to the H^2 norm is not sufficient in the framework of the C^1 norm. Although the control Lyapunov functions method has been introduced to study exponential stability for hyperbolic systems of balance laws, finding a "good" Lyapunov function is the main difficulty, especially when the "natural" control Lyapunov functions do not lead to arbitrarily large exponential decay rates for the original system (see [1], [4, pp. 314 and 361–371]). This phenomenon indeed happens when we deal with Problem (ES) for the inhomogeneous hyperbolic systems (see [6, 7]).

The third one is the "backstepping method", which is now a popular mathematical tool to stabilize the finite-dimensional and infinite-dimensional dynamic systems (see [16, 17, 21, 22, 23]). In [9], a full-state feedback control law, with actuation on only one end of the domain, which achieves H^2 exponential stability of the closed-loop 2×2 linear and quasi-linear hyperbolic system is derived using a backstepping method. Moreover, this method ensures that the linear hyperbolic system vanishes in finite time. Unfortunately, the method presented in [9] cannot be directly extended to $n \times n$ cases, especially when several states convecting in the same direction are controlled (see also [10]). In [14], a first step towards generalization to 3×3 linear hyperbolic systems is addressed in the case where two controlled states are considered. With a similar Volterra transformation, designing an appropriate form of the target system, Hu et al. [15] adopt a classical backstepping controller to handle Problem (ES) for general $n \times n$ linear hyperbolic systems. Well-posedness of the system of kernel equations, which is the main technical challenge, is shown there by an improved successive approximation method. We also mention [11] for the stabilization of first-order hyperbolic system by boundary feedback controls with varying delays.

In this paper, based on the results for the linear case [15], we will use the linearized feedback control to stabilize the nonlinear system as it is mentioned in [9]. Although the target system is different from the one in [9] with a linear term involved in the equations and the kernels are probably piecewise smooth, thanks to the special structure of the nonlocal linear term and the potential discontinuities of the kernels evolving just along their characteristic curves, we show that all the procedures to handle nonlinearities in [9] can be also adapted in this paper with more technical developments. Let us recall some definitions and statements in [9]. Define the norms

$$\|u(t,\cdot)\|_{H^1(0,1)} = \|u(t,\cdot)\|_{L^2(0,1)} + \|u_x(t,\cdot)\|_{L^2(0,1)},$$
$$\|u(t,\cdot)\|_{H^2(0,1)} = \|u(t,\cdot)\|_{H^1(0,1)} + \|u_{xx}(t,\cdot)\|_{L^2(0,1)}$$

in which $\|u(t,\cdot)\|_{L^2(0,1)} = \sqrt{\sum_{i=1}^{n}\int_0^1 u_i^2(t,x)dx}$ and, hereafter, we use $\|u(t,\cdot)\|_{L^2}$ for short.

Our main result is given by the following.

THEOREM 1.1. *Under the assumptions in section* 1*, suppose furthermore that the* C^1 *compatibility conditions are satisfied at the point* $(t,x) = (0,0)$*, then there exists a continuous linear feedback control law* $\mathcal{B} : H^2((0,1);\mathbb{R}^n) \to \mathbb{R}^m$*, satisfying the* C^1 *compatibility conditions at the point* $(t,x) = (0,1)$*, such that for every* $\nu > 0$*, there exist* $\delta > 0$ *and* $c > 0$*, such that the mixed initial boundary value problem* (1.1)*,* (1.5)*,* (1.6)*, and* (1.9) *with* $H(t) = \mathcal{B}(u(t,\cdot))$ *admits a unique* $C^0([0,\infty), H^2((0,1);\mathbb{R}^n))$ *solution* $u = u(t,x)$*, which verifies*

$$\|u(t,\cdot)\|_{H^2(0,1)} \le ce^{-\nu t}\|\phi\|_{H^2(0,1)} \ \forall t > 0, \tag{1.14}$$

provided that $\|\phi\|_{H^2(0,1)} \le \delta$.

Remark 1.3. It should be noticed that since the usual static feedback control laws, i.e., $H_r = G_r(u_1(t,0),\ldots,u_m(t,1),u_{m+1}(t,0),\ldots,u_n(t,0))$ cannot stabilize the general coupled hyperbolic systems (1.1) even in the linear case (see the counterexample in [2, section 5.6]); here we will choose a full-state feedback which has the form shown in (3.19) below.

Remark 1.4. For convenience, we always assume that the feedback controls $H(t) = \mathcal{B}(u(t,\cdot))$ satisfy the C^1 compatibility conditions at the point $(t,x) = (0,1)$. However, if this property fails, one can add some dynamic terms to the controllers (see also Remark 3.1 and [9, section 4]).

The rest of this paper is organized as follows. In section 2, we review our former result on the boundary backstepping controls for an $n \times n$ linear hyperbolic system. Besides, we design a Lyapunov function to stabilize the linear system in the L^2 norm. In section 3, we impose the corresponding linearized closed-loop control to the original nonlinear system and give the feedback control design. In section 4, we prove exponential stability of zero equilibrium with arbitrary decay rate for the quasi-linear system by using the control Lyapunov function method. We finally include two appendices with some technical details.

2. Preliminaries—linear case. In this section, we review the results on stabilization of an $n \times n$ hyperbolic linear system by using the backstepping method (see

[15]). Similar to the situation in [9], this procedure can be applied to locally stabilize the original nonlinear system. Consider the following $n \times n$ hyperbolic systems

$$w_t(t,x) + \Lambda(x) w_x(t,x) = \Sigma(x) w(t,x), \tag{2.1}$$

where $w = (w_1, \dots, w_n)^T$ is a vector function of (t,x), Λ: $[0,1] \to \mathcal{M}_{n,n}(\mathbb{R})$ is an $n \times n$ C^2 diagonal matrix, i.e.,

$$\Lambda(x) = \begin{pmatrix} \Lambda_-(x) & 0 \\ 0 & \Lambda_+(x) \end{pmatrix} \tag{2.2}$$

in which $\Lambda_-(x) := \text{diag}(\lambda_1(x), \dots, \lambda_m(x))$ and $\Lambda_+(x) := \text{diag}(\lambda_{m+1}(x), \dots, \lambda_n(x))$ are diagonal submatrices, without loss of generality, satisfying

$$\lambda_1(x) < \cdots < \lambda_m(x) < 0 < \lambda_{m+1}(x) < \cdots < \lambda_n(x) \ \forall x \in [0,1]. \tag{2.3}$$

On the other hand, $\Sigma : [0,1] \to \mathcal{M}_{n,n}(\mathbb{R})$ is an $n \times n$ matrix with $C^2[0,1]$ entries $\sigma_{ij}(x)\,(1 \leq i \leq n, 1 \leq j \leq n)$. Moreover, for any $i = 1, \dots, n$, without loss of generality (see section 3 below, [9] for $n = 2$, and [14] for $n = 3$), we assume that

$$\sigma_{ii}(x) \equiv 0 \ \ \forall x \in [0,1]. \tag{2.4}$$

The boundary conditions for the linear hyperbolic system (2.1) are given by

$$x = 0: \ \ w_+(t,0) = Q w_-(t,0), \tag{2.5}$$

and

$$x = 1 : w_-(t,1) = U(t), \tag{2.6}$$

where $w_- \in \mathbb{R}^m, w_+ \in \mathbb{R}^{n-m}$ are defined by requiring that $w := (w_-, w_+)^T$, $U = (U_1, \dots, U_m)^T$ are boundary feedback controls, $Q \in \mathcal{M}_{n-m,m}$ is a constant matrix. Our purpose in this section is to find a full-state feedback control law for $U(t)$ to ensure that the closed-loop system (2.1), (2.5)–(2.6) is globally asymptotically stable in the L^2 norm.

2.1. Target system. In section 2.2, it will be shown that we can transform the system (2.1), (2.5)–(2.6) into the following cascade system

$$\gamma_t(t,x) + \Lambda(x)\gamma_x(t,x) = G(x)\gamma(t,0) \tag{2.7}$$

with the boundary conditions

$$x = 0: \quad \gamma_+(t,0) = Q\gamma_-(t,0) \tag{2.8}$$

and

$$x = 1 : \gamma_-(t,1) = 0, \tag{2.9}$$

where $\gamma_- \in \mathbb{R}^m, \gamma_+ \in \mathbb{R}^{n-m}$ are defined by requiring that $\gamma := (\gamma_-, \gamma_+)^T$, G is a lower triangular matrix with the following structure

$$G(x) = \begin{pmatrix} \mathcal{G}_1(x) & 0 \\ \mathcal{G}_2(x) & 0 \end{pmatrix} \tag{2.10}$$

in which $\mathcal{G}_1 \in \mathcal{M}_{m,m}(\mathbb{R})$ is a lower triangular matrix, i.e.,

$$\mathcal{G}_1(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1}(x) & \cdots & g_{m,m-1}(x) & 0 \end{pmatrix} \tag{2.11}$$

and $\mathcal{G}_2(x) \in \mathcal{M}_{n-m,m}(\mathbb{R})$. The coefficients of both $\mathcal{G}_1$ and $\mathcal{G}_2$ are to be determined in section 2.2. Next, we prove that the cascade system (2.7)–(2.9) verifies the following proposition.

Proposition 2.1. *For any given matrix function $G(\cdot) \in C^2[0,1]$, the mixed initial boundary value problem* (2.7)–(2.9) *with initial condition*

$$t = 0 : \gamma(0, x) = \gamma_0(x), \tag{2.12}$$

where $\gamma_0 \in L^2((0,1);\mathbb{R}^n)$ admits a $C^0([0,\infty); L^2((0,1);\mathbb{R}^n)))$ solution $\gamma = \gamma(t,x)$, which is globally exponentially stable in the L^2 norm, i.e., for every $\lambda > 0$, there exists $c > 0$ such that

$$\|\gamma(t,\cdot)\|_{L^2} \leq ce^{-\lambda t}\|\gamma_0\|_{L^2}. \tag{2.13}$$

In fact, this solution vanishes in finite time $t \geq t_F$, where t_F is given by

$$t_F = \int_0^1 \frac{1}{\lambda_{m+1}(s)} + \sum_{r=1}^{m} \frac{1}{|\lambda_r(s)|} ds. \tag{2.14}$$

Proof. Equations (2.7) can be rewritten as

$$\begin{aligned} &\partial_t \gamma_-(t,x) + \Lambda_-(x)\partial_x\gamma_-(t,x) = \mathcal{G}_1(x)\gamma_-(t,0), \\ &\partial_t \gamma_+(t,x) + \Lambda_+(x)\partial_x\gamma_+(t,x) = \mathcal{G}_2(x)\gamma_-(t,0); \end{aligned} \tag{2.15}$$

then consider the following Lyapunov functional with exponential weights,

$$\begin{aligned} V_0(t) = &\int_0^1 e^{-\delta x}\gamma_+(t,x)^T \left(\Lambda_+(x)\right)^{-1} \gamma_+(t,x)dx \\ &- \int_0^1 e^{\delta x}\gamma_-(t,x)^T B \left(\Lambda_-(x)\right)^{-1} \gamma_-(t,x)dx, \end{aligned} \tag{2.16}$$

where $\delta > 0$ is a parameter that will be chosen sufficiently large,

$$B = \operatorname{diag}(b_1, \ldots, b_m) \tag{2.17}$$

with $b_r > 0$ $(r = 1, \ldots, m)$, whose coefficients are to be determined. Obviously, $\sqrt{V_0}$ is a norm equivalent to $\|\gamma(t,\cdot)\|_{L^2}$. Differentiating V_0 with respect to t and integrating by parts yields

$$\dot{V}_0(t) = I + II + III + IV$$

with

$$I = \left[-e^{-\delta x}\gamma_+(t,x)^T\gamma_+(t,x) + e^{\delta x}\gamma_-(t,x)^T B\gamma_-(t,x)\right]_0^1,$$

$$II = -\int_0^1 \delta e^{-\delta x}\gamma_+(t,x)^T\gamma_+(t,x)dx - \int_0^1 \delta e^{\delta x}\gamma_-(t,x)^T B\gamma_-(t,x)dx,$$

$$III = 2\int_0^1 e^{-\delta x}\gamma_+(t,x)^T\left(\Lambda_+(x)\right)^{-1}\mathcal{G}_2(x)\gamma_-(t,0)dx,$$

$$IV = -2\int_0^1 e^{\delta x}\gamma_-(t,x)^T B\left(\Lambda_-(x)\right)^{-1}\mathcal{G}_1(x)\gamma_-(t,0)dx.$$

Noting the boundary conditions (2.8)–(2.9), we have that

$$I = -e^{\delta}\gamma_+(t,1)^T\gamma_+(t,1) - \gamma_-(t,0)^T\Big(B - Q^TQ\Big)\gamma_-(t,0), \tag{2.18}$$

and using Young's inequality we obtain

(2.19)
$$\begin{aligned} III &\le \int_0^1 e^{-\delta x}\gamma_+(t,x)^T\gamma_+(t,x)dx + \gamma_-(t,0)^T\int_0^1 e^{-\delta x}\mathcal{G}_2^T(x)\left(\Lambda_+(x)\right)^{-2}\mathcal{G}_2(x)dx\gamma_-(t,0)\\ &\le \int_0^1 e^{-\delta x}\gamma_+(t,x)^T\gamma_+(t,x)dx + \gamma_-(t,0)^T\int_0^1 \mathcal{G}_2^T(x)\left(\Lambda_+(x)\right)^{-2}\mathcal{G}_2(x)dx\gamma_-(t,0), \end{aligned}$$

(2.20)
$$\begin{aligned} IV &= -2\int_0^1 e^{\delta x}\sum_{m\ge i>j\ge 1}\gamma_i(t,x)\frac{b_i}{\Lambda_i(x)}g_{ij}(x)\gamma_j(t,0)dx\\ &\le -M\int_0^1 e^{\delta x}\sum_{m\ge i>j\ge 1}\frac{b_i}{\Lambda_i(x)}\gamma_i^2(t,x)dx - M\int_0^1 e^{\delta x}\sum_{m\ge i>j\ge 1}\frac{b_i}{\Lambda_i(x)}\gamma_j^2(t,0)dx\\ &\le -M\int_0^1 e^{\delta x}\sum_{m\ge i>j\ge 1}\frac{b_i}{\Lambda_i(x)}\gamma_i^2(t,x)dx + M\mu e^{\delta}\gamma_-(t,0)^T\mathcal{C}\gamma_-(t,0)\\ &\le -mM\int_0^1 e^{\delta x}\sum_{i=2}^{m}\frac{b_i}{\Lambda_i(x)}\gamma_i^2(t,x)dx + M\mu e^{\delta}\gamma_-(t,0)^T\mathcal{C}\gamma_-(t,0)\\ &\le -mM\int_0^1 e^{\delta x}\gamma_-(t,x)^T B\left(\Lambda_-(x)\right)^{-1}\gamma_-(t,x)dx + M\mu e^{\delta}\gamma_-(t,0)^T\mathcal{C}\gamma_-(t,0) \end{aligned}$$

in which

$$M := \|G\|_\infty, \quad \mathcal{C} := \mathrm{diag}(\mathcal{C}_1,\ldots,\mathcal{C}_m) \tag{2.21}$$

with

$$\mathcal{C}_r := \begin{cases} \sum\limits_{j=r+1}^{m} b_j, & 1\le r\le m-1,\\ 0, & r=m, \end{cases} \tag{2.22}$$

and

$$\mu := \max_i\left\{\frac{1}{\|\lambda_i\|_{C^0}}\right\}. \tag{2.23}$$

Let

$$P = Q^TQ + \int_0^1 \mathcal{G}_2^T(x)\left(\Lambda_-(x)\right)^{-2}\mathcal{G}_2(x)dx. \tag{2.24}$$

There exists a diagonal matrix $S = \mathrm{diag}(s_1,\ldots,s_m)$ with $s_r > 0$ $(r = 1,\ldots,m)$ being large enough, such that

$$P \prec S, \tag{2.25}$$

where $P \prec S$ denotes that $S - P$ is a positive-definite matrix. This yields

$$\dot{V}_0(t) \leq -\gamma_-(t,0)^T \Big(B - S - M\mu e^{\delta}\mathcal{C}\Big)\gamma_-(t,0) - (\delta - 1)\int_0^1 e^{-\delta x}\gamma_+(t,x)^T\gamma_+(t,x)dx$$
$$- (\delta - mM\mu)\int_0^1 e^{\delta x}\gamma_-(t,x)^T B\gamma_-(t,x)dx.$$

Thus, for any given $\lambda > 0$, picking

$$\delta > \max\{\lambda\mu + mM\mu, \lambda\mu + 1\}, \tag{2.26}$$

$$b_r > \begin{cases} M\mu e^{\delta} \sum\limits_{j=r+1}^{m} b_j + s_r, & 1 \leq r \leq m-1, \\ s_m, & r = m, \end{cases} \tag{2.27}$$

we have

$$\dot{V}_0 \leq -\lambda V_0, \tag{2.28}$$

where λ can be chosen as large as desired. It is easy to see that parameter matrix B does exist, since one can easily check (2.27) by induction. This shows the exponential stability of the γ system.

To show finite-time convergence to the origin, one can find the explicit solution of (2.7)–(2.9) as follows. Define

$$\phi_i(x) = \int_0^x \frac{1}{|\lambda_i(\xi)|}d\xi, \quad 1 \leq i \leq n. \tag{2.29}$$

Notice that every $\phi_i (1 \leq i \leq n)$ is a monotonically increasing C^2 function of x, and thus invertible. With the same statement in [9] and noting (2.7)–(2.11), one can express the explicit solution of γ_1 by

$$\gamma_1(t,x) = \begin{cases} \gamma_1(0, \phi_1^{-1}(\phi_1(x) + t)) & \text{if } t < \phi_1(1) - \phi_1(x), \\ 0 & \text{if } t \geq \phi_1(1) - \phi_1(x). \end{cases} \tag{2.30}$$

Notice in particular that γ_1 is identically zero for $t \geq \phi_1(1)$. From (2.7) and (2.11), we obtain that $\gamma_2(t,x)$ satisfies the following equation for $t \geq \phi_1(1)$,

$$\partial_t\gamma_2(t,x) + \lambda_2(x)\partial_x\gamma_2(t,x) = 0 \tag{2.31}$$

with

$$\gamma_2(t,1) = 0, \tag{2.32}$$

which ensures the explicit expression of $\gamma_2(t,x)$ to be

$$\gamma_2(t,x) = \begin{cases} \gamma_2(\phi_1(1), \phi_2^{-1}(\phi_2(x) + t)) & \text{if } \phi_1(1) < t < \phi_1(1) + \phi_2(1) - \phi_2(x), \\ 0 & \text{if } t \geq \phi_1(1) + \phi_2(1) - \phi_2(x). \end{cases} \tag{2.33}$$

Therefore, by induction, one has that $\gamma_r(t,x)\,(2 \leq r \leq m)$ satisfies the following equations: for $t > \sum_{k=1}^{r-1} \phi_k(1)$,

$$\partial_t \gamma_r(t,x) + \lambda_r(x)\partial_x \gamma_r(t,x) = 0 \tag{2.34}$$

with the boundary condition

$$\gamma_r(t,1) = 0. \tag{2.35}$$

Thus, when $t > \sum_{k=1}^{r-1} \phi_k(1)$, we have

$$\gamma_r(t,x) = \begin{cases} \gamma_r(\sum\limits_{k=1}^{r-1} \phi_k(1), \phi_r^{-1}(\phi_r(x)+t)) & \text{if } \sum\limits_{k=1}^{r-1} \phi_k(1) < t < \sum\limits_{k=1}^{r} \phi_k(1) - \phi_r(x), \\ 0 & \text{if } t \geq \sum\limits_{k=1}^{r} \phi_k(1) - \phi_r(x). \end{cases} \tag{2.36}$$

This yields that $\gamma_-(t,x) \equiv 0$ ($t > \sum_{k=1}^{m} \phi_k(1)$). From the time $t = \sum_{k=1}^{m} \phi_k(1)$ on, we find γ_+ becomes the solution of the following system,

$$\partial_t \gamma_+(t,x) + \Lambda_+(x)\partial_x \gamma_+(t,x) = 0 \tag{2.37}$$

with

$$x = 0 : \gamma_+(t,0) \equiv 0. \tag{2.38}$$

Since (2.37)–(2.38) is a completely decoupled system, by the characteristic method, after $t = t_F$, where

$$t_F = \phi_{m+1}(1) + \sum_{r=1}^{m} \phi_r(1) = \int_0^1 \frac{1}{\lambda_{m+1}(s)} + \sum_{r=1}^{m} \frac{1}{|\lambda_r(s)|} ds, \tag{2.39}$$

one can see that $\gamma_+(t,x) \equiv 0 (t \geq t_F)$, which concludes the proof of Proposition 2.1. □

2.2. The backstepping transformation and Kernel equations. To map the original system (2.1) into the target system (2.7), we use the following Volterra transformation of the second kind, which is similar to the one in [9, 10]:

$$\gamma(t,x) = w(t,x) - \int_0^x K(x,\xi) w(t,\xi) d\xi \tag{2.40}$$

in which K, defined on $\mathcal{T} = \{(x,\xi) | 0 \leq \xi \leq x \leq 1\}$, is an $n \times n$ matrix of kernels. We point out here that this transformation yields that $w(t,0) \equiv \gamma(t,0)$ ($\forall t > 0$), which is crucial to design our feedback law.

Utilizing (2.1), (2.5) and straightforward computations, one can formally show (see also Appendix A for the validity of the calculations) that

$$\begin{aligned} &\gamma_t + \Lambda(x)\gamma_x \\ &\quad = -\int_0^x \big(K_\xi(x,\xi)\Lambda(\xi) + \Lambda(x)K_x(x,\xi) + K(x,\xi)\Sigma(\xi) + K(x,\xi)\Lambda_\xi(\xi)\big) w(t,\xi) d\xi \\ &\qquad + \big(\Sigma(x) + K(x,x)\Lambda(x) - \Lambda(x)K(x,x)\big) w(t,x) - K(x,0)\Lambda(0) \begin{pmatrix} I & 0 \\ Q & 0 \end{pmatrix} w(t,0). \end{aligned} \tag{2.41}$$

The original system (2.1) is mapped into the target system (2.7) if the kernel satisfies the following equations:

$$\Lambda(x)K_x(x,\xi) + K_\xi(x,\xi)\Lambda(\xi) + K(x,\xi)\Sigma(\xi) + K(x,\xi)\Lambda_\xi(\xi) = 0, \tag{2.42}$$

$$\Sigma(x) + K(x,x)\Lambda(x) - \Lambda(x)K(x,x) = 0, \tag{2.43}$$

$$G(x) = -K(x,0)\Lambda(0)\begin{pmatrix} I & 0 \\ Q & 0 \end{pmatrix}. \tag{2.44}$$

Developing (2.42)–(2.44) leads to the following set of kernel PDEs,

$$\lambda_i(x)\partial_x K_{ij}(x,\xi) + \lambda_j(\xi)\partial_\xi K_{ij}(x,\xi) = -\sum_{k=1}^{n}\big(\sigma_{kj}(\xi) + \delta_{kj}\lambda_j'(\xi)\big)K_{ik}(x,\xi) \tag{2.45}$$

along with the following set of boundary conditions

$$K_{ij}(x,x) = \frac{\sigma_{ij}(x)}{\lambda_i(x)-\lambda_j(x)} \triangleq k_{ij}(x) \qquad \text{for } 1 \le i,j \le n (i \ne j), \tag{2.46}$$

$$K_{ij}(x,0) = -\frac{1}{\lambda_j(0)}\sum_{k=1}^{n-m}\lambda_{m+k}(0)K_{i,m+k}(x,0)q_{k,j} \quad \text{for } 1 \le i \le j \le m. \tag{2.47}$$

To ensure well-posedness of the kernel equations, we add the following artificial boundary conditions for $K_{ij}(m \ge i > j \ge 1, n \ge j > i \ge m+1)$ on $x = 1$:

$$K_{ij}(1,\xi) = k_{ij}^{(1)}(\xi) \text{ for } \ 1 \le j < i \le m \ \cup \ m+1 \le i < j \le n, \tag{2.48}$$

and the boundary conditions for $K_{ij}(n \ge i \ge j \ge m+1)$ on $\xi = 0$:

$$K_{ij}(x,0) = k_{ij}^{(2)}(x) \text{ for } \ m+1 \le j \le i \le n, \tag{2.49}$$

where $k_{ij}^{(1)}$ and $k_{ij}^{(2)}$ are chosen as functions of $C^\infty[0,1]$ satisfying the C^2 compatibility conditions at the points $(x,\xi) = (1,1)$ and $(0,0)$, respectively (see Remark 2.1). The equations evolve in the triangular domain $\mathcal{T} = \{(x,\xi)|0 \le \xi \le x \le 1\}$. By Theorem A.1, one finds that there exists a unique piecewise $C^2(\mathcal{T})$ solution $K(x,\xi)$ to (2.45)–(2.49) with finitely many discontinuities, provided that $\sigma_{ij}(x)$ are $C^2[0,1]$, $\lambda_i(x)$ are $C^2[0,1]$. Moreover, all the possible discontinuous curves have the similar form $\xi = \Omega(x)$, in which $\Omega(\cdot) \in C^2[0,1]$ being a monotonically increasing function with $\Omega(0) = 0$ and $0 < \Omega(x) < x (\forall x \in (0,1])$. The boundary of the Kernel K on $x = \xi$ and $\xi = 0$ are both $C^2[0,1]$ functions, i.e., $K(x,x),\ K(x,0) \in C^2[0,1]$. Thus, G, given by (2.44), is also a $C^2[0,1]$ matrix function under the well-posedness of $K(x,0)$.

Remark 2.1. The C^2 compatibility conditions at the point $(x,\xi) = (1,1)$ are given by

(2.50)

$$k_{ij}^{(1)}(1) = k_{ij}(1) \text{ for } \ 1 \le j < i \le m \ \cup \ m+1 \le i < j \le n,$$

(2.51)

$$(k^{(1)})_{ij}'(1) = \frac{\lambda_i(1)k_{ij}'(1) + \sum_{k=1}^{n}\big(\sigma_{kj}(1) + \delta_{kj}\lambda_j'(1)\big)K_{ik}(1,1)}{\lambda_i(1)-\lambda_j(1)}$$
$$\text{for } 1 \le j < i \le m \cup m+1 \le i < j \le n,$$

$$(k^{(1)})''_{ij}(1) = \frac{\lambda_i(1)\partial_x\left(\frac{\lambda_i(x)k'_{ij}(x)+\sum_{k=1}^n\left(\sigma_{kj}(x)+\delta_{kj}\lambda'_j(x)\right)K_{ik}(x,x)}{\lambda_i(x)-\lambda_j(x)}\right)\Big|_{x=1} + \lambda'_j(1)(k^{(1)})'_{ij}(1)}{\lambda_i(1)-\lambda_j(1)}$$

$$+\frac{\sum_{k=1}^n\left(\sigma'_{kj}(1)+\delta_{kj}\lambda''_j(1)\right)K_{ik}(1,1)+\left(\sigma_{kj}(1)+\delta_{kj}\lambda'_j(1)\right)\partial_\xi K_{ik}(1,1))}{\lambda_i(1)-\lambda_j(1)} \tag{2.52}$$

$$\text{for } 1\le j<i\le m\cup m+1\le i<j\le n,$$

The C^2 compatibility conditions at the point $(x,\xi)=(0,0)$ can be similarly given. It should be mentioned here that in (2.51), the term $K_{ik}(1,1)=k_{ik}(1)$ if $i\neq k$, and $K_{ii}(1,1)$ can be calculated by using the characteristic method with (2.45), (2.46) and the value of $K_{ii}(0,0)$ (i.e., $k_{ii}^{(2)}(0)$). Also, in (2.52), the term $\partial_x K_{ik}(1,1)$ can be calculated by (A.13) and (A.27) if $k\neq i$, otherwise, it can be expressed by using the characteristic method with (A.7), (A.24), (A.13), (A.27), and the value of $\partial_x K_{ii}(0,0)$ (i.e., $(k^{(2)})'_{ii}(0)$); the term $\partial_\xi K_{ik}(1,1)$ can be immediately obtained by (2.45) once $\partial_x K_{ik}(1,1)$, $K_{ik}(1,1)$ are determined.

2.3. The inverse transformation and stabilization for linear systems. Transformation (2.40) is a classical Volterra equation of the second kind; one can check from Theorem A.2 that there exists a unique piecewise $C^2(\mathcal{T})$ matrix function $L(x,\xi)$ with finitely many discontinuities which are $C^2[0,1]$ monotonically increasing functions passing through the point $(x,\xi)=(0,0)$, such that

$$w(t,x)=\gamma(t,x)+\int_0^x L(x,\xi)\gamma(t,\xi)d\xi. \tag{2.53}$$

From the transformation (2.40) evaluated at $x=1$, noting that $\gamma_-(t,1)\equiv 0$ (see (2.9)), one obtains the following feedback control laws for the linearized system (2.1) with the boundary conditions (2.5)

$$U_i(t)=\int_0^1\sum_{j=1}^n K_{ij}(1,\xi)w_j(t,\xi)d\xi \quad (i=1,\dots,m) \tag{2.54}$$

in which $U_i (i=1,\dots,m)$ are the elements of U in (2.6). This immediately leads to our feedback stabilization result for the linear system as follows.

THEOREM 2.1. *The mixed initial boundary value problem* (2.1) *with the boundary conditions* (2.5)*, the feedback control law* (2.54)*, and initial condition*

$$t=0: w(0,x)=w_0(x), \tag{2.55}$$

in which $w_0\in L^2((0,1);\mathbb{R}^n)$, admits an $L^2((0,1);\mathbb{R}^n)$ solution $w=w(t,x)$. Moreover, for every $\eta>0$, there exists $c>0$ such that

$$\|w(t,\cdot)\|_{L^2}\le ce^{-\eta t}\|w_0\|_{L^2}. \tag{2.56}$$

In fact, w vanishes in finite time $t\ge t_F$, where t_F is given by (2.14).

Remark 2.2. If we focus on the linear problem, Λ and Σ can be assumed to be $C^1([0,1])$ and $C^0([0,1])$ functions. The corresponding kernels K and L are then both functions of $L^\infty(\mathcal{T})$ (see [15]).

3. Backstepping boundary control design for the nonlinear system. As mentioned in [9], we wish the linear controller (2.54) designed by the backstepping method to work locally for the corresponding nonlinear system. Let us show that this is indeed the case. Introduce

$$\varphi_i(x) := \exp\left(-\int_0^x \frac{f_{ii}(s)}{\Lambda_i(s)}ds\right), \quad i = 1, \ldots, n, \tag{3.1}$$

where f_{ii} and Λ_i $(i = 1, c\ldots, n)$ are defined in (1.3) and (1.4), respectively. One can make the following coordinates transformation

$$w(t,x) = \begin{pmatrix} \varphi_1(x) & & \\ & \ddots & \\ & & \varphi_n(x) \end{pmatrix} u(t,x) = \Phi(x)u(t,x). \tag{3.2}$$

Then the original control system (1.1) with respect to u is transformed into the following system expressed in the new coordinates:

$$w_t(t,x) + \overline{A}(x,w)w_x(t,x) = \widetilde{F}(x,w) \tag{3.3}$$

in which

$$\overline{A}(x,w) = \Phi(x)A(x,\Phi^{-1}(x)w)\Phi^{-1}(x), \tag{3.4}$$

$$\widetilde{F}(x,w) = \Phi(x)F(x,\Phi^{-1}(x)w) - \overline{A}(x,w)\begin{pmatrix} \frac{f_{11}(x)}{\Lambda_1(x)} & & \\ & \ddots & \\ & & \frac{f_{nn}(x)}{\Lambda_n(x)} \end{pmatrix} w. \tag{3.5}$$

Obviously, one can check that

$$\widetilde{F}(x,0) = 0, \tag{3.6}$$

$$\overline{A}(x,0) = \Phi(x)A(x,0)\Phi^{-1}(x) = A(x,0). \tag{3.7}$$

Moreover, defining

$$\Sigma(x) = \left.\frac{\partial \widetilde{F}(x,w)}{\partial w}\right|_{w=0}, \tag{3.8}$$

we have that

$$\Sigma_{ij}(x) = \begin{cases} \frac{\varphi_i(x)}{\varphi_j(x)} f_{ij}(x), i \neq j, \\ 0, i = j. \end{cases} \tag{3.9}$$

Therefore, we may rewrite (3.3) as a linear system with the same structure as (2.1) plus nonlinear terms:

$$w_t(t,x) + \Lambda(x)w_x(t,x) = \Sigma(x)w(t,x) + \Lambda_{NL}(x,w)w_x(t,x) + f_{NL}(x,w), \tag{3.10}$$

where

$$\Lambda(x) = A(x,0) \tag{3.11}$$

and

$$\Lambda_{NL}(x,w) = \Lambda(x) - \overline{A}(x,w), \;\; f_{NL}(x,w) = \widetilde{F}(x,w) - \Sigma(x)w(t,x). \tag{3.12}$$

For the boundary conditions of the system (3.10), defining

$$Q = \left(\frac{\partial G_s}{\partial u_r}\right)_{(n-m)\times m}\Bigg|_{u=0} \quad \text{and } G_{NL}(w_-(t,0)) = G(w_-(t,0)) - Qw_-(t,0), \tag{3.13}$$

one obtains that

$$x = 0 : w_+(t,0) = Qw_-(t,0) + G_{NL}(w_-(t,0)) \tag{3.14}$$

and

$$x = 1 : w_-(t,1) = U(t), \tag{3.15}$$

where

$$U(t) = \begin{pmatrix} \varphi_1(1) & & \\ & \ddots & \\ & & \varphi_m(1) \end{pmatrix} H(t) = \Phi(1)H(t). \tag{3.16}$$

It is easily verified that

$$\Lambda_{NL}(x,0) = 0, \quad f_{NL}(x,0) = \frac{\partial f_{NL}}{\partial w}(x,0) = 0, \tag{3.17}$$

and

$$G_{NL}(0) = \frac{\partial G_{NL}}{\partial w}(0) = 0. \tag{3.18}$$

Thus, the feedback control law in (1.6) can be chosen as

$$h_r(t) = \varphi_r^{-1}(1)U_r(t) = \varphi_r^{-1}(1)\int_0^1 \sum_{j=1}^n K_{rj}(1,\xi)\varphi_j(\xi)u_j(t,\xi)d\xi, \quad r = 1,\ldots,m, \tag{3.19}$$

where the kernels are computed from (2.45)–(2.49) with the coefficients $\Sigma(x)$ and $\Lambda(x)$ obtained from (3.9) and (3.11). One easily verifies that under the assumptions of section 1, both Σ and Λ are functions of C^2.

Remark 3.1. The C^1 compatibility conditions at the point $(t,x) = (0,1)$ for system (1.1) with boundary conditions (3.15) are

$$\phi_r(1) = \sum_{j=1}^n \int_0^1 \tilde{k}_{rj}(\xi)\phi_j(\xi)d\xi, \quad r = 1,\ldots,m, \tag{3.20}$$

$$f_r(1,\phi(1)) - \sum_{j=1}^n a_{rj}(1,\phi(1))\phi_j'(1) = \sum_{k=1}^n \int_0^1 \tilde{k}_{rk}(\xi)\left(f_k(1,\phi(1)) - \sum_{j=1}^n a_{kj}(1,\phi(1))\phi_j'(1)\right), \quad r = 1,\ldots,m, \tag{3.21}$$

where $\tilde{k}_{rk}(\xi)$ are the elements of the matrix $\widetilde{K}(\xi)$ with

$$\widetilde{K}(\xi) = \Phi^{-1}(1)K(1,\xi)\Phi(\xi). \tag{3.22}$$

Notice that (3.20)–(3.21) depend on the feedback control design, however, there are no physical reasons that the initial data should satisfy them. In order to guarantee the initial conditions independent of these artificial conditions, following [9], we modify the boundary controls on $x = 1$ as

$$x = 1: \quad u_r = h_r(t) + a_r(t) + b_r(t), \quad r = 1, \ldots, m, \tag{3.23}$$

where a_r and b_r are the state of the following dynamic systems

$$\dot{a}_r(t) = -d_r a_r(t), \quad \dot{b}_r(t) = -\tilde{d}_r b_r(t), \quad r = 1, \ldots, m, \tag{3.24}$$

with $d_r > 0, \tilde{d}_r > 0$, and $d_r \neq \tilde{d}_r, r = 1, \ldots, m$. By the modified control designs (3.23), the compatibility conditions on $x = 1$ are rewritten by

(3.25)

$$\phi_r(1) = \sum_{j=1}^{n} \int_0^1 \tilde{k}_{rj}(\xi)\phi_j(\xi)d\xi + a_r(0) + b_r(0), \quad r = 1, \ldots, m,$$

$$f_r(1, \phi(1)) - \sum_{j=1}^{n} a_{rj}(1, \phi(1))\phi_j'(1)$$

(3.26)

$$= \sum_{k=1}^{n} \int_0^1 \tilde{k}_{rk}(\xi) \left(f_k(1, \phi(1)) - \sum_{j=1}^{n} a_{kj}(1, \phi(1))\phi_j'(1) \right) - d_r a_r(0) - \tilde{d}_r b_r(0),$$

$$r = 1, \ldots, m.$$

For any $1 \leq r \leq m$, call

$$\mathcal{P}_r(\phi) = \phi_r(1) - \sum_{j=1}^{n} \int_0^1 \tilde{k}_{rj}(\xi)\phi_j(\xi)d\xi, \tag{3.27}$$

$$\begin{aligned} \mathcal{M}_r(\phi) = {} & f_r(1, \phi(1)) - \sum_{j=1}^{n} a_{rj}(1, \phi(1))\phi_j'(1) \\ & - \sum_{k=1}^{n} \int_0^1 \tilde{k}_{rk}(\xi) \left(f_k(1, \phi(1)) - \sum_{j=1}^{n} a_{kj}(1, \phi(1))\phi_j'(1) \right). \end{aligned} \tag{3.28}$$

Picking

$$a_r(0) = -\frac{\mathcal{M}_r(\phi) + \tilde{d}_r \mathcal{P}_r(\phi)}{d_r - \tilde{d}_r}, \quad b_r(0) = \frac{d_r \mathcal{P}_r(\phi) + \mathcal{M}_r(\phi)}{d_r - \tilde{d}_r}, \tag{3.29}$$

the compatibility conditions are automatically verified. Similar stabilization results to Theorem 1.1 are still valid for the closed–loop system (1.1), (1.5), and (3.23) (see [9, Theorem 4.1]). In fact, this dynamic extension is designed to avoid restriction for artificial boundary conditions due to the compatibility conditions at the points $(t, x) = (0, 1)$, and it has been introduced in [3] to deal with the stabilization of the Euler equations of incompressible fluids (see also [24]).

4. Proof of Theorem 1.1. In this section, we will show the exponential stability for the system (1.1), (1.5), and (1.6) with arbitrary decay rate under the boundary feedback controls (3.19) by the control Lyapunov function method. Because of the coordinate transformation (3.2), it suffices to prove the same property for the system (3.3), (3.14)–(3.16). The related proof can be divided into the following steps. Roughly speaking, using the backstepping transformation (2.40), we first map the initial nonlinear system (3.3), (3.14)–(3.16) into another nonlinear target system but with cascade zero-order terms, which has the same stability property as the initial system. The rapid exponential stability of the target system, thanks to its special structure, can be realized by constructing the strict Lyapunov function as mentioned in [7, 6, 9].

4.1. Definitions. For $\gamma(x) := (\gamma_1(x), \ldots, \gamma_n(x)) \in \mathbb{R}^n$, we first define some notations:

$$|\gamma(x)| = \sum_{i=1}^{n} |\gamma_i(x)|,\ \|\gamma\|_\infty := \operatorname*{ess\,sup}_{x\in[0,1]} |\gamma(x)|, \tag{4.1}$$

$$\|\gamma\|_{L^p} := \left(\int_0^1 |\gamma(\xi)|^p d\xi\right)^{\frac{1}{p}},\ 1 \le p < +\infty.$$

For an $n \times n$ matrix, denote

$$|M| := \max\{|M\gamma| : \gamma \in \mathbb{R}^n, |\gamma| = 1\}. \tag{4.2}$$

For a piecewise kernel matrix $K(x, \xi)$, which is a continuous function on each domain $D_i (i = 1, \ldots, \mathcal{S} < +\infty)$, respectively, with

$$\mathcal{T} = \bigcup_{i=1}^{\mathcal{S}} D_i, \tag{4.3}$$

$$meas(D_i \cap D_j) = 0, (i \neq j), \tag{4.4}$$

where $meas(\cdot)$ denotes the measure of the corresponding measurable set. Let

$$\|K\|_\infty := \max_i \sup_{(x,\xi)\in D_i} |K(x, \xi)|. \tag{4.5}$$

As before, we recall the following symbols of [9] for simplicity:

$$\mathcal{K}[\gamma](t, x) = \gamma(t, x) - \int_0^x K(x, \xi)\gamma(t, \xi)d\xi, \tag{4.6}$$

$$\mathcal{L}[\gamma](t, x) = \gamma(t, x) + \int_0^x L(x, \xi)\gamma(t, \xi)d\xi, \tag{4.7}$$

$$\mathcal{K}_1[\gamma](t, x) = -K(x, x)\gamma(t, x) + \int_0^x K_\xi(x, \xi)\gamma(t, \xi)d\xi + E_1(x)\gamma(t, \Omega(x)), \tag{4.8}$$

$$\mathcal{K}_2[\gamma](t, x) = -K(x, x)\gamma(t, x) - \int_0^x K_x(x, \xi)\gamma(t, \xi)d\xi + E_2(x)\gamma(t, \Omega(x)), \tag{4.9}$$

$$\mathcal{L}_1[\gamma](t, x) = L(x, x)\gamma(t, x) + \int_0^x L_x(x, \xi)\gamma(t, \xi)d\xi + E_3(x)\gamma(t, \Omega(x)) \tag{4.10}$$

in which $E_i(x)\gamma(t, \Omega(x))\,(i = 1, 2, 3)$ involves all the possible jumps when we use integrations by parts for the term $\int_0^x K(x, \xi)\gamma_\xi(t, \xi)d\xi$ and take the partial derivative with respect to x on $\mathcal{K}[\gamma]$ and $\mathcal{L}[\gamma]$, respectively (see, in particular, (A.17)).

Define $F_1[\gamma]$, $F_2[\gamma]$, and $G_{bou}[\gamma]$ as

$$F_1[\gamma] := \Lambda_{NL}(x, \mathcal{L}[\gamma]), \quad F_2[\gamma] := f_{NL}(x, \mathcal{L}[\gamma]), \tag{4.11}$$

$$G_{bou}[\gamma](t) := K(x,0)\Lambda(0) \begin{pmatrix} 0 & 0 \\ G_{NL}(\gamma_-(t,0)) & 0 \end{pmatrix} \tag{4.12}$$

in which G_{NL} is given by (3.13).Obviously, by (3.18), there exist positive constants δ_1, C_1, C_2, and C_3 such that if $\|\gamma\|_\infty < \delta_1$ then for every $v_- \in \mathbb{R}^m$,

$$\left|G_{NL}(\gamma_-(t,0))\right| \le C_1 |\gamma_-(t,0)|^2, \tag{4.13}$$

$$\left|\frac{\partial G_{NL}(\gamma_-(t,0))}{\partial \gamma_-}\right| \le C_2 |\gamma_-(t,0)|, \tag{4.14}$$

$$\left|\frac{\partial^2 G_{NL}(\gamma_-(t,0))}{\partial \gamma_-^2} v_-\right| \le C_3 |v_-|. \tag{4.15}$$

Here and hereafter, for $i = 1, 2, \ldots, C_i$, denote positive constants, which are independent of γ, ζ, and θ (the latter two variables will be defined in the next subsections).

To prove our result, we notice that if we apply the (inverse) backstepping transformation (2.40) to the nonlinear system (3.10), we obtain the following transformed system

$$\begin{aligned} &\gamma_t(t,x) + \Lambda(x)\gamma_x(t,x) - G(x)\gamma(t,0) \\ &\quad = \mathcal{K}[\Lambda_{NL}(x,w)w_x] + \mathcal{K}[f_{NL}(x,w)] + G_{bou}[\gamma] \\ &\quad = \mathcal{K}[\Lambda_{NL}(x,w)\gamma_x] + \mathcal{K}[\Lambda_{NL}(x,w)\mathcal{L}_1[\gamma]] + \mathcal{K}[f_{NL}(x,w)] + G_{bou}[\gamma] \\ &\quad = F_3[\gamma,\gamma_x] + F_4[\gamma] + G_{bou}[\gamma], \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} F_3 &= \mathcal{K}[F_1[\gamma]\gamma_x], \\ F_4 &= \mathcal{K}[F_1[\gamma]\mathcal{L}_1[\gamma] + F_2[\gamma]]. \end{aligned}$$

The boundary conditions are

$$x = 0 : \gamma_+(t,0) = Q\gamma_-(t,0) + G_{NL}(\gamma_-(t,0)) \tag{4.17}$$

and

$$x = 1 : \gamma_-(t,1) = 0. \tag{4.18}$$

Notice that here we may lose the regularity on the point $(0,0)$ for the kernels K and L, which leads both of them to be discontinuous (see [15]). However, by the assumptions on the coefficients and applying Theorems A.1 and A.2, the direct and inverse transformations (2.40) and (2.53) have C^2 piecewise kernels functions with finitely many discontinuities, which have the form $\xi = \Omega(x)$ being functions in $C^2[0,1]$ with $\Omega(0) = 0$ and $0 < \Omega(x) < x$ ($\forall x \in (0,1]$). Fortunately, differentiating twice with respect to x in these transformations, by similar arguments to [9] and [24, Proposition 3.1] as well the additive property of the integral and

$$\|\gamma(t,\Omega)\|_{L^2} \le C_4 \|\gamma\|_{L^2}, \tag{4.19}$$

it can be shown that the H^2 norm of γ is equivalent to the H^2 norm of w. Thus, if we show H^2 local stability of the origin for (4.16)–(4.18), the same holds for w, i.e., u.

In order to get the desired H^2 estimation for γ, we next estimate the growth of $\|\gamma\|_{L^2}$, $\|\gamma_t\|_{L^2}$, and $\|\gamma_{tt}\|_{L^2}$, respectively.

4.2. Analyzing the growth of $\|\gamma\|_{L^2}$. Let

$$F_3[\gamma,\gamma_x] = \left(F_3^-[\gamma,\gamma_x], F_3^+[\gamma,\gamma_x]\right)^T, \tag{4.20}$$

$$F_4[\gamma] = \left(F_4^-[\gamma], F_4^+[\gamma]\right)^T, G_{bou}[\gamma] = \left(G_{bou}^-[\gamma], G_{bou}^+[\gamma]\right)^T, \tag{4.21}$$

where F_3^-, F_4^- and $G_{bou}^- \in \mathbb{R}^m$, F_3^+ and F_4^+ and $G_{bou}^+ \in \mathbb{R}^{n-m}$.

For $\delta > 0$, define

$$\begin{aligned} V_1(t) = &\int_0^1 e^{-\delta x}\gamma_+(t,x)^T\left(\Lambda_+(x)\right)^{-1}\gamma_+(t,x)dx \\ &- \int_0^1 e^{\delta x}\gamma_-(t,x)^T B\left(\Lambda_-(x)\right)^{-1}\gamma_-(t,x)dx \end{aligned} \tag{4.22}$$

in which the matrix B is given by (2.17), but the coefficients will be reinitialized. Differentiating V_1 with respect to time and integrating by parts yields

$$\dot{V}_1(t) = V + VI + VII + VIII + IX + X$$

with

$$\begin{aligned} V &= \left[-e^{-\delta x}\gamma_+(t,x)^T\gamma_+(t,x) + e^{\delta x}\gamma_-(t,x)^T B\gamma_-(t,x)\right]_0^1, \\ VI &= -\int_0^1 \delta e^{-\delta x}\gamma_+(t,x)^T\gamma_+(t,x)dx - \int_0^1 \delta e^{\delta x}\gamma_-(t,x)^T B\gamma_-(t,x)dx, \\ VII &= 2\int_0^1 e^{-\delta x}\gamma_+(t,x)^T\left(\Lambda_+(x)\right)^{-1}\mathcal{G}_2(x)\gamma_-(t,0)dx, \\ VIII &= -2\int_0^1 e^{\delta x}\gamma_-(t,x)^T B\left(\Lambda_-(x)\right)^{-1}\mathcal{G}_1(x)\gamma_-(t,0)dx, \\ IX &= 2\int_0^1 e^{-\delta x}\gamma_+(t,x)^T\left(\Lambda_+(x)\right)^{-1}\left(F_3^+[\gamma,\gamma_x] + F_4^+[\gamma] + G_{bou}^+[\gamma]\right)dx, \\ X &= -2\int_0^1 e^{\delta x}\gamma_-(t,x)^T B\left(\Lambda_-(x)\right)^{-1}\left(F_3^-[\gamma,\gamma_x] + F_4^-[\gamma] + G_{bou}^-[\gamma]\right)dx. \end{aligned}$$

By the same arguments as in [9] and noting Lemma B.2, we have

$$\begin{aligned} &\int_0^1 e^{-\delta x}\gamma_+(t,x)^T\left(\Lambda_+(x)\right)^{-1}\left(F_3^+[\gamma,\gamma_x] + F_4^+[\gamma]\right)dx \\ &\quad + \int_0^1 e^{\delta x}\gamma_-(t,x)^T B\left(\Lambda_-(x)\right)^{-1}\left(F_3^-[\gamma,\gamma_x] + F_4^-[\gamma]\right)dx \\ &\leq C_4\int_0^1 |\gamma|(|F_3[\gamma,\gamma_x]| + |F_4[\gamma]|)dx \\ &\leq C_5\left(\|\gamma_x\|_\infty V_1 + V_1^{\frac{3}{2}}\right) \end{aligned} \tag{4.23}$$

which, combining with (4.13), yields that

$$IX + X \leq C_6(\|\gamma_x\|_\infty V_1 + V_1^{\frac{3}{2}} + \|\gamma\|_\infty|\gamma_-(t,0)|^2). \tag{4.24}$$

Moreover, by (4.13) and (4.17), for $\|\gamma\|_\infty < \delta_1$, one has

$$\begin{aligned} V &= -e^{-\delta}\gamma_+(t,1)^T\gamma_+(t,1) + e^{\delta}\gamma_-(t,1)^T B\gamma_-(t,1) \\ &\quad + \gamma_+(t,0)^T\gamma_+(t,0) - \gamma_-(t,0)^T B\gamma_-(t,0) \\ &\le -\gamma_-(t,0)^T\Big(B - Q^TQ - (C_1^2\|\gamma\|_\infty^2 + 2C_1\|\gamma\|_\infty|Q|)I_m\Big)\gamma_-(t,0). \end{aligned} \tag{4.25}$$

By (2.19) and (2.20), for $\|\gamma\|_\infty < \delta_1$, one immediately obtains

$$\begin{aligned} \dot V_1(t) &\le -\gamma_-(t,0)^T\Big(B - S - M\mu e^{\delta}\mathcal{C} - C_6\|\gamma\|_\infty I_m\Big)\gamma_-(t,0) \\ &\quad - (\delta-1)\int_0^1 e^{-\delta x}\gamma_+(t,x)^T\gamma_+(t,x)dx \\ &\quad - (\delta - mM\mu)\int_0^1 e^{\delta x}\gamma_-(t,x)^T B\gamma_-(t,x)dx + C_6\big(V_1^{\frac{3}{2}} + \|\gamma_x\|_\infty V_1\big), \end{aligned} \tag{4.26}$$

where M, $\mathcal{C}, \mu$ are given by (2.21) and (2.23), while S is stated in (2.25).

Thus, for any given $\lambda > 0$, picking

$$\delta > \max\{\lambda\mu + mM\mu, \lambda\mu + 1\}, \tag{4.27}$$

$$b_r := \begin{cases} M\mu e^{\delta}\sum\limits_{j=r+1}^{m} b_j + \tilde s_r, & 1 \le r \le m-1, \\ \tilde s_m, & r = m, \end{cases} \tag{4.28}$$

then if $\|\gamma\|_\infty$ is suitably small, there exists $C_7 > 0$ such that

$$C_7 I_m \prec B - \widetilde{S} - M\mu e^{\tilde\delta}\mathcal{C} - C_6\|\gamma\|_\infty I_m. \tag{4.29}$$

This yields the following proposition.

Proposition 4.1. *For any given $\lambda > 0$, there exists $\delta_2 > 0, C_8 > 0$, and $C_9 > 0$, such that*

$$\dot V_1 \le -\lambda V_1 + C_8\big(V_1^{\frac{3}{2}} + \|\gamma_x\|_\infty V_1\big) - C_9|\gamma_-(t,0)|^2, \tag{4.30}$$

provided $\|\gamma\|_\infty \le \delta_2$.

4.3. Analyzing the growth of $\|\gamma_t\|_{L^2}$. Let $\zeta = \gamma_t$. Taking the partial derivative with respect to t in (4.16) yields

$$\zeta_t(t,x) + (\Lambda(x) - F_1[\gamma])\zeta_x(t,x) - G(x)\zeta(t,0) = F_5[\gamma,\gamma_x,\zeta] + F_6[\gamma,\zeta] + G'_{bou}[\gamma](t), \tag{4.31}$$

where

$$\begin{aligned} F_5 &= \mathcal{K}_1[F_1[\gamma]\zeta] + \int_0^x K(x,\xi)F_{12}[\gamma,\gamma_x]\zeta(\xi)d\xi \\ &\quad + K(x,0)\Lambda_{NL}(0,\gamma(0))\zeta(0) + \mathcal{K}[F_{11}[\gamma,\zeta]\gamma_x], \end{aligned} \tag{4.32}$$

$$F_6 = \mathcal{K}[F_{11}[\gamma,\zeta]\mathcal{L}_1[\gamma]] + \mathcal{K}[F_1[\gamma]\mathcal{L}_1[\zeta]] + \mathcal{K}[F_{21}[\gamma,\zeta]], \tag{4.33}$$

$$G'_{bou}[\gamma](t) = K(x,0)\Lambda(0)\begin{pmatrix} 0 & 0 \\ \frac{\partial G_{NL}(\gamma_-(t,0))}{\partial\gamma_-}\zeta_-(t,0)) & 0 \end{pmatrix} \tag{4.34}$$

with

$$\begin{aligned} F_{11} &= \frac{\partial \Lambda_{NL}}{\partial w}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta], \\ F_{12} &= \frac{\partial \Lambda_{NL}}{\partial w}(x, \mathcal{L}[\gamma])(\gamma_x + \mathcal{L}_1[\gamma]) + \frac{\partial \Lambda_{NL}}{\partial x}(x, \mathcal{L}[\gamma]), \\ F_{21} &= \frac{\partial f_{NL}}{\partial w}(x, \mathcal{L}[\gamma])\mathcal{L}[\zeta]. \end{aligned} \tag{4.35}$$

Remark 4.1. In fact, here F_{11}, F_{12}, and F_{21} denote $\frac{\partial \Lambda_{NL}(x,\mathcal{L}[\gamma])}{\partial t}$, $\frac{\partial \Lambda_{NL}(x,\mathcal{L}[\gamma])}{\partial x}$, and $\frac{\partial f_{NL}(x,\mathcal{L}[\gamma])}{\partial t}$, respectively.

The boundary conditions are given by

$$x = 0 : \zeta_+(t, 0) = Q\zeta_-(t, 0) + \frac{\partial G_{NL}}{\partial \gamma_-}\Big(\gamma_-(t, 0)\Big)\zeta_-(t, 0) \tag{4.36}$$

and

$$x = 1 : \zeta_-(t, 1) = 0 \tag{4.37}$$

in which $\zeta_- \in \mathbb{R}^m, \zeta_+ \in \mathbb{R}^{n-m}$ are defined by requiring that $\zeta := (\zeta_-, \zeta_+)^T$.

Similarly to [9], we need the following lemma in order to find a strict Lyapunov function for $\zeta(t, x)$.

LEMMA 4.2. *There exists $\delta_3 > 0$ such that, for any $\|\gamma\|_\infty \leq \delta_3$, there exists a symmetric positive-definite matrix $R[\gamma]$ satisfying the identity*

$$R[\gamma](\Lambda(x) - F_1[\gamma]) - (\Lambda(x) - F_1[\gamma])^T R[\gamma] = 0. \tag{4.38}$$

Moreover, we have that

$$|R[\gamma](x)| \leq c_1 + c_2\|\gamma\|_\infty, \tag{4.39}$$

$$\left|\Big(\Theta[\gamma]\Lambda(x)\Big)_x\right| \leq c_2(\|\gamma\|_\infty + \|\gamma_x\|_\infty), \tag{4.40}$$

$$|(R[\gamma])_t| \leq c_3(|\zeta| + \|\zeta\|_{L^1}), \tag{4.41}$$

where c_1, c_2, and c_3 are positive constants, and

$$\Theta[\gamma] = R[\gamma] - D(x), \quad D(x) = \begin{pmatrix} -e^{\delta x} B(\Lambda_-(x))^{-1} & 0 \\ 0 & e^{-\delta x}(\Lambda_+(x))^{-1} \end{pmatrix}. \tag{4.42}$$

Proof. Denote $\mathcal{D}_n(x)$ as the set of $n \times n$ diagonal matrices with C^1 elements. Let $\Lambda(x) := \mathrm{diag}(\Lambda_1(x), \dots, \Lambda_n(x)) \in \mathcal{D}_n(x)$ be such that $\Lambda_i(x) \neq \Lambda_j(x)$ $(i \neq j\ \forall x \in [0, 1])$ holds. Notice that $D \in \mathcal{D}_n(x)$. Based on the proof in [6, Lemma 4.1], one can easily see that there exist a positive real number η and a map

$$\mathcal{N} : \{M \in \mathcal{M}_{n,n}(\mathbb{R}; x); \|M(x) - \Lambda(x)\|_{C^1} < \eta\} \to \mathcal{S}_n$$

of class C^∞ such that

$$\mathcal{N}(\Lambda(x)) = D(x) \tag{4.43}$$

and

$$\mathcal{N}(M)M - M^T\mathcal{N}(M) = 0 \quad \forall M \in \mathcal{M}_{n,n}(\mathbb{R}; x), \quad \|M(x) - \Lambda(x)\|_{C^1} < \eta. \tag{4.44}$$

It then suffices to define $R[\gamma]$ by

$$R[\gamma] = \mathcal{N}(\Lambda(x) - F_1[\gamma]). \tag{4.45}$$

Moreover, by the regularity of $\mathcal{N}$ and Lemmas B.2–B.3, one can show that

$$\begin{aligned} |R[\gamma]| &\le |D(x)| + |R[\gamma] - D(x)| \\ &\le c_4 + c_5|F_1[\gamma]| \\ &\le c_4 + c_6\|\gamma\|_\infty, \end{aligned} \tag{4.46}$$

$$\begin{aligned} \left|\left((R[\gamma] - D(x))\Lambda(x)\right)_x\right| &\le |(R[\gamma] - D(x))_x\Lambda(x)| + |(R[\gamma] - D(x))\Lambda_x(x)| \\ &\le c_7|F_{12}| + c_8|F_1| \\ &\le c_9(\|\gamma\|_\infty + \|\gamma_x\|_\infty), \end{aligned} \tag{4.47}$$

and

$$|R[\gamma]_t| \le c_{10}\left|\frac{\partial F_1[\gamma]}{\partial t}\right| \tag{4.48}$$

$$\le c_{10}|F_{11}[\gamma, \zeta]| \tag{4.49}$$

$$\le c_{11}(|\zeta| + \|\zeta\|_{L^1}). \tag{4.50}$$

Here $c_i\,(i = 4, 5, \ldots, 10)$ are positive constants. This concludes the proof of Lemma 4.1. ∎

Define

$$V_2(t) = \int_0^1 \zeta^T(t,x)R[\gamma]\zeta(t,x)dx. \tag{4.51}$$

Using (4.38) and noting $R[\gamma]$ is symmetric, by straightforward computations, one can show that

$$\dot V_2(t) = XI + XII + XIII + XIV + XV$$

with

$$\begin{aligned} XI &= \int_0^1 \zeta^T(t,x)(R[\gamma](\Lambda(x) - F_1[\gamma]))_x\zeta(t,x)dx, \\ XII &= -[\zeta^T(t,x)R[\gamma](\Lambda(x) - F_1[\gamma])\zeta(t,x)]_{x=0}^{x=1}, \\ XIII &= \int_0^1 \zeta(t,x)(R[\gamma])_t\zeta(t,x)dx, \\ XIV &= 2\int_0^1 \zeta^T(t,x)R[\gamma]F_5[\gamma,\gamma_x,\zeta,\zeta_x]dx + 2\int_0^1 \zeta^T(t,x)R[\gamma]F_6[\gamma,\zeta]dx, \\ XV &= 2\int_0^1 \zeta^T(t,x)R[\gamma]G(x)\zeta(t,0)dx + 2\int_0^1 \zeta^T(t,x)R[\gamma]G'_{bou}[\gamma](t)dx. \end{aligned}$$

For XII and XV, by the boundary conditions (4.36)–(4.37) and similar computations as in (2.19) and (2.20), under the assumption that $\|\gamma\|_\infty$ is suitably small, we have

(4.52)
$$\begin{aligned} XII + XV &= -[\zeta^T(t,x)R[\gamma](\Lambda(x) - F_1[\gamma])\zeta(t,x)]_{x=0}^{x=1} \\ &\quad + 2\int_0^1 \zeta^T(t,x)R[\gamma]G(x)\zeta(t,0)dx + 2\int_0^1 \zeta^T(t,x)R[\gamma]G'_{bou}[\gamma](t)dx \\ &= -[\zeta^T(t,x)(D(x)\Lambda(x) + \Theta[\gamma]\Lambda(x) - D(x)F_1[\gamma] - \Theta[\gamma]F_1[\gamma])\zeta(t,x)]_{x=0}^{x=1} \\ &\quad + 2\int_0^1 \zeta^T(t,x)D(x)G(x)\zeta(t,0)dx + 2\int_0^1 \zeta^T(t,x)\Theta[\gamma]G(x)\zeta(t,0)dx \\ &\quad + 2\int_0^1 \zeta^T(t,x)R[\gamma]G'_{bou}[\gamma](t)dx \\ &\le -\zeta_-(t,0)^T\Big(B - S - M\mu e^\delta \mathcal{C} - C_{10}\|\gamma\|_\infty I_m\Big)\zeta_-(t,0) \\ &\quad + \int_0^1 e^{-\delta x}\zeta_+(t,x)^T\zeta_+(t,x)dx + mM\mu\int_0^1 e^{\delta x}\zeta_-(t,x)^T B\zeta_-(t,x)dx \\ &\quad + C_{11}\|\gamma\|_\infty V_2 \end{aligned}$$

with S in (2.25).

On the other hand, by using Lemmas 4.2 and B.3 (see also [9]), there exists δ_3, for $\|\gamma\|_\infty < \delta_3$, such that one has

$$\begin{aligned} XI &\le -\delta\int_0^1 e^{-\delta x}\zeta_+(t,x)^T\zeta_+(t,x)dx - \delta\int_0^1 e^{\delta x}\zeta_-(t,x)^T B\zeta_-(t,x)dx \\ &\quad + C_{12}\|\zeta\|_{L^2}^2(\|\gamma\|_\infty + \|\gamma_x\|_\infty), \end{aligned} \tag{4.53}$$

$$XIII \le C_{13}\|\zeta\|_{L^2}^2\|\zeta\|_\infty, \tag{4.54}$$

$$\begin{aligned} XIV &\le C_{14}\Big(\|\zeta\|_{L^2}^2(\|\gamma\|_\infty + \|\gamma_x\|_\infty) + \|\zeta\|_{L^2}|\zeta(t,0)||\gamma(t,0)|\Big) \\ &\le C_{15}\big(\|\zeta\|_{L^2}^2(\|\gamma\|_\infty + \|\gamma_x\|_\infty) + \|\zeta\|_{L^2}(|\zeta(t,0)|^2 + |\gamma(t,0)|^2)\big) \\ &\le C_{16}\big(\|\zeta\|_{L^2}^2(\|\gamma\|_\infty + \|\gamma_x\|_\infty) + \|\zeta\|_{L^2}(|\zeta_-(t,0)|^2 + |\gamma_-(t,0)|^2)\big). \end{aligned} \tag{4.55}$$

Combining all the calculations (4.52)–(4.55) and noting $\|\zeta\|_{L^2} \le C_{17}\sqrt{V_2}$, we obtain the following.

PROPOSITION 4.3. *For any given $\lambda > 0$, choosing B given by* (4.28)*, there exists $\delta_4 > 0$, such that if $\|\gamma\|_\infty + \|\zeta\|_{L^2} < \delta_4$, one has*

$$\dot V_2(t) \le -\lambda V_2 + C_{18}\Big((\|\gamma\|_\infty + \|\zeta\|_\infty)V_2 + \|\zeta\|_{L^2}|\gamma_-(t,0)|^2\Big) - C_{19}|\zeta_-(t,0)|^2. \tag{4.56}$$

4.4. Analyzing the growth of $\|\gamma_{tt}\|_{L^2}$. We next deal with $\|\gamma_{tt}\|_{L^2}$. Define $\theta = \gamma_{tt}$. Taking a partial derivative with respect to t for (4.31), one obtains an equation of θ:

(4.57)
$$\theta_t + [\Lambda(x) - F_1[\gamma]]\theta_x = G(x)\theta(t,0) + F_7[\gamma,\gamma_x,\zeta,\zeta_x,\theta] + F_8[\gamma,\zeta,\theta] + G''_{bou}[\gamma](t),$$

where

(4.58)
$$F_7 = \mathcal{K}_1[F_{11}[\gamma,\zeta]\zeta] + \int_0^x K(x,\xi)F_{12}[\gamma,\gamma_x]\theta(\xi)d\xi + \mathcal{K}_1[F_1[\gamma]\theta] + \int_0^x K(x,\xi)F_{14}[\gamma,\gamma_x,\zeta,\zeta_x]\zeta(\xi)d\xi + K(x,0)\frac{\partial\Lambda_{NL}}{\partial\gamma}(0,\gamma(0))\zeta(0)\zeta(0) + K(x,0)\Lambda_{NL}(0,\gamma(0))\theta(0) + \mathcal{K}[F_{11}[\gamma,\zeta]\zeta_x] + \mathcal{K}[F_{13}[\gamma,\zeta,\theta]\gamma_x] + F_{11}[\gamma]\zeta_x,$$

(4.59)
$$F_8 = 2\mathcal{K}[F_{11}[\gamma,\zeta]\mathcal{L}_1[\zeta]] + \mathcal{K}[F_1[\gamma]\mathcal{L}_1[\theta]] + \mathcal{K}[F_{13}[\gamma,\zeta,\theta]\mathcal{L}_1[\gamma]] + \mathcal{K}[F_{22}[\gamma,\zeta,\theta]],$$

(4.60)
$$G'_{bou}[\gamma](t) = K(x,0)\Lambda(0)\begin{pmatrix} 0 & 0 \\ \frac{\partial^2 G_{NL}(\gamma_-(t,0))}{\partial\gamma_-^2}\zeta_-(t,0)\zeta_-(t,0)) & 0\end{pmatrix} + K(x,0)\Lambda(0)\begin{pmatrix} 0 & 0 \\ \frac{\partial G_{NL}(\gamma_-(t,0))}{\partial\gamma_-}\zeta_-(t,0)\theta_-(t,0)) & 0\end{pmatrix}$$

with

$$F_{13} = \frac{\partial\Lambda_{NL}^2}{\partial w^2}(x,\mathcal{L}[\gamma])\mathcal{L}[\zeta]\mathcal{L}[\zeta] + \frac{\partial\Lambda_{NL}}{\partial w}(x,\mathcal{L}[\gamma])\mathcal{L}[\theta], \tag{4.61}$$

$$F_{14} = \frac{\partial\Lambda_{NL}^2}{\partial w^2}(x,\mathcal{L}[\gamma])\mathcal{L}[\zeta](\gamma_x + \mathcal{L}_1[\gamma]) + \frac{\partial\Lambda_{NL}}{\partial w}(x,\mathcal{L}[\gamma])(\zeta_x + \mathcal{L}_1[\zeta]) + \frac{\partial^2\Lambda_{NL}}{\partial x\partial w}(x,\mathcal{L}[\gamma])\mathcal{L}[\zeta], \tag{4.62}$$

$$F_{22} = \frac{\partial^2 f_{NL}}{\partial w^2}(x,\mathcal{L}[\gamma])\mathcal{L}[\zeta]\mathcal{L}[\zeta] + \frac{\partial f_{NL}}{\partial w}(x,\mathcal{L}[\gamma])\mathcal{L}[\theta]. \tag{4.63}$$

The boundary conditions of θ are given by

$$x = 0 : \theta_+(t,0) = Q\theta_-(t,0) + \frac{\partial G_{NL}}{\partial\gamma_-}\Big(\gamma_-(t,0)\Big)\theta_-(t,0) + \frac{\partial^2 G_{NL}}{\partial\gamma_-^2}\Big(\gamma_-(t,0)\Big)\zeta_-(t,0)\zeta_-(t,0) \tag{4.64}$$

and

$$x = 1 : \theta_-(t,1) = 0, \tag{4.65}$$

where $\theta_- \in \mathbb{R}^m, \theta_+ \in \mathbb{R}^{n-m}$ are defined by requiring that $\theta := (\theta_-,\theta_+)^T$.

In order to control $\|\theta\|_{L^2}$, we introduce

$$V_3(t) = \int_0^1 \theta^T(t,x)R[\gamma]\theta(t,x)dx, \tag{4.66}$$

then it is easy to see that

$$\dot{V}_3(t) = XVI + XVII + XVIII + XIX + XX \tag{4.67}$$

with

$$XVI = \int_0^1 \theta^T(t,x)(R[\gamma](\Lambda(x) - F_1[\gamma]))_x\theta(t,x)dx,$$

$$XVII = -[\theta^T(t,x)R[\gamma](x)(\Lambda(x) - F_1[\gamma](x))\theta(t,x)]_{x=0}^{x=1},$$
$$XVIII = \int_0^1 \theta^T(t,x)(R[\gamma])_t\theta(t,x)dx,$$
$$XIX = 2\int_0^1 \theta^T(t,x)R[\gamma]F_7[\gamma,\gamma_x,\zeta,\zeta_x,\theta]dx + 2\int_0^1 \theta^T(t,x)R[\gamma]F_8[\gamma,\zeta,\theta]dx,$$
$$XX = 2\int_0^1 \theta^T(t,x)R[\gamma]G(x)\theta(t,0)dx + 2\int_0^1 \theta^T(t,x)R[\gamma]G''_{bou}[\gamma](t)dx.$$

Let us first look at the second and the last terms of (4.67) (i.e., XVII and XX), by some straight computations; noting that $\|\gamma\|_\infty + \|\zeta\|_\infty$ is suitably small, one gets

$$\begin{aligned}
&XVII + XX \\
&\leq -\theta_-(t,0)^T\Big(B - S - M\mu e^\delta\mathcal{C} - C_{20}(\|\zeta\|_\infty + \|\theta\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_\infty)I_m\Big)\theta_-(t,0) \\
&\quad + C_{21}\|\theta\|_{L^2}|\zeta_-(t,0)|^2 + C_{22}|\zeta_-(t,0)|^4 + C_{23}|\theta_-(t,0)||\zeta_-(t,0)|^2 \\
&\quad + \int_0^1 e^{-\delta x}\theta_+(t,x)^T\theta_+(t,x)dx + mM\mu\int_0^1 e^{\delta x}\theta_-(t,x)^TB\theta_-(t,x)dx + C_{24}\|\gamma\|_\infty V_3 \\
&\leq -\theta_-(t,0)^T\Big(B - S - M\mu e^\delta\mathcal{C} - C_{25}(\|\zeta\|_\infty + \|\theta\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_\infty)I_m\Big)\theta_-(t,0) \\
&\quad + C_{26}(\|\theta\|_{L^2} + \|\zeta\|_\infty)|\zeta_-(t,0)|^2 + C_{22}|\zeta_-(t,0)|^4 + \int_0^1 e^{-\delta x}\theta_+(t,x)^T\theta_+(t,x)dx \\
&\quad + mM\mu\int_0^1 e^{\delta x}\theta_-(t,x)^TB\theta_-(t,x)dx + C_{25}\|\gamma\|_\infty V_3.
\end{aligned} \tag{4.68}$$

On the other hand, applying Lemmas B.4–B.8, one has

$$\begin{aligned}
&XIX \\
&\leq C_{26}(1 + \|\gamma\|_\infty + \|\gamma_x\|_\infty)\|\zeta\|_\infty\|\theta\|_{L^2}\|\zeta\|_{L^2} + C_{27}(1 + \|\gamma\|_\infty + \|\gamma_x\|_\infty)\|\theta\|_{L^2}\|\zeta\|_{L^2}^2 \\
&+ C_{28}(\|\zeta\|_{L^\infty}\|\theta\|_{L^2}\|\zeta_x\|_{L^2} + \|\zeta\|_{L^2}^2\|\theta\|_{L^2} + \|\theta\|_{L^2}\|\zeta\|_{L^2}\|\zeta_x\|_{L^2} + \|\theta\|_{L^2}^2(\|\gamma\|_\infty + \|\gamma_x\|_\infty)) \\
&+ C_{29}(\|\gamma\|_{L^2} + \|\gamma\|_\infty)\|\theta\|_{L^2}^2 + C_{30}\|\theta\|_{L^2}(|\zeta(t,0)|^2 + |\gamma(t,0)||\theta(t,0)|) \\
&\leq C_{31}(\|\zeta\|_\infty\|\theta\|_{L^2}\|\zeta\|_{L^2} + \|\theta\|_{L^2}\|\zeta\|_{L^2}^2) + C_{32}\|\theta\|_{L^2}(|\zeta_-(t,0)|^2 + |\gamma_-(t,0)|^2 + |\theta_-(t,0)|^2) \\
&+ C_{33}(\|\zeta\|_{L^\infty}\|\theta\|_{L^2}\|\zeta_x\|_{L^2} + \|\zeta\|_{L^2}^2\|\theta\|_{L^2} + \|\theta\|_{L^2}\|\zeta\|_{L^2}\|\zeta_x\|_{L^2} + \|\theta\|_{L^2}^2(\|\gamma\|_\infty + \|\gamma_x\|_\infty)) \\
&\leq C_{34}(V_1^{\frac{1}{2}}V_3^{\frac{1}{2}}V_2^{\frac{1}{2}} + V_3^{\frac{1}{2}}V_2 + V_2^{\frac{1}{2}}V_3 + V_1V_3^{\frac{1}{2}} + V_3V_1^{\frac{1}{2}} + V_3^{\frac{3}{2}}) \\
&+ C_{35}\|\theta\|_{L^2}(|\zeta_-(t,0)|^2 + |\gamma_-(t,0)|^2 + |\theta_-(t,0)|^2).
\end{aligned} \tag{4.69}$$

Then by similar procedures in section 4.3 for XI, we have

$$\begin{aligned}
XVI \leq &-\delta\int_0^1 e^{-\delta x}\theta_+(t,x)^T\theta_+(t,x)dx - \delta\int_0^1 e^{\delta x}\theta_-(t,x)^TB\theta_-(t,x)dx \\
&+ C_{36}\|\theta\|_{L^2}^2(\|\gamma\|_\infty + \|\gamma_x\|_\infty),
\end{aligned} \tag{4.70}$$

$$XVIII \leq C_{37}\|\theta\|_{L^2}^2\|\zeta\|_\infty \tag{4.71}$$

which, together with (4.68)–(4.69), yields the following.

Proposition 4.4. *For any given $\lambda > 0$, there exists $\delta_5 > 0$, such that*

$$\begin{aligned}\dot{V}_3 \leq & -\lambda V_3 + C_{38}(V_1^{\frac{1}{2}} V_3^{\frac{1}{2}} V_2^{\frac{1}{2}} + V_3^{\frac{1}{2}} V_2 + V_2^{\frac{1}{2}} V_3 + V_1 V_3^{\frac{1}{2}} + V_3 V_1^{\frac{1}{2}} + V_3^{\frac{3}{2}}) \\ & + C_{39}(\|\theta\|_{L^2} + \|\zeta\|_\infty)(|\zeta_-(t,0)|^2 + |\gamma_-(t,0)|^2) \\ & + C_{40}(\|\gamma\|_\infty + \|\zeta\|_\infty + \|\gamma_x\|_\infty) V_3,\end{aligned} \tag{4.72}$$

provided that $\|\gamma\|_\infty + \|\zeta\|_\infty + \|\theta\|_{L^2} \leq \delta_5$.

4.5. Proof of the H^2 stability for γ. In this subsection, we analyze the fast decay of $\|\gamma(t,\cdot)\|_{H^2(0,1)}$ as $t \to +\infty$, which is sufficient to prove Theorem 1.1 because of the equivalence of the H^2 norm between γ and u. Similar proofs can also be found in [6, 7, 13]. Denote $W = V_1 + V_2 + V_3$, by Propositions 4.1, 4.3, and 4.4 as well as Lemma B.8, one can show that for any given $\lambda > 0$, there exists $\delta_6 > 0$ and $C_{41} > 0$, such that

$$\dot{W} \leq -\lambda W + C_{41} W^{\frac{3}{2}}, \tag{4.73}$$

provided that $\|\gamma\|_\infty + \|\zeta\|_\infty + \|\theta\|_{L^2} \leq \delta_6$. Then, for any ν with $0 < \nu < \lambda$, there exists δ_7 such that

$$C_{41} W^{\frac{3}{2}} < (\lambda - \nu) W \quad \forall W < \delta_7, \tag{4.74}$$

which, combining with (4.73), yields

$$\dot{W} \leq -\nu W \leq 0 \ \ \forall W \leq \delta_7, \tag{4.75}$$

i.e.,

$$W(t) \leq e^{-\nu t} W(0) \ \ \forall W \leq \delta_7 \tag{4.76}$$

under the assumption that both $\|\gamma\|_\infty$ and $\|\zeta\|_\infty$ are small enough. Therefore, let $T^* > 0$ be sufficiently large, $\exists \tilde{\delta} > 0$, such that for any given $\gamma_0 \in H^2(0,1;\mathbb{R}^n)$ with $\|\gamma_0\|_{H^2(0,1)} \leq \tilde{\delta}$, which also satisfies the C^1 compatibility conditions at the points $(t,x) = (0,0)$ and $(0,1)$, by noting (4.16)–(4.18) and using Lemma 1.1 (see especially [2, Appendix B]), one finds that there exists $\mathcal{Q} = \mathcal{Q}(\tilde{\delta}) > 0$ with $\lim_{\tilde{\delta} \to 0^+} \mathcal{Q}(\tilde{\delta}) = 0$ such that there exists a unique solution $\gamma = \gamma(t,x)$ in $C^0([0,T^*); H^2((0,1);\mathbb{R}^n))$ satisfying

$$\|\gamma(t,\cdot)\|_{H^2(0,1)} \leq \mathcal{Q}(\tilde{\delta}) \ \forall t \in [0,T^*]. \tag{4.77}$$

Since by the classical Sobolev's inequality (B.33), one has

$$\|\gamma(t,\cdot)\|_\infty \leq C_{42} \|\gamma(t,\cdot)\|_{H^2(0,1)} \ \forall t \in [0,T^*], \tag{4.78}$$

one can choose $\tilde{\delta}$ small enough such that $\|\gamma(t,\cdot)\|_\infty$ is sufficiently small. Then by Lemma B.6 and Sobolev's inequality (B.33) again, one has

$$\|\zeta(t,\cdot)\|_\infty \leq C_{43} \|\gamma(t,\cdot)\|_{H^2(0,1)} \ \forall t \in [0,T^*] \tag{4.79}$$

which implies that Lemmas B.6–B.8 and Proposition B.5 valid at $t = 0$ and that

$$\|\theta(t,\cdot)\|_{L^2} \leq C_{44} \|\gamma_0\|_{H^2(0,1)} \ \forall t \in [0,T^*]$$

which is also small enough if $\|\gamma_0\|_{H^2(0,1)}$ is chosen to be small. On the other hand, noting Proposition B.5 and Lemmas B.6–B.8, it is easy to see that there exists $C_{45} \geq 1$ such that ($\|\gamma\|_\infty + \|\zeta\|_\infty$ should be small enough, which is indeed the case because of (4.78)–(4.79))

$$\frac{1}{C_{45}} W(t) \leq \|\gamma(t,\cdot)\|_{H^2(0,1)} \leq C_{45} W(t). \tag{4.80}$$

Thus, by (4.80), we have

$$W(t) \leq C_{45} \|\gamma(t,\cdot)\|_{H^2(0,1)} \ \forall t \in [0, T^*] \tag{4.81}$$

which implies $W(t) \leq \delta_7$ if we choose $\tilde{\delta}$ small enough. Combining with (4.76) and (4.80), we obtain

$$\|\gamma(t,\cdot)\|_{H^2(0,1)} \leq C_{45}^2 e^{-\nu t} \|\gamma_0\|_{H^2(0,1)} \ \forall t \in [0, T^*]. \tag{4.82}$$

Then, if $T^* > 0$ is chosen such that

$$C_{45}^2 e^{-\nu T^*} \leq 1, \tag{4.83}$$

i.e.,

$$T^* \geq 2 \frac{\ln C_{45}}{\nu}, \tag{4.84}$$

it is easy to repeat the above procedures on the time interval $[T^* 2T^*]$, $[2T^* 3T^*], \ldots$, which yields a unique global solution of γ in $C^0([0,+\infty), H^2((0,1);\mathbb{R}^n))$. In addition, noticing that (4.77) holds, one has

$$\|\gamma(t,\cdot)\|_{H^2(0,1)} \leq \mathcal{Q}(\tilde{\delta}) \ \forall t \in [0, +\infty). \tag{4.85}$$

Then, applying the above arguments (4.77)–(4.82) (just change the time interval $[0, T^*]$ to $[0, +\infty)$) a second time, we get

$$\|\gamma(t,\cdot)\|_{H^2(0,1)} \leq C_{45}^2 e^{-\nu t} \|\gamma_0\|_{H^2(0,1)} \ \forall t \in [0, +\infty). \tag{4.86}$$

Since the H^2 norms of u and γ are also equivalent (see the transformation (2.40) and its inverse (2.53)), we immediately obtain the conclusion of Theorem 1.1. ∎

Appendix A. In this section, we will show the well-posedness and piecewise smoothness of the kernels K and L. Let

$$\rho_i^s(x) := \phi_s^{-1}(\phi_i(x)) \ \forall 1 \leq i \leq s \leq m, \tag{A.1}$$

$$\rho_i^{m+1}(x) := 0 \ \forall i = 1, \ldots, m \tag{A.2}$$

with ϕ_i $(1 \leq i \leq n)$ defined in (2.29). It is easy to see that

$$0 = \rho_i^{m+1}(x) < \rho_i^m(x) < \cdots < \rho_i^i(x) = x, i = 1, \ldots, m, \tag{A.3}$$

and

$$\frac{d\rho_i^s(x)}{dx} = \frac{\lambda_s(\rho_i^s(x))}{\lambda_i(x)},\ x \in [0,1],\ \forall 1 \le i \le s \le m, \tag{A.4}$$

provided that $\Lambda \in C^1[0,1]$.

Define also

$$\mathcal{T}_i^s = \{(x,\xi)|0 \le x \le 1, \rho_i^{s+1}(x) \le \xi \le \rho_i^s(x)\}\ \forall 1 \le i \le s \le m; \tag{A.5}$$

then $\mathcal{T} = \bigcup_{s=i}^m \mathcal{T}_i^s$ ($\forall 1 \le i \le m$) and $meas(\mathcal{T}_i^s \bigcap \mathcal{T}_i^k) = 0$ ($\forall 1 \le i \le s, k \le m, s \ne k$). Then one has the following theorems.

THEOREM A.1. *Let $N \in \mathbb{N}^+$. Under the assumption that $\sigma_{ij} \in C^N[0,1]$, $\lambda_i \in C^N[0,1]$ $(i,j = 1,\ldots,n)$, there exists a unique piecewise $C^N(\mathcal{T})$ solution K to the hyperbolic system* (2.45)–(2.49) *with $K(x,0) \in C^N[0,1]$. Moreover,*

(1) *for the case $1 \le i \le m$, $1 \le j \le n$, suppose that the C^N compatibility conditions at the point $(x,\xi) = (1,1)$ are satisfied, then K_{ij} are C^N functions on each $\mathcal{T}_i^s$ ($\forall 1 \le i \le s \le m$) and satisfy the continuous conditions on the curves $\xi = \rho_i^s(x)\,(s \ne j)$,*

$$K_{ij}(x,\rho_i^s(x)+0) = K_{ij}(x,\rho_i^s(x)-0), \forall 1 \le i < s \le m, 1 \le j \le n, \text{and } j \ne s; \tag{A.6}$$

(2) *for the case $m+1 \le i \le n$, $1 \le j \le n$, K_{ij} are $C^N(\mathcal{T})$ functions, provided that the C^N compatibility conditions at the points $(x,\xi) = (1,1)$ and $(0,0)$ are satisfied, respectively.*

Proof. We divide the proof into two parts. We first prove (2). For this, we only prove the case $N = 1$. For $N \ge 1$, the results can be obtained by induction. On the case $N = 1$, one can, in fact, refer to [15] and Remark A.2 to find unique $C^0(\mathcal{T})$ kernels $K_{ij}\,(i = m+1,\ldots,n, j = 1,\ldots,n)$ for the boundary problem (2.45), (2.46), (2.48) and (2.49) with $m+1 \le i \le n, 1 \le j \le n$, provided that the C^0 compatibility conditions are satisfied at the the points $(x,\xi) = (1,1)$ (see, in particular, (2.50) with $m+1 \le i < j \le n$) and $(0,0)$, respectively. Though only constant coupling coefficients and transport velocities are considered, the method in [15] straightforwardly extends to spatially varying coefficients with more involved technical developments.

Next, we will improve the regulality of $K_{ij}\,(m+1 \le i \le n, 1 \le j \le n)$. Let $\mathcal{H}_{ij} = \partial_x K_{ij}(x,\xi)$ and $\mathcal{Y}_{ij} = \partial_\xi K_{ij}(x,\xi)$. By differentiating with respect to x in (2.45), one can show that

$$\begin{aligned} &\lambda_i(x)\partial_x\mathcal{H}_{ij}(x,\xi) + \lambda_j(\xi)\partial_\xi\mathcal{H}_{ij}(x,\xi) \\ &\quad = -\sum_{k=1}^n \left(\sigma_{kj}(\xi) + \delta_{kj}\lambda_j'(\xi)\right)\mathcal{H}_{ik}(x,\xi) - \lambda_i'(x)\mathcal{H}_{ij}(x,\xi). \end{aligned} \tag{A.7}$$

Differentiating the boundary conditions in (2.46), we have

$$\mathcal{H}_{ij}(x,x) + \mathcal{Y}_{ij}(x,x) = k_{ij}'(x) \qquad \text{for } m+1 \le i \le n, 1 \le j \le n. \tag{A.8}$$

Next, differentiating the boundary conditions in (2.48)–(2.49), we have

$$\mathcal{Y}_{ij}(1,\xi) = (k_{ij}^{(1)})'(\xi) \text{ for } \ m+1 \le i < j \le n \tag{A.9}$$

and the boundary conditions for $\mathcal{H}_{ij}$ ($n \geq i \geq j \geq m+1$) on $\xi = 0$:

$$\mathcal{H}_{ij}(x,0) = (k_{ij}^{(2)})'(x) \text{ for } m+1 \leq j \leq i \leq n. \tag{A.10}$$

In view of (2.45), it is easy to see that

$$\lambda_i(x)\mathcal{H}_{ij}(x,x) + \lambda_j(x)\mathcal{Y}_{ij}(x,x) = -\sum_{k=1}^{n} \big(\sigma_{kj}(x) + \delta_{kj}\lambda_j'(x)\big)K_{ik}(x,x), \tag{A.11}$$

$$\lambda_i(1)\mathcal{H}_{ij}(1,\xi) + \lambda_j(\xi)\mathcal{Y}_{ij}(1,\xi) = -\sum_{k=1}^{n} \big(\sigma_{kj}(\xi) + \delta_{kj}\lambda_j'(\xi)\big)K_{ik}(1,\xi). \tag{A.12}$$

Combining (A.8) and (A.11), we have

$$\mathcal{H}_{ij}(x,x) = \frac{\lambda_j(x)k_{ij}'(x) + \sum_{k=1}^{n} \big(\sigma_{kj}(x) + \delta_{kj}\lambda_j'(x)\big)K_{ik}(x,x)}{\lambda_j(x) - \lambda_i(x)} \quad \text{for } m+1 \leq i \leq n, 1 \leq j \leq n(i \neq j). \tag{A.13}$$

Similarly, plugging (A.9) into (A.12), one immediately obtains, for $m+1 \leq i < j \leq n$, we have

$$\mathcal{H}_{ij}(1,\xi) = -\frac{1}{\lambda_i(1)}\left(\sum_{k=1}^{n} \big(\sigma_{kj}(\xi) + \delta_{kj}\lambda_j'(\xi)\big)K_{ik}(1,\xi) + \lambda_j(\xi)(k^{(1)})_{ij}'(\xi)\right), \tag{A.14}$$

which is a $C^0[0,1]$ function. By the theory in [15] (see also Remark A.2), we can prove that there exists a unique $\mathcal{H} \in C^0(\mathcal{T})$ for the boundary value problem (A.7), (A.10) and (A.13)–(A.14), provided that the corresponding C^0 compatibility conditions are satisfied at the points $(x,\xi) = (1,1)$ (see, in particular, (2.51) with $m+1 \leq i < j \leq n$) and $(0,0)$, respectively. Noting (2.45), we know that $\mathcal{Y}$ shares the same regularity as $\mathcal{H}$. $K_{ij}(x,0)$ and $K_{ij}(x,x)\,(i = m+1,\dots,n, j = 1,\dots,n)$ are $C^1[0,1]$ functions. This then finishes the proof of (2).

We are now in the position to prove (1). Similarly, we only prove the case $N = 1$. For $N \geq 1$, the results can be obtained by induction. For the case $1 \leq i \leq m, 1 \leq j \leq n$, the corresponding compatibility conditions on the point $(x,\xi) = (0,0)$ can not be satisfied beforehand (see, in particular, the boundary conditions (2.46) and (2.47) on the point $(x,\xi) = (0,0)$ with $1 \leq i < j \leq m$). Therefore, jumps may happen when, for example, taking the derivatives of space on the transformation (2.40) and using integrations by parts. Here, we point out that this is not an issue thanks to the fact that the possible discontinuity of $K_{ij}(x,\xi)\,(1 \leq i < j \leq m)$ is just along its characteristic curve, respectively. Suppose that the transformation (2.40) with $1 \leq i \leq m$ is given by

$$\gamma_i(t,x) = w_i(t,x) - \sum_{j=1}^{n}\sum_{s=i}^{m} \int_{\rho_i^{s+1}(x)}^{\rho_i^s(x)} K_{ij}^s(x,\xi)w_j(t,\xi)d\xi, \quad i = 1,\dots,m, \tag{A.15}$$

where K_{ij}^s ($s = i,\dots,m$) are assumed to be suitable smooth functions on the domain $\mathcal{T}_i^s = \{(x,\xi) | 0 \leq x \leq 1, \rho_i^{s+1}(x) \leq \xi \leq \rho_i^s(x)\}$. Thus the computations for (2.40) with

$1 \leq i \leq m$ should be

(A.16)

$$\begin{aligned} \partial_t \gamma_i(t,x) &= -\lambda_i(x)\partial_x w_i(t,x) + \sum_{j=1}^{n} \sigma_{ij}(x) w_j(t,x) \\ &\quad - \sum_{j=1}^{n}\sum_{s=i}^{m} \int_{\rho_i^{s+1}(x)}^{\rho_i^s(x)} K_{ij}^s(x,\xi)\left(-\lambda_j(\xi)\partial_\xi w_j(t,\xi) + \sum_{k=1}^{n} \sigma_{jk}(\xi) w_k(t,\xi)\right) d\xi \\ &= -\lambda_i(x)\partial_x w_i(t,x) + \sum_{j=1}^{n} \sigma_{ij}(x) w_j(t,x) \\ &\quad + \sum_{j=1}^{n}\sum_{s=i}^{m} K_{ij}^s(x,\rho_i^s(x))\lambda_j(\rho_i^s(x)) w_j(t,\rho_i^s(x)) \\ &\quad - \sum_{j=1}^{n}\sum_{s=i}^{m} K_{ij}^s(x,\rho_i^{s+1}(x))\lambda_j(\rho_i^{s+1}(x)) w_j(t,\rho_i^{s+1}(x)) \\ &\quad - \sum_{j=1}^{n}\sum_{s=i}^{m} \int_{\rho_i^{s+1}(x)}^{\rho_i^s(x)} \partial_\xi K_{ij}^s(x,\xi)\lambda_j(\xi) w_j(t,\xi) \\ &\quad - \sum_{j=1}^{n}\sum_{s=i}^{m} \int_{\rho_i^{s+1}(x)}^{\rho_i^s(x)} K_{ij}^s(x,\xi)\lambda_j'(\xi) w_j(t,\xi) \\ &\quad - \sum_{j=1}^{n}\sum_{s=i}^{m}\sum_{k=1}^{n} \int_{\rho_i^{s+1}(x)}^{\rho_i^s(x)} K_{ij}^s(x,\xi)\sigma_{jk}(\xi) w_k(t,\xi) d\xi \end{aligned}$$

and

$$\begin{aligned} \partial_x \gamma_i(t,x) &= \partial_x w_i(t,x) - \sum_{j=1}^{n}\sum_{s=i}^{m} \frac{\lambda_s(\rho_i^s(x))}{\lambda_i(x)} K_{ij}^s(x,\rho_i^s(x)) w_j(t,\rho_i^s(x)) \\ &\quad + \sum_{j=1}^{n}\sum_{s=i}^{m-1} \frac{\lambda_{s+1}(\rho_i^{s+1}(x))}{\lambda_i(x)} K_{ij}^s(x,\rho_i^{s+1}(x)) w_j(t,\rho_i^{s+1}(x)) \\ &\quad - \sum_{j=1}^{n}\sum_{s=i}^{m} \int_{\rho_i^{s+1}(x)}^{\rho_i^s(x)} \partial_x K_{ij}^s(x,\xi) w_j(t,\xi) d\xi. \end{aligned} \tag{A.17}$$

Then

$$\partial_t \gamma_i(t,x) + \lambda_i(x)\partial_x \gamma_i(t,x) = \sum_{j=1}^{i-1} g_{ij}(x)\gamma(t,0) \tag{A.18}$$

yields the kernels equations

$$\begin{aligned} &\lambda_i(x)\partial_x K_{ij}^s(x,\xi) + \lambda_j(\xi)\partial_\xi K_{ij}^s(x,\xi) \\ &\quad = -\sum_{k=1}^{n} (\sigma_{kj}(\xi) + \delta_{kj}\lambda_j'(\xi)) K_{ik}^s(x,\xi) \ \forall 1 \leq i \leq s \leq m, 1 \leq j \leq n \end{aligned} \tag{A.19}$$

and the boundary conditions

$$K_{ij}^i(x,x) = \frac{\sigma_{ij}(x)}{\lambda_i(x) - \lambda_j(x)}, \ j \neq i, \tag{A.20}$$

$$K_{ij}^m(x,0) = -\frac{1}{\lambda_j(0)} \sum_{k=1}^{n-m} \lambda_{m+k}(0) K_{i,m+k}^m(x,0) q_{k,j} \ \forall 1 \le i \le j \le m \tag{A.21}$$

with continuous conditions on the curves $\xi = \rho_i^s(x)\,(s \neq j)$

$$K_{ij}^{s-1}(x, \rho_i^s(x)) = K_{ij}^s(x, \rho_i^s(x)) \ \forall 1 \le i < s \le m, 1 \le j \le n, \text{ and } j \neq s. \tag{A.22}$$

The artificial boundary conditions (see (2.48)) for K_{ij}^s are given by

$$K_{ij}^s(1,\xi) = k_{ij}^{(1)}(\xi), \xi \in [\rho_i^{s+1}(1), \rho_i^s(1)], \ s = i, \dots, m, 1 \le j < i \le m \tag{A.23}$$

which satisfy the desired compatibility conditions at the points $s = \rho_i^{i+1}(1), \dots, \rho_i^m(1)$, respectively. To prove the continuity of the kernel equations

$$K_{ij}^s\,(i = 1, \dots, m, s = i, \dots, m, j = 1, \dots, n)$$

on each piece $\mathcal{T}_i^s$, we can classically transform the differential equations (A.19) into the integral equations. Thanks to the continuous conditions (A.6) for K_{ij}^s with $s \neq j$ along $\xi = \rho_i^s(x)$, such integral equations have the same form as the one given in [15, section VI.A]. Therefore, by using the method of successive approximations and the same argument in [15, section VI] (see also Remark A.2), one can obtain the C^0 kernels K_{ij}^s on each $\mathcal{T}_i^s$ which satisfies (A.19)–(A.23).

Next, we will improve the regularity of $K_{ij}^s\,(1 \le i \le s \le m, 1 \le j \le n)$. Let $\mathcal{H}_{ij}^s = \partial_x K_{ij}^s(x,\xi)$ and $\mathcal{Y}_{ij}^s = \partial_\xi K_{ij}^s(x,\xi)$. By differentiating with respect to x in (A.19), one can show that $\forall i = 1, \dots, m, \ s = i, \dots, m, j = 1, \dots, n$,

$$\begin{aligned} &\lambda_i(x)\partial_x \mathcal{H}_{ij}^s(x,\xi) + \lambda_j(\xi)\partial_\xi \mathcal{H}_{ij}^s(x,\xi) \\ &\quad = -\sum_{k=1}^{n} (\sigma_{kj}(\xi) + \delta_{kj}\lambda_j'(\xi))\mathcal{H}_{ik}^s(x,\xi) - \lambda_i'(x)\mathcal{H}_{ij}^s(x,\xi). \end{aligned} \tag{A.24}$$

Differentiating the boundary conditions in (A.20) and (A.21), we have

$$\mathcal{H}_{ij}^i(x,x) + \mathcal{Y}_{ij}^i(x,x) = k_{ij}'(x) \qquad \text{for } 1 \le i \le m, 1 \le j \le n\,(i \neq j), \tag{A.25}$$

$$\mathcal{H}_{ij}^m(x,0) = -\frac{1}{\lambda_j(0)} \sum_{k=1}^{n-m} \lambda_{m+k}(0)\mathcal{H}_{i,m+k}^m(x,0) q_{k,j} \quad \text{for } 1 \le i \le j \le m. \tag{A.26}$$

By noting (A.19), one has

$$\mathcal{H}_{ij}^i(x,x) = \frac{\lambda_j(x)k_{ij}'(x) + \sum_{k=1}^{n} \big(\sigma_{kj}(x) + \delta_{kj}\lambda_j'(x)\big)K_{ik}^i(x,x)}{\lambda_j(x) - \lambda_i(x)} \quad \text{for } 1 \le i \le m, 1 \le j \le n\,(i \neq j). \tag{A.27}$$

Next, differentiating the continuous conditions on $\xi = \rho_i^s(x)\,(s \neq j)$ in (A.6), for $s = i+1, \dots, m$, $m \ge i \ge j \ge 1$ or $s = i+1, \dots, j-1, j+1, \dots, m, m \ge j > i \ge 1$ or $1 \le i \le s \le m$, $m+1 \le j \le n$ (i.e., $s \neq j$), we have

$$\mathcal{H}_{ij}^{s-1}(x,\rho_i^s(x)) + \frac{\lambda_s(\rho_i^s(x))}{\lambda_i(x)}\mathcal{Y}_{ij}^{s-1}(x,\rho_i^s(x)) = \mathcal{H}_{ij}^s(x,\rho_i^s(x)) + \frac{\lambda_s(\rho_i^s(x))}{\lambda_i(x)}\mathcal{Y}_{ij}^s(x,\rho_i^s(x)), \tag{A.28}$$

which, combining with (A.19), yields that

(A.29)
$$\mathcal{H}_{ij}^{s-1}(x,\rho_i^s(x)) = \mathcal{H}_{ij}^{s}(x,\rho_i^s(x)) + \frac{\lambda_s(\rho_i^s(x))\sigma_{sj}(\rho_i^s(x))(K_{is}^{s-1}(x,\rho_i^s(x)) - K_{is}^{s}(x,\rho_i^s(x)))}{\lambda_i(x)(\lambda_j(\rho_i^s(x)) - \lambda_s(\rho_i^s(x)))}.$$

Similarly, plugging (A.23) into (A.19), one immediately obtains, for $s = i+1,\ldots,m$, $1 \le j < i \le m$, we have

$$\text{(A.30)} \quad \mathcal{H}_{ij}^{s}(1,\xi) = -\frac{1}{\lambda_i(1)}\left(\sum_{k=1}^{n}(\sigma_{kj}(\xi) + \delta_{kj}\lambda_j'(\xi))K_{ik}^{s}(1,\xi) - \lambda_j(\xi)\partial_\xi k_{ij}^{(1)}(\xi)\right), \quad \xi \in [\rho_i^{s+1}(1), \rho_i^s(1)].$$

Then by following the same argument in [15, section VI], one can obtain the C^0 kernels $\mathcal{H}_{ij}^s$ $(1 \le i \le s \le m, 1 \le j \le n)$ on each $\mathcal{T}_i^s$ which satisfy (A.24) and the boundary conditions (A.26), (A.27), (A.29), and (A.30). Noting (A.19), we know that $\mathcal{Y}^s$ shares the same regularity as $\mathcal{H}^s$. Once $K_{ij}^s\,(i = 1,\ldots,m, s = i,\ldots,m, j = 1,\ldots,n)$ exist in $C^1(\mathcal{T}_i^s)$, respectively, one can easily see that $K_{ij}(x,0) \equiv K_{ij}^m(x,0)$ and $K_{ij}(x,x) \equiv K_{ij}^i(x,x)\,(1 \le i \le m, 1 \le j \le n)$ are $C^1[0,1]$ functions. These complete the proof of (1). □

Remark A.1. Due to the potential jump of K_{is}^{s-1} and K_{is}^s along $\xi = \rho_i^s(x)$, one has to notice that we also have a discontinuity for $\mathcal{H}_{is}^{s-1}$ and $\mathcal{H}_{is}^s$ along $\xi = \rho_i^s(x)$ even with $s \ne j$, which is different form (A.6).

Remark A.2. It is worth mentioning that in [15], we only prove $K \in L^\infty(\mathcal{T})$ and do not clarify the regularity of the kernel because of brevity purposes. However, since the solutions of the kernel equations can be expressed by a series

$$\text{(A.31)} \qquad K_{ij}(x,\xi) = \sum_{n=0}^{\infty} \Delta K_{ij}^n(x,\xi)\,(m+1 \le i \le n, 1 \le j \le n),$$

and the initialization ΔK_{ij}^0 is continuous thanks to the C^0 compatibility conditions on the points $(x,\xi) = (1,1)$ and $(0,0)$, with almost the same procedure in [9, section A.3] and [15], one can prove that $\Delta K_{ij}^n(x,\xi)\,(n \ge 0)$ is continuous (since it is an integral of ΔK_{n-1}, which can be assumed to be continuous by induction) and

$$\text{(A.32)} \qquad \Delta K_{ij}^n(x,\xi) \le \bar{\phi}\frac{M^n(\phi_i(x) - (1-\epsilon)\phi_i(\xi))^n}{n!} \quad (n \ge 0)$$

in which $\bar{\phi}$, M, and $0 < \epsilon < 1$ are some positive constants (see [15] for the case with constant matrix Λ), which yields that there exists unique $C^0(\mathcal{T})$ solutions $K_{ij}\,(i = m+1,\ldots,n, j = 1,\ldots,n)$ satisfying the boundary problem (2.45), (2.46), (2.48), and (2.49), provided that $\sigma_{ij} \in C^0[0,1]$, $\lambda_i \in C^1[0,1]$ with $m+1 \le i \le n, 1 \le j \le n$, and the C^0 compatibility conditions (2.50) are satisfied at the the points $(x,\xi) = (1,1)$ and $(0,0)$, respectively.

Considering the regularity of inverse kernels, we have the following theorem.

Theorem A.2. *Under the assumptions of Theorem* A.1*, for any $N \in \mathbb{N}^+$, there exists a unique piecewise $C^N(\mathcal{T})$ kernel L with finitely many discontinuities. Moreover, all the possible discontinuous curves have the similar form $\xi = \Omega(x)$ in which $\Omega(\cdot) \in C^N[0,1]$ is a monotonically increasing function with $\Omega(0) = 0$ and $0 < \Omega(x) < x\,(\forall x \in (0,1])$.*

Proof. We only prove the case $N = 1$. Other cases can be easily proved by induction. Since the inverse kernels L satisfy the solution of the following Volterra equations

$$L(x,\xi) = K(x,\xi) + \int_{\xi}^{x} K(x,s)L(s,\xi)ds, \tag{A.33}$$

we next prove that $L(x,\xi)$ is a piecewise continuous function which has the same potential discontinuities as the kernel K on $\mathcal{T}$ (see Appendix A.1). In fact, $L(x,\xi)$ can be expressed by a series

$$L(x,\xi) = \sum_{n=0}^{\infty} \Delta L_n(x,\xi) \tag{A.34}$$

in which

$$\Delta L_0 = K(x,\xi) \tag{A.35}$$

is a piecewise continuous matrix function and ΔL_n $(n \geq 1)$ satisfy the following iteration,

$$\Delta L_n(x,\xi) = \int_{\xi}^{x} K(x,s)\Delta L_{n-1}(s,\xi)ds. \tag{A.36}$$

Due to (A.6), the only possible discontinuity of the kernel K_{ij} with $1 \leq i < j \leq m$ is along the C^1 monotonically increasing curve $\xi = \rho_i^j(x)$. Then it is easy to see that $\Delta L_1(x,\xi) = \int_{\xi}^{x} K(x,s)K(s,\xi)ds$ is a continuous matrix function on $\mathcal{T}$. By induction, one immediately obtains that ΔL_n $(n \geq 1)$ are C^0 functions on $\mathcal{T}$ and

$$|\Delta L_n(x,\xi)| \leq \frac{\|K\|_{\infty}^{n+1}(x-\xi)^n}{n!}, \quad n \geq 0. \tag{A.37}$$

Therefore, $L(x,\xi) = \sum_{n=0}^{\infty} \Delta L_n(x,\xi)$ is uniformly convergent on $\mathcal{T}$, which is piecewise continuous (since $\Delta L_0 = K$ is a piecewise continuous function). The potential discontinuities of L are the same as the one which appeared on K (see, in particular, Theorem A.1).

Once L is determined on $\mathcal{T}$, one can, according to (A.33) and

$$L(x,\xi) = K(x,\xi) + \int_{\xi}^{x} L(x,s)K(s,\xi)ds, \tag{A.38}$$

calculate L_x and L_ξ , respectively, which are piecewise continuous functions on $\mathcal{T}$ with finitely many discontinuities. Moreover, all the possible discontinuous curves have similar form $\xi = \Omega(x)$ in which $\Omega(\cdot) \in C^N[0,1]$ is a monotonically increasing function with $\Omega(0) = 0$ and $0 < \Omega(x) < x$ $(\forall x \in (0,1])$. ∎

Appendix B. In this appendix, we first sketch out four useful lemmas (the details can be found in [9]), but here we also have to take into account the jump terms due to the piecewise discontinuity of the kernels. Letting c_i $(i = 1, 2\ldots)$ denote positive constants, then one has the following.

LEMMA B.1.

$$|\mathcal{K}[\gamma]| + |\mathcal{L}[\gamma]| \leq c_1(|\gamma| + \|\gamma\|_{L^1}), \tag{B.1}$$

$$|\mathcal{K}_1[\gamma]| + |\mathcal{K}_2[\gamma]| + |\mathcal{L}_1[\gamma]| \leq c_2(|\gamma| + \|\gamma\|_{L^1} + |\gamma(\Omega(x))|). \tag{B.2}$$

Proof. (B.1) can be found in [9], and (B.2) can be easily obtained by (4.8)–(4.10). ∎

LEMMA B.2. *Suppose $\|\gamma\|_\infty$ is suitably small, so that one can see that*

$$|F_1| \le c_3(|\gamma| + \|\gamma\|_{L^1}), \tag{B.3}$$

$$|F_2| \le c_4(|\gamma|^2 + \|\gamma\|_{L^1}^2), \tag{B.4}$$

$$|F_3| \le c_5(|\gamma| + \|\gamma\|_{L^2})(\|\gamma_x\|_{L^2} + |\gamma_x|), \tag{B.5}$$

$$|F_4| \le c_6(|\gamma|^2 + \|\gamma\|_{L^2}^2 + |\gamma(\Omega(x))|^2). \tag{B.6}$$

Proof. Equations (B.3)–(B.5) can be found in [9], thus we only prove (B.6). By noting Lemma B.1, (B.3)–(B.4), one has

$$\begin{aligned} |F_4| \le c_7\Bigg(&(|\gamma| + \|\gamma\|_{L^1})(|\gamma| + \|\gamma\|_{L^1} + |\gamma(\Omega(x))|) + |\gamma| + \|\gamma\|_{L^1} \\ &+ \Big\|(|\gamma| + \|\gamma\|_{L^1})(|\gamma| + \|\gamma\|_{L^1} + |\gamma(\Omega(x))|) + |\gamma| + \|\gamma\|_{L^1}\Big\|_{L^1}\Bigg), \end{aligned} \tag{B.7}$$

since Ω is a strictly increasing function on $C^2[0,1]$ with $\Omega(0) = 0$ and $0 < \Omega(x) \le x\,(\forall 0 < x \le 1)$; then we can see that

$$\|\gamma(\Omega)\|_{L^2} \le c_8\|\gamma\|_{L^2}, \tag{B.8}$$

together with (B.7) and the Hölder inequality, immediately yields (B.6). ∎

The next two lemmas follow from Lemmas B.1–B.2, (B.8), and straightforward computations.

LEMMA B.3. *Suppose $\|\gamma\|_\infty$ is suitably small, so that one can see that*

$$|F_{11}| \le c_9(|\zeta| + \|\zeta\|_{L^1}), \tag{B.9}$$

$$|F_{12}| \le c_{10}(|\gamma_x| + |\gamma| + \|\gamma\|_{L^1} + |\gamma(\Omega(x))|), \tag{B.10}$$

$$|F_{21}| \le c_{11}(|\gamma| + \|\gamma\|_{L^1})(|\zeta| + \|\zeta\|_{L^1}), \tag{B.11}$$

$$\begin{aligned} |F_5| \le c_{12}\Big(&|\zeta| + \|\zeta\|_{L^2})(|\gamma| + \|\gamma\|_{L^2}) + (|\zeta| + \|\zeta\|_{L^2})(|\gamma_x| + \|\gamma_x\|_{L^2}) \\ &+ |\gamma(0)||\zeta(0)| + |\gamma(\Omega(x))||\zeta(\Omega(x))|\Big), \end{aligned} \tag{B.12}$$

$$|F_6| \le c_{13}(|\gamma| + \|\gamma\|_{L^2} + |\gamma(\Omega(x))|)(|\zeta| + \|\zeta\|_{L^2} + |\zeta(\Omega(x))|). \tag{B.13}$$

LEMMA B.4. *Suppose $\|\gamma\|_\infty$ is suitably small, so that one can see that*

$$|F_{13}| \le c_{14}(|\zeta|^2 + \|\zeta\|_{L^1}^2 + |\theta| + \|\theta\|_{L^1}), \tag{B.14}$$

$$\begin{aligned} |F_{14}| \le c_{15}\Big(&(|\zeta| + \|\zeta\|_{L^1})(1 + |\gamma_x| + |\gamma| + \|\gamma\|_{L^1} + |\gamma(\Omega(x))|) \\ &+ |\zeta| + |\zeta_x| + \|\zeta\|_{L^1} + |\zeta(\Omega(x))|\Big), \end{aligned} \tag{B.15}$$

$$|F_{22}| \le c_{16}\Big((|\gamma| + \|\gamma\|_{L^1})(|\theta| + \|\theta\|_{L^1}) + |\zeta|^2 + \|\zeta\|_{L^1}^2\Big), \tag{B.16}$$

$$|F_7| \le c_{17}(|\zeta|^2 + \|\zeta\|_{L^2}^2)(1 + \|\gamma\|_\infty + \|\gamma_x\|_\infty)$$

$$
\begin{aligned}
&+ c_{18}(|\zeta(\Omega(x))|^2 + \|\zeta\|_{L^1}|\zeta(\Omega(x))| + |\gamma(\Omega(x))||\theta(\Omega(x))| + \|\gamma\|_{L^1}|\theta(\Omega(x))|) \\
&+ c_{19}(|\zeta| + \|\zeta\|_{L^2})(|\zeta_x| + \|\zeta_x\|_{L^2} + \|\zeta\|_{L^2}) \\
&+ c_{20}(|\gamma| + \|\gamma\|_{L^2} + \|\gamma_x\|_{\infty})(|\theta| + \|\theta\|_{L^2}) + c_{21}(|\zeta(0)|^2 + |\gamma(0)||\theta(0)|),
\end{aligned} \tag{B.17}
$$

$$
\begin{aligned}
|F_8| \le c_{22}\Big(&(|\zeta|^2 + \|\zeta\|_{L^2}^2)(1 + \|\gamma\|_{\infty}) + (|\gamma| + \|\gamma\|_{L^2})(|\theta| + \|\theta\|_{L^2}) + |\zeta(\Omega(x))|^2 \\
&+ |\gamma||\theta(\Omega(x))| + \|\gamma\|_{L^1}|\theta(\Omega(x))| + |\theta||\gamma(\Omega(x))| + \|\theta\|_{L^1}|\gamma(\Omega(x))|\Big).
\end{aligned} \tag{B.18}
$$

Next, we show the following proposition which is also mentioned in [9], however, here more technical developments are involved.

PROPOSITION B.5. *There exists $\delta > 0$ such that for any $\|\gamma\|_{\infty} + \|\zeta\|_{\infty} \le \delta$, one has*

$$\|\theta\|_{\infty} \le c_{23}(\|\gamma_{xx}\|_{\infty} + \|\gamma_x\|_{\infty} + \|\gamma\|_{\infty}), \tag{B.19}$$

$$\|\theta\|_{L^2} \le c_{24}(\|\gamma_{xx}\|_{L^2} + \|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}), \tag{B.20}$$

$$\|\gamma_{xx}\|_{\infty} \le c_{25}(\|\theta\|_{\infty} + \|\zeta\|_{\infty} + \|\gamma\|_{\infty}), \tag{B.21}$$

$$\|\gamma_{xx}\|_{L^2} \le c_{26}(\|\theta\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}). \tag{B.22}$$

Proof. We prove the next three lemmas to get Proposition B.5.

LEMMA B.6. *There exists δ such that, if $\|\gamma\|_{\infty} \le \delta$, then the following inequalities hold:*

$$\|\zeta\|_{\infty} \le c_{27}(\|\gamma_x\|_{\infty} + \|\gamma\|_{\infty}), \tag{B.23}$$

$$\|\zeta\|_{L^2} \le c_{28}(\|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}), \tag{B.24}$$

$$\|\gamma_x\|_{\infty} \le c_{29}(\|\zeta\|_{\infty} + \|\gamma\|_{\infty}), \tag{B.25}$$

$$\|\gamma_x\|_{L^2} \le c_{30}(\|\zeta\|_{L^2} + \|\gamma\|_{L^2}). \tag{B.26}$$

Proof. Noting (4.16), one can easily see that

$$\zeta(t,x) + \Lambda(x)\gamma_x(t,x) - G(x)\gamma(t,0) = F_3[\gamma,\gamma_x] + F_4[\gamma] + G_{bou}[\gamma](t). \tag{B.27}$$

The difference between our proof and the proof in [9, Lemma B.6] is the appearance of the term $G(x)\gamma(t,0)$ in (B.27). Noting (2.44) and Theorem A.1, we have $G(\cdot) \in C^1[0,1]$. Then since one can show that

$$\|G(\cdot)\gamma(t,0)\|_{L^2} \le c_{31}\|G(\cdot)\gamma(t,0)\|_{\infty} \le c_{32}\|\gamma\|_{\infty} \le c_{33}(\|\gamma_x\|_{L^2} + \|\gamma\|_{L^2}), \tag{B.28}$$

which yields, by the same arguments as in [9, Lemma B.6], (B.23)–(B.25).

On the other hand, by the special structure of $G(x)$, we have

$$\|\partial_x\gamma_1\|_{L^2} \le c_{34}(\|\zeta\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_{\infty} + \|\gamma\|_{L^2}\|\gamma\|_{\infty} + \|\gamma\|_{\infty}^2), \tag{B.29}$$

$$\|\partial_x\gamma_2\|_{L^2} \le c_{35}(\|\zeta\|_{L^2} + \|\gamma_1\|_{\infty} + \|\gamma_x\|_{L^2}\|\gamma\|_{\infty} + \|\gamma\|_{L^2}\|\gamma\|_{\infty} + \|\gamma\|_{\infty}^2), \tag{B.30}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\|\partial_x\gamma_m\|_{L^2} \le c_{m+33}\left(\|\zeta\|_{L^2} + \sum_{r=1}^{m-1}\|\gamma_r\|_{\infty} + \|\gamma_x\|_{L^2}\|\gamma\|_{\infty} + \|\gamma\|_{L^2}\|\gamma\|_{\infty} + \|\gamma\|_{\infty}^2\right), \tag{B.31}$$

(B.32)
$$\|\partial_x\gamma_s\|_{L^2} \le c_{s+33}\left(\|\zeta\|_{L^2} + \sum_{r=1}^{m}\|\gamma_r\|_\infty + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_\infty^2\right),$$

in which $s = m+1, \ldots, n$. Noting the classical Sobolev inequality

$$\|\gamma\|_{L^\infty} \le \widetilde{C}_1\Big(\|\gamma\|_{L^2} + \|\gamma_x\|_{L^2}\Big) \le \widetilde{C}_2\|\gamma\|_{H^1}, \tag{B.33}$$

one gets that

(B.34)
$$\|\partial_x\gamma_1\|_{L^2} \le \widetilde{C}_3(\|\zeta\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty),$$

(B.35)
$$\|\partial_x\gamma_2\|_{L^2} \le \widetilde{C}_4(\|\zeta\|_{L^2} + \|\gamma\|_{L^2} + \|\partial_x\gamma_1\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty),$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

(B.36)
$$\|\partial_x\gamma_m\|_{L^2} \le \widetilde{C}_{m+2}\left(\|\zeta\|_{L^2} + \|\gamma\|_{L^2} + \sum_{r=1}^{m-1}\|\gamma_r\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty\right),$$

(B.37)
$$\|\partial_x\gamma_s\|_{L^2} \le \widetilde{C}_{s+2}\left(\|\zeta\|_{L^2} + \|\gamma\|_{L^2} + \sum_{r=1}^{m}\|\gamma_r\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_{L^\infty} + \|\gamma\|_{L^2}\|\gamma\|_\infty\right),$$

where $s = m+1, \ldots, n$. Then, we can easily obtain by induction that

$$\|\gamma_x\|_{L^2} \le \widetilde{C}(\|\zeta\|_{L^2} + \|\gamma_x\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}\|\gamma\|_\infty + \|\gamma\|_{L^2}), \tag{B.38}$$

which concludes (B.26), under the assumption that $\|\gamma\|_\infty$ is small enough. ∎

Combining the same technical approach as in [9, Lemmas B.7 and B.8] and an analogous argument used in the proof of Lemma B.6 and noting $G \in C^2[0,1]$, the details of which we omit, one can show the next two lemmas.

Lemma B.7. *There exists δ such that, if $\|\gamma\|_\infty \le \delta$, then the following inequalities hold:*

$$\|\gamma_{xx}\|_\infty \le \tilde{c}_1(\|\zeta_x\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty), \tag{B.39}$$
$$\|\gamma_{xx}\|_{L^2} \le \tilde{c}_2(\|\zeta_x\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \tag{B.40}$$
$$\|\zeta_x\|_\infty \le \tilde{c}_3(\|\gamma_{xx}\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty), \tag{B.41}$$
$$\|\zeta_x\|_{L^2} \le \tilde{c}_4(\|\gamma_{xx}\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \tag{B.42}$$

where $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$, and $\tilde{c}_4$ are positive constants.

Lemma B.8. *There exists δ such that, if $\|\gamma\|_\infty + \|\zeta\|_\infty \le \delta$, then the following inequalities hold:*

$$\|\theta\|_\infty \le \tilde{c}_5(\|\zeta_x\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty), \tag{B.43}$$
$$\|\theta\|_{L^2} \le \tilde{c}_6(\|\zeta_x\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \tag{B.44}$$
$$\|\zeta_x\|_\infty \le \tilde{c}_7(\|\theta\|_\infty + \|\zeta\|_\infty + \|\gamma\|_\infty), \tag{B.45}$$
$$\|\zeta_x\|_{L^2} \le \tilde{c}_8(\|\theta\|_{L^2} + \|\zeta\|_{L^2} + \|\gamma\|_{L^2}), \tag{B.46}$$

where $\tilde{c}_5$, $\tilde{c}_6$, $\tilde{c}_7$, and $\tilde{c}_8$ are positive constants.

The above three Lemmas B.6–B.8 immediately yield Proposition B.5. ∎

Acknowledgments. The authors would like to thank Jean-Michel Coron for his encouragement and fruitful discussions, and also thank Guillaume Olive and Frédéric Marbach for interesting remarks.

REFERENCE

[1] G. Bastin and J.-M. Coron, *On boundary feedback stabilization of non-uniform linear* 2×2 *hyperbolic systems over a bounded interval*, Systems Control Lett., 60 (2011), pp. 900–906, https://doi.org/10.1016/j.sysconle.2011.07.008.

[2] G. Bastin and J.-M. Coron, *Stability and Boundary Stabilization of* 1*-D Hyperbolic Systems*, Birkhäuser Progr. Nonlinear Differential Equations Appl. Subser. Control 1, Cham, Switzerland, 2016.

[3] J.-M. Coron, *On the null asymptotic stabilization of the two-dimensional incompressible Euler equations in a simply connected domain*, SIAM J. Control Optim., 37 (1999), pp. 1874–1896, https://doi.org/10.1137/S036301299834140X.

[4] J.-M. Coron, *Control and Nonlinearity*, Math. Surveys Monogr. 126, American Mathematical Society, Providence, RI, 2007.

[5] J.-M. Coron and G. Bastin, *Dissipative boundary conditions for one-dimensional quasi-linear hyperbolic systems: Lyapunov stability for the* C^1*-norm*, SIAM J. Control Optim., 53 (2015), pp. 1464–1483, https://doi.org/10.1137/14097080X.

[6] J.-M. Coron, G. Bastin, and B. d'Andréa-Novel, *Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems*, SIAM J. Control Optim., 47 (2008), pp. 1460–1498, https://doi.org/10.1137/070706847.

[7] J.-M. Coron, B. d'Andréa Novel, and G. Bastin, *A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws*, IEEE Trans. Automat. Control, 52 (2007), pp. 2–11, https://doi.org/10.1109/TAC.2006.887903.

[8] J.-M. Coron and H.-M. Nguyen, *Dissipative boundary conditions for nonlinear 1-D hyperbolic systems: Sharp conditions through an approach via time-delay systems*, SIAM J. Math. Anal., 47 (2015), pp. 2220–2240, https://doi.org/10.1137/140976625.

[9] J.-M. Coron, R. Vazquez, M. Krstic, and G. Bastin, *Local exponential* H^2 *stabilization of a* 2×2 *quasilinear hyperbolic system using backstepping*, SIAM J. Control Optim., 51 (2013), pp. 2005–2035, https://doi.org/10.1137/120875739.

[10] F. Di Meglio, R. Vazquez, and M. Krstic, *Stabilization of a system of* $n+1$ *coupled first-order hyperbolic linear PDEs with a single boundary input*, IEEE Trans. Automat. Control, 58 (2013), pp. 3097–3111, https://doi.org/10.1109/TAC.2013.2274723.

[11] M. Dick, M. Gugat, and G. Leugering, *Feedback stabilization of quasilinear hyperbolic systems with varying delays*, in 2012 17th International Conference on Methods Models in Automation Robotics (MMAR), IEEE, Piscataway, NJ, 2012, pp. 125–130, https://doi.org/10.1109/MMAR.2012.6347931.

[12] J. M. Greenberg and T. Li, *The effect of boundary damping for the quasilinear wave equation*, J. Differential Equations, 52 (1984), pp. 66–75, https://doi.org/10.1016/0022-0396(84)90135-9.

[13] M. Gugat, G. Leugering, and K. Wang, *Neumann boundary feedback stabilization for a nonlinear wave equation: A strict* H^2*-Lyapunov function*, Math. Control Relat. Fields, 7 (2017), pp. 419–448, https://doi.org/10.3934/mcrf.2017015.

[14] L. Hu and F. Di Meglio, *Finite-time backstepping stabilization of* 3×3 *hyperbolic systems*, in IEEE European Control Conference, IEEE, Piscataway, NJ, 2015, pp. 67–72.

[15] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstic, *Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs*, IEEE Trans. Automat. Control, 61 (2016), pp. 3301–3314, https://doi.org/10.1109/TAC.2015.2512847.

[16] M. Krstic and A. Smyshlyaev, *Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays*, Systems Control Lett., 57 (2008), pp. 750–758, https://doi.org/10.1016/j.sysconle.2008.02.005.

[17] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs*, Adv. Des. Control 16, SIAM, Philadelphia, 2008.

[18] T.-T. Li, *Global Classical Solutions for Quasilinear Hyperbolic Systems*, Appl. Rech. Math. 32, Masson, Paris, 1994.

[19] T.-T. Li and B.-P Rao, *Exact boundary controllability for quasi-linear hyperbolic systems*, SIAM J. Control Optim., 41 (2003), pp. 1748–1755, https://doi.org/10.1137/S0363012901390099.

[20] T. H. Qin, *Global smooth solutions of dissipative boundary value problems for first order quasilinear hyperbolic systems*, Chinese Ann. Math. Ser. B, 6 (1985), pp. 289–298.

[21] A. Smyshlyaev and M. Krstic, *Adaptive Control of Parabolic PDEs*, Princeton University Press, Princeton, NJ, 2010, https://doi.org/10.1515/9781400835362.

[22] R. Vazquez and M. Krstic, *Control of* 1*-D parabolic PDEs with Volterra nonlinearities, Part* I*. Design*, Automatica J. IFAC, 44 (2008), pp. 2778–2790, https://doi.org/10.1016/j.automatica.2008.04.013.

[23] R. Vazquez and M. Krstic, *Control of* 1*D parabolic PDEs with Volterra nonlinearities, Part* II*. Analysis*, Automatica J. IFAC, 44 (2008), pp. 2791–2803, https://doi.org/10.1016/j.automatica.2008.04.007.

[24] R. Vázquez, E. Trélat, and J.-M. Coron, *Control for fast and stable laminar-to-high-Reynolds-numbers transfer in a* 2*D Navier-Stokes channel flow*, Discrete Contin. Dyn. Syst. Ser. B, 10 (2008), pp. 925–956, https://doi.org/10.3934/dcdsb.2008.10.925.