Marcum Q-functions and explicit kernels for stabilization of 2 × 2 linear hyperbolic systems with constant coefficients

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ABSTRACT

We find the exact analytical solution to a Goursat PDE system governing the kernels of a backstepping-based boundary control law that stabilizes a constant-coefficient 2 × 2 system of first-order hyperbolic linear PDEs. The solution to the Goursat system is related to the solution of a simpler, explicitly solvable Goursat system through a suitable infinite series of powers of partial derivatives which is summed explicitly in terms of special functions, including Bessel functions and the generalized Marcum Q-functions of the first order. The Marcum functions are common in certain applications in communications but have not appeared previously in control design problems. The dependence of the explicit solutions with respect to system parameters is analyzed through several examples, including the stabilization of a constant equilibrium for a quasi-linear plant.

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1. Introduction

The class of 2 × 2 systems of first-order hyperbolic linear PDEs has attracted considerable attention due to the many examples of physical systems that can be modeled by the class, such as open channels [1–4], transmission lines [5], gas flow pipelines [6] or road traffic models [7]. Given the range of applications, many techniques for stabilization of these systems have been proposed in the literature. Such methods include the use of the explicit evolution of the Riemann invariants along the characteristics [8] and the use of control Lyapunov functions [9] (which are also extensible to $n \times n$ systems [10]). Other approaches include [11–13] (which use a Lyapunov method), [14,15,1] (using a Riemann invariants approach), and [16].

Recently, a new design approach based on the backstepping method [17,18] has been developed [19] for 2 × 2 hyperbolic linear systems. The method allows the design of full-state boundary control laws, boundary observers and output-feedback control laws, which guarantee L^2 stability of the closed-loop system and convergence of the state estimates. The results have been extended to include the quasi-linear case [20] (making the closed-loop system locally exponentially stable in the H^2 sense), a disturbance rejection problem [21], and the case of an underactuated hyperbolic

system consisting of n rightward-convecting equations coupled with one leftward-convecting equation [22].

In this paper we derive exact analytical expressions for the problem of boundary stabilization for constant-coefficient 2 × 2 system of first-order hyperbolic linear PDEs, with actuation at only one of the boundaries.

Having explicit stabilizing feedback-laws for infinite-dimensional systems is rare; it allows a better understanding of the structure of the control law and its dependence with respect to the different parameters of the system. Also, having an analytical expression makes implementation simpler and more precise. Most importantly, when the PDE plant parameters are unknown, the only way to design implementable adaptive controllers [23–28] is when the control gains are available as explicit functions of plant parameters. However, explicit laws are seldom found in the literature, even for 1-D constant-coefficient systems, except in the simplest of cases. The ability to produce explicit control laws for many non-trivial systems has been the distinguishing quality of the backstepping approach (see [17] for examples). However, explicit controllers have heretofore not been available for first-order hyperbolic linear 2 × 2 systems. For these systems, the class of constant-coefficient parameters gives a very wide range of plants, with 7 distinct parameters that can have any value, with only the speeds of propagation being restricted to be positive. This class of hyperbolic systems contains linearizations of quasi-linear hyperbolic system around constant equilibria.

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The control kernels obtained in this paper are given in terms of modified Bessel functions of the first kind (frequently seen in explicit controllers designed by using backstepping), and also in terms of the generalized Marcum Q-function of first order. This function was developed by Marcum for radar analysis [29] and arises in performance analysis of partially coherent, differentially coherent, and noncoherent communications [30,31] and also in statistics [32], but to the best of the authors' knowledge it is the first time that it appears in control theory outside specific applications in communications.

To derive the explicit solutions, we start from the backstepping design of [19]. The kernels used in the feedback law are found by solving a well-posed 2×2 system of first-order hyperbolic linear PDEs in a triangular domain (known as the kernel equations). When the plant model has constant coefficients, the resulting kernel equations have a very specific structure which can be exploited to obtain an explicit solution in terms of special functions. The procedure to find analytical solutions is as follows. First, we apply scaling to reduce the number of constant parameters in the kernel equations from seven to two. The resulting reduced equation is transformed (by using a series approach) to an infinite set of equations which are simpler and can be explicitly solved recursively. Finding the sum of the series solution and reverting the scaling transformations, we finally obtain explicit expressions for the kernels. The appearance of the generalized Marcum Q-function, which did not show up previously in explicit expressions of backstepping controllers, is mainly due to a coupling of the two PDEs in the 2×2 hyperbolic plant.

The paper is organized as follows. In Section 2 we state our main result—the exact analytical expressions for these feedback laws. The proof of this result is detailed in Section 3. Next we present some examples in Section 4. We finish in Section 5 with some concluding remarks. We also include an Appendix with some technical lemmas.

2. Stabilization of constant coefficient 2 × 2 linear hyperbolic systems

2.1. Main result

Consider the following system

$$u_t = -\epsilon_1 u_x + c_1 u + c_2 v, \tag{1}$$

$$v_t = \epsilon_2 v_x + c_3 u + c_4 v, \tag{2}$$

evolving in $x \in [0, 1]$, $t > 0$, with boundary conditions

$$u(0, t) = qv(0, t), \tag{3}$$

$$v(1, t) = U(t), \tag{4}$$

where $U(t)$ is the actuation variable. The initial conditions, denoted as u_0 and v_0, are assumed to belong to $L^2([0, 1])$.

In (1)–(2), ϵ_1, ϵ_2 are assumed to be positive-valued constants. There are no further assumptions about the constant coefficients q, c_1, c_2, c_3 and c_4. When the coefficients of (1)–(4) are such that the system is open-loop unstable, it is necessary to design a feedback law for $U(t)$ that results in a stable closed-loop system.

System (1)–(4) is the most general possible *heterodirectional* 2×2 linear hyperbolic system (without including integral or boundary terms). In this context, "heterodirectional" means that the two state variable ($u(x, t)$ and $v(x, t)$) evolve in *opposite* spatial directions (with speeds ϵ_1 and ϵ_2, respectively) as time moves forward. For this reason, the boundary conditions (3)–(4) are at opposite boundaries of the domain.

In Section 1 a number of procedures to design feedback laws for 2×2 linear hyperbolic systems have been reviewed. However, only backstepping is able to deal with (1)–(4) for arbitrary values of the coefficients. Thus, following [19], we apply the backstepping method, which allows to find a stabilizing linear full-state feedback law as follows

$$U(t) = \int_0^1 k_u(\xi) u(\xi, t) d\xi + \int_0^1 k_v(\xi) v(\xi, t) d\xi, \tag{5}$$

where $k_u(\xi)$ and $k_v(\xi)$ are the *control kernels*, which are found by solving an auxiliary set of partial differential equations. We next state the main result of this paper, which gives explicit formulae for $k_u(\xi)$ and $k_v(\xi)$ that stabilize the closed-loop system.

Theorem 1. *Consider the system* (1)–(4) *with initial conditions u_0 and v_0 and control law* (5), *where the control kernels of $k_u(\xi)$ and $k_v(\xi)$ are explicitly given for $q = 0$ by*

$$k_u(\xi) = c_3 H(\xi) \left(I_0 \left[\frac{2\bar{c}}{\epsilon_1 + \epsilon_2} \mu(\xi) \right] + \eta^2(\xi) I_2 \left[\frac{2\bar{c}}{\epsilon_1 + \epsilon_2} \mu(\xi) \right] \right), \tag{6}$$

$$k_v(\xi) = \bar{c} \left(1 + \frac{\epsilon_2}{\epsilon_1} \right) H(\xi) \frac{\xi}{\mu(\xi)} I_1 \left[\frac{2\bar{c}}{\epsilon_1 + \epsilon_2} \mu(\xi) \right], \tag{7}$$

and for $q \neq 0$ by

$$k_u(\xi) = \frac{\epsilon_2}{q\epsilon_1} H(\xi) \left[F_0(\xi) + F_\Pi(\xi) + \eta(\xi) F_1(\xi) \right], \tag{8}$$

$$k_v(\xi) = H(\xi) \left[F_0(\xi) + F_\Pi(\xi) + \frac{1}{\eta(\xi)} F_1(\xi) \right], \tag{9}$$

where

$$H(\xi) = \frac{-1}{\epsilon_1 + \epsilon_2} \exp \left[\frac{(c_1 - c_4)(1 - \xi)}{\epsilon_1 + \epsilon_2} \right], \tag{10}$$

$$\eta(\xi) = \sqrt{\frac{1 - \xi}{1 + \frac{\epsilon_2}{\epsilon_1} \xi}}, \tag{11}$$

$$\mu(\xi) = \sqrt{(1 - \xi) \left(1 + \frac{\epsilon_2}{\epsilon_1} \xi \right)}, \tag{12}$$

$$F_0(\xi) = \frac{c_2}{q} I_0 \left[\frac{2\bar{c}}{\epsilon_1 + \epsilon_2} \mu(\xi) \right], \tag{13}$$

$$F_1(\xi) = \bar{c} I_1 \left[\frac{2\bar{c}}{\epsilon_1 + \epsilon_2} \mu(\xi) \right], \tag{14}$$

$$F_\Pi(\xi) = \left(c_3 q \frac{\epsilon_1}{\epsilon_2} - \frac{c_2}{q} \right) \Pi \left[\frac{q\epsilon_1 c_3}{\epsilon_2(\epsilon_1 + \epsilon_2)} (1 - \xi), \right.$$
$$\left. \frac{c_2}{q(\epsilon_1 + \epsilon_2)} \left(1 + \frac{\epsilon_2}{\epsilon_1} \xi \right) \right], \tag{15}$$

$$\Pi(x, y) = e^{x+y} Q_1 \left(\sqrt{2x}, \sqrt{2y} \right), \tag{16}$$

$$\bar{c} = \sqrt{c_2 c_3 \frac{\epsilon_1}{\epsilon_2}}, \tag{17}$$

with I_n denoting the modified Bessel function of the first kind (of order n), and Q_1 the generalized Marcum Q-function of first order [29], which is given by

$$Q_1 \left(\sqrt{2x}, \sqrt{2y} \right) = 1 - ye^{-x} \int_0^1 e^{-sy} I_0 \left(2\sqrt{sxy} \right) ds$$

$$= e^{-(x+y)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^n y^{n+m}}{n!(n+m)!}. \tag{18}$$

Then, for any value of the coefficients c_1, c_2, c_3, c_4 and q, and under the assumption that $u_0, v_0 \in L^2([0, 1])$ and $\epsilon_1, \epsilon_2 > 0$, the equilibrium $u \equiv v \equiv 0$ is exponentially stable in the L^2 sense. Moreover, the equilibrium is reached in finite time $t = \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}$.

The proof of the stability part of Theorem 1 is based on [19] and it is given in detail in Section 3.

The formulae (6), (7) for the case $q = 0$ are found by letting $q \to 0$ in (8), (9) and by applying Lemma A.3, thus finding

$$F_\Pi(\xi) = -F_0(\xi) - \eta(\xi)F_1(\xi) + qc_3 \frac{\epsilon_1}{\epsilon_2}\left(I_0\left[\frac{2\bar{c}}{\epsilon_1 + \epsilon_2}\mu(\xi)\right]\right.$$
$$\left. + \eta^2(\xi)I_2\left[\frac{2\bar{c}}{\epsilon_1 + \epsilon_2}\mu(\xi)\right]\right) + O(q^2), \qquad (19)$$

from which (6), (7) follow.

The same kernels of Theorem 1 can be used to locally stabilize (in terms of the H^2 norm) equilibria of *quasi-linear* 2×2 *systems* if the resulting linearization around the equilibrium has constant coefficients (see [20] for additional details). In particular, this situation happens when stabilizing *constant* equilibrium profiles of the nonlinear plant (since linearization results in a constant coefficient system). In Section 4.1 we give a quasi-linear example.

The result stated in Theorem 1 requires full-state knowledge. In [19] a dual method to design an anti-collocated boundary observer is presented (which was extended to the quasi-linear case in [33]). The observer only needs measurements of $u(0, t)$ and uses a copy of the plant driven by output injection to obtain estimates of the state. The design procedure requires solving a set of partial differential equations to obtain the output injection gain kernels. In the constant-coefficient case, it can be shown that the observer kernel equations can be directly transformed into the same set of PDEs that produce the control kernels (8)–(9). Thus, one finds explicit expressions for the observer kernels which are structurally very similar to (8)–(9). Combining this explicit observer with the explicit full-state controller of Theorem 1 one can obtain a stabilizing explicit output-feedback controller which only needs measurements of $u(0, t)$.

2.2. Who was Marcum?

The Marcum Q-function was developed by Jess Marcum to deal with radar communications just after the second world war [29]. In subsequent years the class of special functions that carries his name has been frequently used in the analysis of partially coherent, differentially coherent, and noncoherent communications [30,31] and also in statistics [32]. There are several papers in the literature devoted to the computation of these functions, see for instance [34], and commands for computing the Marcum functions are available in Mathematica and Matlab.

Jess Ira Marcum was a mathematically gifted electrical engineer working at the RAND Corporation in Santa Monica, California, in the late 1940s. He developed the eponymous class of special functions for the purpose of analytically calculating radar backscatter from a steady target in a noisy background. He then spent the 1950s devoting himself to casino gambling, developing the first ever "card-counting" method for blackjack. Marcum used analytical calculations only, a decade before the same methods were reinvented with the help of computer calculations. After successfully and successively applying his method he was first banned from all casinos in Las Vegas and then from all other casino sites in the US and pre-revolution Havana. Around 1970 Marcum returned to employment in research but continued with a parallel career as a consultant to casinos, passing away in 1992 [35].

3. Proof of Theorem 1

To prove Theorem 1 we introduce three steps. First, we follow the backstepping method as outlined in [19], arriving at a control law that requires the solution of a set of kernel PDEs. Next, we simplify and reduce the equations as much as possible arriving at a

set of *reduced equations* that contain a minimal set of parameters. Then, we solve the reduced equations by posing a series solution which allows us to formulate the problem as an infinite chain of PDEs whose solution can be formally stated as derivatives of an initial function (itself the solution of a simpler PDE, explicitly solvable). Finally, we also apply some technical lemmas (contained in the Appendix) to express the result in terms of the generalized Marcum Q-function of first order instead of as a series.

3.1. Backstepping stabilization laws for 2×2 linear hyperbolic systems

Following [19], to apply the backstepping method, system (1)–(4) needs first to be put in the proper (anti-diagonal) form, which requires eliminating the c_1 term in (1) and the c_4 term in (2). For that, define new variables y and z by exponentially scaling u and v as follows

$$y(x, t) = e^{-c_1/\epsilon_1 x}u(x, t), \qquad (20)$$

$$z(x, t) = e^{c_4/\epsilon_2 x}v(x, t). \qquad (21)$$

Computing the spatial derivative of these new variables, we find

$$y_x(x, t) = -\frac{c_1}{\epsilon_1}e^{-c_1/\epsilon_1 x}u(x, t) + e^{-c_1/\epsilon_1 x}u_x(x, t), \qquad (22)$$

$$z_x(x, t) = \frac{c_4}{\epsilon_2}e^{c_4/\epsilon_2 x}v(x, t) + e^{c_4/\epsilon_2 x}v_x(x, t). \qquad (23)$$

Thus, we obtain

$$y_t + \epsilon_1 y_x = e^{-c_1/\epsilon_1 x}(u_t + \epsilon_1 u_x) - c_1 e^{-c_1/\epsilon_1 x}u(x, t)$$
$$= e^{-c_1/\epsilon_1 x}c_2 v = c_2 e^{-\left(\frac{c_1}{\epsilon_1} + \frac{c_4}{\epsilon_2}\right)x}z(x, t), \qquad (24)$$

$$z_t - \epsilon_2 z_x = e^{c_4/\epsilon_2 x}(v_t - \epsilon_2 v_x) - c_4 e^{c_4/\epsilon_2 x}v(x, t)$$
$$= e^{c_4/\epsilon_2 x}c_3 u = c_3 e^{\left(\frac{c_1}{\epsilon_1} + \frac{c_4}{\epsilon_2}\right)x}y(x, t). \qquad (25)$$

Using the boundary conditions (3)–(4) to find boundary conditions for y and z, we finally reach the system expressed in y-z variables

$$y_t = -\epsilon_1 y_x + c_2 e^{-\left(\frac{c_1}{\epsilon_1} + \frac{c_4}{\epsilon_2}\right)x}z(x, t), \qquad (26)$$

$$z_t = \epsilon_2 z_x + c_3 e^{\left(\frac{c_1}{\epsilon_1} + \frac{c_4}{\epsilon_2}\right)x}y(x, t), \qquad (27)$$

$$y(0, t) = qz(0, t), \qquad (28)$$

$$z(1, t) = V(t), \qquad (29)$$

where we have defined

$$V(t) = e^{c_4/\epsilon_2}U(t). \qquad (30)$$

The backstepping method can now be applied to (26)–(29), and following [19], the closed-loop system is exponentially stable if we set

$$V(t) = \int_0^1 K^{vu}(1, \xi)y(\xi, t)d\xi + \int_0^1 K^{vv}(1, \xi)z(\xi, t)d\xi, \qquad (31)$$

where $K^{vu}(x, \xi)$ and $K^{vv}(x, \xi)$ are the solution to the following *kernel equations*

$$\epsilon_2 K_x^{vu}(x, \xi) - \epsilon_1 K_\xi^{vu}(x, \xi) = c_3 e^{\left(\frac{c_1}{\epsilon_1} + \frac{c_4}{\epsilon_2}\right)\xi}K^{vv}(x, \xi), \qquad (32)$$

$$\epsilon_2 K_x^{vv}(x, \xi) + \epsilon_2 K_\xi^{vv}(x, \xi) = c_2 e^{-\left(\frac{c_1}{\epsilon_1} + \frac{c_4}{\epsilon_2}\right)\xi}K^{vu}(x, \xi), \qquad (33)$$

evolving in the triangular domain $\mathcal{T} = \{(x, \xi) : 0 \le \xi \le x \le 1\}$, with boundary conditions

$$K^{vu}(x, x) = -\frac{c_3 e^{\left(\frac{c_1}{\epsilon_1} + \frac{c_4}{\epsilon_2}\right)x}}{\epsilon_1 + \epsilon_2}, \qquad (34)$$

$$K^{vv}(x, 0) = \frac{q\epsilon_1}{\epsilon_2}K^{vu}(x, 0). \qquad (35)$$

The system (32)–(35) is guaranteed by [19] to have a unique solution. Once the solution is computed, we recover the feedback law for $U(t)$ in terms of the original variables u and v just by using (20)–(21) and (30), reaching (5) with

$$k_u(\xi) = K^{vu}(1, \xi)e^{-c_4/\epsilon_2 - c_1/\epsilon_1 \xi}, \tag{36}$$
$$k_v(\xi) = K^{vv}(1, \xi)e^{c_4/\epsilon_2(\xi - 1)}.$$

3.2. Reducing the kernel equations

Next we simplify the system of PDEs (32)–(35). In the first place, looking at the structure of the equations, we find that it is not a constant coefficient system. However it is possible to transform it to a constant coefficient case by defining new kernel variables; as before, this is carried out by using an exponential scaling, as follows

$$G^{vu}(x, \xi) = e^{-\left(\frac{c_1}{\epsilon_1} + \frac{c_4}{\epsilon_2}\right)\left(\frac{\epsilon_1 x + \epsilon_2 \xi}{\epsilon_1 + \epsilon_2}\right)} K^{vu}(x, \xi), \tag{37}$$

$$G^{vv}(x, \xi) = e^{-\left(\frac{c_1}{\epsilon_1} + \frac{c_4}{\epsilon_2}\right)\left(\frac{\epsilon_1 x - \epsilon_1 \xi}{\epsilon_1 + \epsilon_2}\right)} K^{vv}(x, \xi). \tag{38}$$

Using (32)–(35) we arrive at a new set of kernel equations for G^{vu} and G^{vv}

$$\epsilon_2 G_x^{vu}(x, \xi) - \epsilon_1 G_\xi^{vu}(x, \xi) = c_3 G^{vv}(x, \xi), \tag{39}$$

$$\epsilon_2 G_x^{vv}(x, \xi) + \epsilon_2 G_\xi^{vv}(x, \xi) = c_2 G^{vu}(x, \xi), \tag{40}$$

with boundary conditions

$$G^{vu}(x, x) = -\frac{c_3}{\epsilon_1 + \epsilon_2}, \tag{41}$$

$$G^{vv}(x, 0) = \frac{q\epsilon_1}{\epsilon_2} G^{vu}(x, 0). \tag{42}$$

Defining now $c = \frac{\sqrt{c_2 c_3}}{\epsilon_2}$, $\bar{G}^{vu} = \sqrt{\frac{c_2}{c_3}} G^{vu}$, $\epsilon = \frac{\epsilon_1}{\epsilon_2}$, and $\hat{q} = \sqrt{\frac{c_3}{c_2}} \frac{q\epsilon_1}{\epsilon_2}$, we further simplify the equations, arriving at

$$G_x^{vv}(x, \xi) + G_\xi^{vv}(x, \xi) = c\bar{G}^{vu}(x, \xi), \tag{43}$$

$$\bar{G}_x^{vu}(x, \xi) - \epsilon \bar{G}_\xi^{vu}(x, \xi) = cG^{vv}(x, \xi), \tag{44}$$

$$\bar{G}^{vu}(x, x) = -\frac{c}{1 + \epsilon}, \tag{45}$$

$$G^{vv}(x, 0) = \hat{q}\bar{G}^{vu}(x, 0). \tag{46}$$

Finally, define $\hat{x} = cx$, $\hat{\xi} = c\xi$, $\hat{G}^{vu} = -\bar{G}^{vu} \frac{1 + \epsilon}{c}$, $\hat{G}^{vv} = -G^{vv} \frac{1 + \epsilon}{c}$. Then we obtain the following expression

$$\hat{G}_x^{vv}(\hat{x}, \hat{\xi}) + \hat{G}_\xi^{vv}(\hat{x}, \hat{\xi}) = \hat{G}^{vu}(\hat{x}, \hat{\xi}), \tag{47}$$

$$\hat{G}_x^{vu}(\hat{x}, \hat{\xi}) - \epsilon \hat{G}_\xi^{vu}(\hat{x}, \hat{\xi}) = \hat{G}^{vv}(\hat{x}, \hat{\xi}), \tag{48}$$

$$\hat{G}^{vu}(\hat{x}, \hat{x}) = 1, \tag{49}$$

$$\hat{G}^{vv}(\hat{x}, 0) = \hat{q}\hat{G}^{vu}(\hat{x}, 0), \tag{50}$$

which has to be solved in a new triangular domain $\mathcal{T}_c = \{(\hat{x}, \hat{\xi}) : 0 \leq \xi \leq x \leq c\}$, and only depends on two parameters, ϵ and $\hat{q}$. We call this new set of equations the *reduced kernel equations*.

3.3. Solving the reduced kernel equations

To solve (47)–(50), we pose a solution as a series in the parameter $\hat{q}$, in the following way

$$\hat{G}^{vu}(x, \xi) = \sum_{n=0}^{\infty} \hat{q}^n A_n(x, \xi), \qquad \hat{G}^{vv}(x, \xi) = \sum_{n=0}^{\infty} \hat{q}^n B_n(x, \xi). \tag{51}$$

The convergence of this series will be checked when A_n and B_n are found; we assume for the meantime that the series is convergent.

Then, the equations verified by the coefficients A_n and B_n are:

$$(\partial_x - \epsilon \partial_\xi) A_n(x, \xi) = B_n(x, \xi), \tag{52}$$

$$(\partial_x + \partial_\xi) B_n(x, \xi) = A_n(x, \xi), \tag{53}$$

with boundary conditions

$$A_0(x, x) = 1, \tag{54}$$

$$A_n(x, x) = 0, \quad \forall n > 0 \tag{55}$$

$$B_0(x, 0) = 0, \tag{56}$$

$$B_n(x, 0) = A_{n-1}(x, 0), \quad \forall n > 0. \tag{57}$$

We need to solve this system of equation for all values of n. To do so, we decouple the system by finding the equation verified by $B_n(x, \xi)$. For that we need the following Lemma.

Lemma 3.1. *Denote $b_n(x) = B_n(x, x)$. Then*

$$b_n(x, x) = \begin{cases} x, & n = 0, \\ 1, & n = 1, \\ 0, & n > 1. \end{cases} \tag{58}$$

Proof. Using (53) with $\xi = x$ we notice that

$$b_n(x)' = A_n(x, x) = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases} \tag{59}$$

Thus we obtain

$$b_n(x) = \begin{cases} x + B_0(0, 0), & n = 0, \\ B_n(0, 0), & n > 0, \end{cases} \tag{60}$$

and from boundary condition (56) we obtain $b_0(x) = x$ and from boundary condition (57) we obtain $b_n(x) = A_{n-1}(0, 0)$; then, from (55), we finally obtain $b_1(x) = 1$ and, for $n > 1$, $b_n(x) = 0$. □

Combining (52) and (53) and Lemma 3.1 we reach the following set of partial differential equations

$$(\partial_x + \partial_\xi)(\partial_x - \epsilon \partial_\xi) B_n(x, \xi) = B_n(x, \xi), \tag{61}$$

$$B_n(x, x) = \begin{cases} x, & n = 0, \\ 1, & n = 1, \\ 0, & n > 1, \end{cases} \tag{62}$$

$$B_n(x, 0) = \begin{cases} 0, & n = 0 \\ A_{n-1}(x, 0), & n > 0. \end{cases} \tag{63}$$

The plan to solve this set of differential equations is the following.

1. First solve for $B_0(x, \xi)$ (since it is an autonomous equation).
2. Obtain $A_0(x, \xi)$ from (53) by setting $A_0(x, \xi) = (\partial_x + \partial_\xi) B_0(x, \xi)$.
3. Iterate this procedure for $n \geq 1$, first solving for B_n (which requires using the previously computed A_{n-1} to use the value of $A_{n-1}(x, 0)$ for boundary condition (63)) and then obtaining $A_n(x, \xi)$ from (53) by setting $A_n(x, \xi) = (\partial_x + \partial_\xi) B_n(x, \xi)$.

The following result shows that this plan can be carried out explicitly.

Lemma 3.2. *Denote by $\Phi(x, \xi)$ the (smooth) function that verifies:*

$$(\partial_x + \partial_\xi)(\partial_x - \epsilon \partial_\xi)\Phi(x, \xi) = \Phi(x, \xi), \tag{64}$$

$$\Phi(x, x) = x, \tag{65}$$

$$\Phi(x, 0) = 0. \tag{66}$$

Then, defining A_n and B_n as

$$B_n = (\partial_x + \partial_\xi)^n \Phi(x, \xi), \tag{67}$$

$$A_n = (\partial_x + \partial_\xi)^{n+1} \Phi(x, \xi), \tag{68}$$

the set of equation (52)–(57) is verified for all values of $n \geq 0$.

Proof. We show the result by direct substitution (it could also be proved by induction). The function $\Phi(x, \xi)$ is smooth (see Lemma A.1 from the Appendix) so the definitions (67)–(68) make sense. Notice that (53) is automatically verified. Let us see that the

(B_n) set of equations given by (61)–(63) is verified. Obviously (61) is verified due to the fact that Φ verifies (64) and using interchangeability of partial derivatives. It is also easy to see that (62) is verified. Finally, (63) is obviously verified for $n = 0$, but also for $n \geq 0$ due to the fact that $A_{n-1}(x, \xi) = B_n(x, \xi)$. □

Substituting the result of Lemma 3.2 in (51) we get

$$\hat{G}^{vv}(x, \xi) = \sum_{n=0}^{\infty} \hat{q}^n (\partial_x + \partial_\xi)^n \Phi(x, \xi). \tag{69}$$

Noting that from (69) it follows that

$$\hat{G}^{vv} - \hat{q}\left(\hat{G}_x^{vv} + \hat{G}_\xi^{vv}\right) = \Phi \tag{70}$$

from (47) it follows that

$$\hat{G}^{vv} - \hat{q}\hat{G}^{vu} = \Phi \tag{71}$$

which means that $\hat{G}^{vu}$ can also be obtained either from (47) by differentiation or as

$$\hat{G}^{vu} = \frac{1}{\hat{q}}\left(\hat{G}^{vv} - \Phi\right). \tag{72}$$

3.4. Obtaining explicit formulae for the solution of the reduced kernel equations

We use Lemma A.1 from the Appendix to solve the equation for Φ, obtaining

$$\Phi(x, \xi) = (1 + \epsilon)\xi \frac{I_1\left(\frac{2}{1+\epsilon}\sqrt{(x - \xi)(\epsilon x + \xi)}\right)}{\sqrt{(x - \xi)(\epsilon x + \xi)}}. \tag{73}$$

Expanding $\Phi(x, \xi)$ as a power series by using Lemma A.2 for $n = 1$

$$\Phi(x, \xi) = \frac{(1 + \epsilon)\xi}{x - \xi} \sum_{k=0}^{\infty} \left(\frac{1}{1+\epsilon}\right)^{2k+1} \frac{(x - \xi)^{k+1}}{(k+1)!} \frac{(\epsilon x + \xi)^k}{k!}$$

$$= \xi \sum_{k=0}^{\infty} \left(\frac{1}{1+\epsilon}\right)^{2k} \frac{(x - \xi)^k}{(k+1)!} \frac{(\epsilon x + \xi)^k}{k!}, \tag{74}$$

and writing $\xi = \frac{\epsilon x + \xi}{1+\epsilon} - \epsilon \frac{x - \xi}{1+\epsilon}$, we get the following power series expression for Φ

$$\Phi(x, \xi) = \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^k}{(k+1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^{k+1}}{k!} - \epsilon \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^{k+1}}{(k+1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^k}{k!}. \tag{75}$$

To find an explicit formula for (69), we have to compute $(\partial_x + \partial_\xi)^n \Phi(x, \xi)$, which is easier to carry out for Φ expressed as in (75). The following result holds.

Lemma 3.3. *For all $n > 0$*

$$(\partial_x + \partial_\xi)^n \Phi(x, \xi) = \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^{k+n-1}}{(k+n-1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^k}{k!}$$

$$\times \left(1 - \epsilon \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^2}{(k+n)(k+n+1)}\right). \tag{76}$$

Proof. We prove the lemma by induction. First, for $n = 1$, we compute the derivative $(\partial_x + \partial_\xi)\Phi(x, \xi)$ obtaining

$(\partial_x + \partial_\xi)\Phi(x, \xi)$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^k}{k!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^k}{k!} - \epsilon \sum_{k=1}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^{k+1}}{(k+1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^{k-1}}{(k-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^k}{k!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^k}{k!} \left(1 - \epsilon \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^2}{(k+1)(k+2)}\right). \tag{77}$$

Now, assume the result is correct for n. Then, for $n + 1$, we obtain

$(\partial_x + \partial_\xi)^{n+1}\Phi(x, \xi) = (\partial_x + \partial_\xi)\left[(\partial_x + \partial_\xi)^n \Phi(x, \xi)\right]$

$$= \sum_{k=1}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^{k+n-1}}{(k+n-1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^{k-1}}{(k-1)!}$$

$$\times \left(1 - \epsilon \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^2}{(k+n)(k+n+1)}\right)$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^{k+n}}{(k+n)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^k}{k!}$$

$$\times \left(1 - \epsilon \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^2}{(k+n+1)(k+n+2)}\right), \tag{78}$$

where in the last step we have shifted k by one to start the sum at $k = 0$. Thus the result is proven. □

Applying Lemma 3.3 in (69) we get absolute convergence of the series, and if we substitute the derivatives, we get, for $\hat{G}^{vv}$

$$\hat{G}^{vv}(x, \xi) = \Phi(x, \xi) + \sum_{n=1}^{\infty} \hat{q}^n (\partial_x + \partial_\xi)^n \Phi(x, \xi)$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^k}{(k+1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^{k+1}}{k!} - \epsilon \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^{k+1}}{(k+1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^k}{k!}$$

$$+ \sum_{n=1}^{\infty} \hat{q}^n \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^{k+n-1}}{(k+n-1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^k}{k!}$$

$$\times \left(1 - \epsilon \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^2}{(k+n)(k+n+1)}\right)$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^k}{(k+1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^{k+1}}{k!}$$

$$+ \sum_{n=1}^{\infty} \hat{q}^n \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^{k+n-1}}{(k+n-1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^k}{k!}$$

$$- \epsilon \sum_{n=0}^{\infty} \hat{q}^n \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^{k+n+1}}{(k+n+1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^k}{(k+1)!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^{k+1}}{k!} + \frac{\epsilon}{\hat{q}} \sum_{k=0}^{\infty} \frac{\left(\frac{x-\xi}{1+\epsilon}\right)^k}{k!} \frac{\left(\frac{\epsilon x+\xi}{1+\epsilon}\right)^k}{k!}$$

$$+ \left(\hat{q} - \frac{\epsilon}{\hat{q}}\right) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left(\hat{q}\frac{x-\xi}{1+\epsilon}\right)^{k+n}}{(k+n)!} \frac{\left(\frac{\epsilon x+\xi}{\hat{q}(1+\epsilon)}\right)^k}{k!}$$

$$= \sqrt{\frac{\epsilon x + \xi}{x - \xi}} I_1\left(\frac{2}{1+\epsilon}\sqrt{(x - \xi)(\epsilon x + \xi)}\right)$$

$$+ \frac{\epsilon}{\hat{q}} I_0\left(\frac{2}{1+\epsilon}\sqrt{(x - \xi)(\epsilon x + \xi)}\right)$$

$$+ \left(\hat{q} - \frac{\epsilon}{\hat{q}}\right) \Pi\left(\frac{\hat{q}(x - \xi)}{1+\epsilon}, \frac{\epsilon x + \xi}{\hat{q}(1+\epsilon)}\right), \tag{79}$$

where we have used Lemma A.2 to express the power series as modified Bessel functions, and defined the function

$$\Pi(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^n y^{n+m}}{n!(n+m)!}, \tag{80}$$

which can also be written in terms of the generalized Marcum Q-function of first order, as shown in Lemma A.3. Next, using (47) and

differentiating $\hat{G}^{vv}$, or by using (70), we obtain $\hat{G}^{vu}$ as follows

$$\hat{G}^{vu}(x, \xi) = \frac{\epsilon}{\hat{q}^2} I_0 \left(\frac{2}{1+\epsilon} \sqrt{(x-\xi)(\epsilon x + \xi)} \right)$$
$$+ \frac{\epsilon}{\hat{q}} \sqrt{\frac{x-\xi}{\epsilon x + \xi}} I_1 \left(\frac{2}{1+\epsilon} \sqrt{(x-\xi)(\epsilon x + \xi)} \right)$$
$$+ \left(1 - \frac{\epsilon}{\hat{q}^2} \right) \Pi \left(\frac{\hat{q}(x-\xi)}{1+\epsilon}, \frac{\epsilon x + \xi}{\hat{q}(1+\epsilon)} \right). \quad (81)$$

Undoing the various scaling transformations that were made to arrive at the reduced equation, we get the K^{vu} and K^{vv} backstepping kernels in term of the original coefficients, as follows

$$K^{uv}(x, \xi) = \frac{-1}{q(\epsilon_1 + \epsilon_2)} \exp \left[\left(\frac{c_1}{\epsilon_1} + \frac{c_4}{\epsilon_2} \right) \frac{\epsilon_1 x + \epsilon_2 \xi}{\epsilon_1 + \epsilon_2} \right]$$
$$\times \left\{ \frac{c_2 \epsilon_2}{q \epsilon_1} I_0 \left[\frac{2\sqrt{c_2 c_3}}{\epsilon_1 + \epsilon_2} \sqrt{(x-\xi)(\epsilon_1/\epsilon_2 x + \xi)} \right] \right.$$
$$+ \sqrt{c_2 c_3 \frac{x-\xi}{\epsilon_1/\epsilon_2 x + \xi}}$$
$$\times I_1 \left[\frac{2\sqrt{c_2 c_3}}{\epsilon_1 + \epsilon_2} \sqrt{(x-\xi)(\epsilon_1/\epsilon_2 x + \xi)} \right]$$
$$+ \frac{c_3 q^2 \epsilon_1 - c_2 \epsilon_2}{q \epsilon_1}$$
$$\left. \times \Pi \left[\frac{q \epsilon_1 c_3}{\epsilon_2} \frac{x-\xi}{\epsilon_1 + \epsilon_2}, \frac{c_2}{q \epsilon_1} \frac{\epsilon_1 x + \epsilon_2 \xi}{\epsilon_1 + \epsilon_2} \right] \right\}, \quad (82)$$

$$K^{vv}(x, \xi) = \frac{-1}{\epsilon_1 + \epsilon_2} \exp \left[\left(\frac{c_1}{\epsilon_1} + \frac{c_4}{\epsilon_2} \right) \frac{\epsilon_1 x - \epsilon_1 \xi}{\epsilon_1 + \epsilon_2} \right]$$
$$\times \left\{ \frac{c_2}{q} I_0 \left[\frac{2\sqrt{c_2 c_3}}{\epsilon_1 + \epsilon_2} \sqrt{(x-\xi)(\epsilon_1/\epsilon_2 x + \xi)} \right] \right.$$
$$+ \sqrt{c_2 c_3 \frac{\epsilon_1/\epsilon_2 x + \xi}{x-\xi}}$$
$$\times I_1 \left[\frac{2\sqrt{c_2 c_3}}{\epsilon_1 + \epsilon_2} \sqrt{(x-\xi)(\epsilon_1/\epsilon_2 x + \xi)} \right]$$
$$+ \frac{c_3 q^2 \epsilon_1 - c_2 \epsilon_2}{q \epsilon_2}$$
$$\left. \times \Pi \left[\frac{q \epsilon_1 c_3}{\epsilon_2} \frac{x-\xi}{\epsilon_1 + \epsilon_2}, \frac{c_2}{q \epsilon_1} \frac{\epsilon_1 x + \epsilon_2 \xi}{\epsilon_1 + \epsilon_2} \right] \right\}. \quad (83)$$

Finally, applying (36) to compute the control kernels, an defining the intermediate functions (10)–(16) to simplify the resulting expression, we verify (8)–(9). Finally we apply [19] to finally arrive at the result of Theorem 1.

4. Examples

In this section we present several examples to graphically show how the control kernels depend on the different plant coefficients. We also illustrate the application of the explicit control law (5) in a nonlinear plant inspired in a combustion instability model.

4.1. Stabilization of a constant equilibrium profile for a nonlinear plant

Consider the following system with quadratic nonlinearities

$$u_t + u_x = \frac{c}{2}(u - v^2), \quad (84)$$

$$v_t - v_x = \frac{c}{2}(v - u^2), \quad (85)$$

with boundary conditions

$$u(0, t) = v(0, t), \qquad v(1, t) = U(t). \quad (86)$$

This example is inspired by models of thermoacoustic combustion instabilities in elongated combustion chambers with momentum-dependent heat release [36,37].

If $U(t) = 1$, there is an equilibrium at $u \equiv v \equiv 1$, and the objective is to stabilize the system around this equilibrium. Define error variables $\tilde{u}(x, t) = u(x, t) - 1$ and $\tilde{v}(x, t) = v(x, t) - 1$, and $\tilde{U}(t) = U(t) - 1$. The error system is

$$\tilde{u}_t + \tilde{u}_x = \frac{c}{2}((\tilde{u} + 1) - (\tilde{v} + 1)^2), \quad (87)$$

$$\tilde{v}_t - \tilde{v}_x = \frac{c}{2}((\tilde{v} + 1) - (\tilde{u} + 1)^2), \quad (88)$$

with boundary conditions

$$\tilde{u}(0, t) = \tilde{v}(0, t), \qquad \tilde{v}(1, t) = \tilde{U}(t). \quad (89)$$

Linearizing system (87)–(88) around the origin, we obtain

$$\tilde{u}_t + \tilde{u}_x = \frac{c}{2}(\tilde{u} - 2\tilde{v}), \quad (90)$$

$$\tilde{v}_t - \tilde{v}_x = \frac{c}{2}(\tilde{v} - 2\tilde{u}), \quad (91)$$

which is a constant coefficient system; following the notation of Section 2, we get $c_1 = c/2, c_2 = -c, c_3 = -c, c_4 = c/2, \epsilon_1 = \epsilon_2 = 1, q = 1$.

To see that this is a potentially unstable system, take both a time and a space derivative in (90) and subtract the resulting expressions. One obtains

$$\tilde{u}_{tt} - \tilde{u}_{xx} = c\tilde{u}_t + \frac{3c^2}{4}\tilde{u}, \quad (92)$$

which is a wave equation with in-domain antidamping (for positive values of c) and anti-stiffness, which is well-known to yield instability [38]. Numerical simulations of the open-loop nonlinear system (Fig. 2) show the system becoming unstable for large enough c.

The explicit solution for the control kernels is, in this case,

$$k_u(\xi) = \frac{c}{2} \left\{ I_0 \left[c\sqrt{1-\xi^2} \right] - \sqrt{\frac{1-\xi}{1+\xi}} I_1 \left[c\sqrt{1-\xi^2} \right] \right\}, \quad (93)$$

$$k_v(\xi) = \frac{c}{2} \left\{ I_0 \left[c\sqrt{1-\xi^2} \right] - \sqrt{\frac{1+\xi}{1-\xi}} I_1 \left[c\sqrt{1-\xi^2} \right] \right\}. \quad (94)$$

In Fig. 1 we give the values of the control kernels for different values of c. Notice the exponential increase as c grows.

To find the numerical solution of the open-loop and closed-loop system we use the HPDE solver for Matlab [39], which is well-known and has been tested for many types of hyperbolic systems. In the numerical simulations, we use the following initial conditions, which are close to the equilibrium profile:

$$u(x, 0) = 1 + 0.2\frac{x^2 + \sin(6x)}{2}, \quad (95)$$

$$v(x, 0) = 1 - 0.2\frac{x^2 + 2\sin(3x)}{2}. \quad (96)$$

With these initial conditions and the value $c = 6$, the open-loop system is unstable as shown in Fig. 2. The application of the control law makes the origin of the closed-loop system asymptotically stable as shown in Fig. 3.
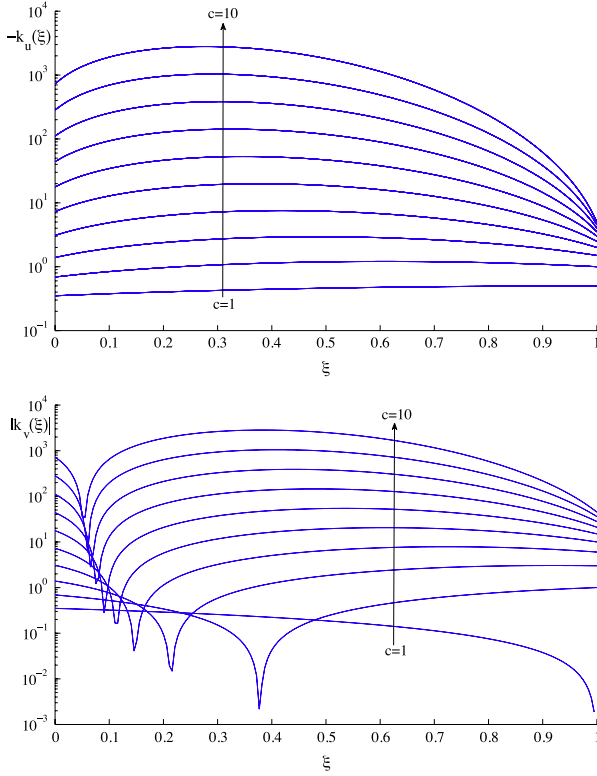
Fig. 1. Control kernels $k_u(\xi)$ and $k_v(\xi)$ for the example of Section 4.1, for different values of c.

4.2. A plant with varying q coefficient

Consider

$$u_t = -u_x + cv, \tag{97}$$
$$v_t = v_x + cv, \tag{98}$$

with boundary conditions

$$u(0, t) = qv(0, t), \tag{99}$$
$$v(1, t) = U(t), \tag{100}$$

Then the control kernels are

$$k_u(\xi) = -\frac{c}{2q}\left\{\frac{1}{q}I_0\left[c\sqrt{1-\xi^2}\right] + \sqrt{\frac{1-\xi}{1+\xi}}I_1\left[c\sqrt{1-\xi^2}\right]\right.$$
$$+ \left(q - \frac{1}{q}\right)\exp\left[\frac{c}{2}\left(q(1-\xi) + \frac{1+\xi}{q}\right)\right]$$
$$\left.\times Q_1\left[\sqrt{cq(1-\xi)}, \sqrt{\frac{c}{q}(1+\xi)}\right]\right\}, \tag{101}$$

$$k_v(\xi) = -\frac{c}{2}\left\{\frac{1}{q}I_0\left[c\sqrt{1-\xi^2}\right] + \sqrt{\frac{1+\xi}{1-\xi}}I_1\left[c\sqrt{(1-\xi^2)s}ds\right]\right.$$
$$+ \left(q - \frac{1}{q}\right)\exp\left[\frac{c}{2}\left(q(1-\xi) + \frac{1+\xi}{q}\right)\right]$$
$$\left.\times Q_1\left[\sqrt{cq(1-\xi)}, \sqrt{\frac{c}{q}(1+\xi)}\right]\right\} \tag{102}$$

or for $q = 0$:

$$k_u(\xi) = -\frac{c}{2}\left(I_0\left[c\sqrt{1-\xi^2}\right] + \frac{1-\xi}{1+\xi}I_2\left[c\sqrt{1-\xi^2}\right]\right), \tag{103}$$

$$k_v(\xi) = \frac{-c\xi}{\sqrt{1-\xi^2}}I_1\left[c\sqrt{1-\xi^2}\right]. \tag{104}$$

In Fig. 4 we represent these kernels for $q \in [0, 5]$ and for fixed $c = 6$. We find that both kernels grow exponentially with q, particularly on the boundary $\xi = 0$.

4.3. A plant with varying propagation speeds

Consider

$$u_t = -\epsilon u_x + cv, \tag{105}$$
$$v_t = v_x + cv, \tag{106}$$

with boundary conditions

$$u(0, t) = v(0, t), \tag{107}$$
$$v(1, t) = U(t). \tag{108}$$

Then the control kernels are

$$k_u(\xi) = \frac{-c}{\epsilon(1+\epsilon)}\left\{I_0\left[\frac{2c}{1+\epsilon}\sqrt{(1-\xi)(\epsilon+\xi)}\right]\right.$$
$$+ \epsilon\sqrt{\frac{1-\xi}{\epsilon+\xi}}I_1\left[\frac{2c}{1+\epsilon}\sqrt{(1-\xi)(\epsilon+\xi)}\right]$$
$$+ (\epsilon - 1)\exp\left[\frac{c}{1+\epsilon}\left(\epsilon(1-\xi) + \frac{\epsilon+\xi}{\epsilon}\right)\right]$$
$$\left.\times Q_1\left[\sqrt{\frac{2c\epsilon(1-\xi)}{1+\epsilon}}, \sqrt{\frac{2c(\epsilon+\xi)}{\epsilon(1+\epsilon)}}\right]\right\}, \tag{109}$$

$$k_v(\xi) = \frac{-c}{1+\epsilon}\left\{I_0\left[\frac{2c}{1+\epsilon}\sqrt{(1-\xi)(\epsilon+\xi)}\right]\right.$$
$$+ \sqrt{\frac{\epsilon+\xi}{1-\xi}}I_1\left[\frac{2c}{1+\epsilon}\sqrt{(1-\xi)(\epsilon+\xi)}\right]$$
$$+ (\epsilon - 1)\exp\left[\frac{c}{1+\epsilon}\left(\epsilon(1-\xi) + \frac{\epsilon+\xi}{\epsilon}\right)\right]$$
$$\left.\times Q_1\left[\sqrt{\frac{2c\epsilon(1-\xi)}{1+\epsilon}}, \sqrt{\frac{2c(\epsilon+\xi)}{\epsilon(1+\epsilon)}}\right]\right\}. \tag{110}$$

In Fig. 5 we represent these kernels for $\epsilon \in [0.2, 10]$ and for fixed $c = 6$. We find that $k_v(\xi)$ grows rapidly with ϵ, particularly on the boundary $\xi = 0$. On the other hand the behavior of $k_u(\xi)$ is non-monotone with respect to ϵ, initially growing on the boundary $\xi = 0$ but then decreasing for ϵ larger than unity, and at the same time growing in the center of the domain.

5. Conclusions

In this work we have derived an explicit control law to solve the problem of boundary stabilization for constant-coefficient 2×2 system of first-order hyperbolic linear PDEs. The control law is found by applying the backstepping method and then solving the resulting control kernel equations, which has required the development of a method that expresses their solution as the sum of solutions of an infinite set of explicitly solvable equations.

The resulting explicit expressions contain not only modified Bessel functions of the first kind (frequently seen in explicit controllers previously found by using backstepping) but also generalized Marcum Q-functions of first order. This it is the first time that this special function appears in control theory outside specific applications in communications. It is expected that this function might appear in other explicit controllers designed by the backstepping method.

To analyze the dependence of the explicit solutions with respect to system parameters, several cases have been analyzed, including the stabilization of a constant equilibrium for a quasi-linear plant. The dependence is found to be mostly of exponential type, and particularly complex with respect to the speeds of propagation.
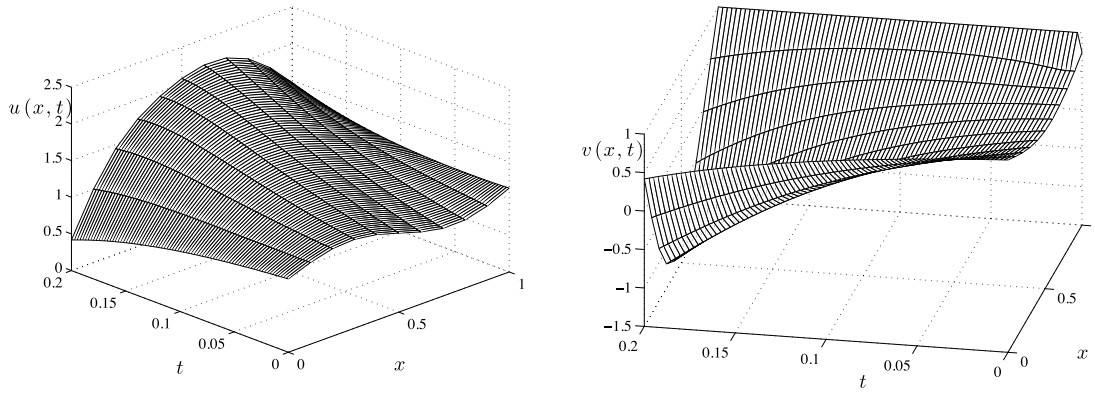
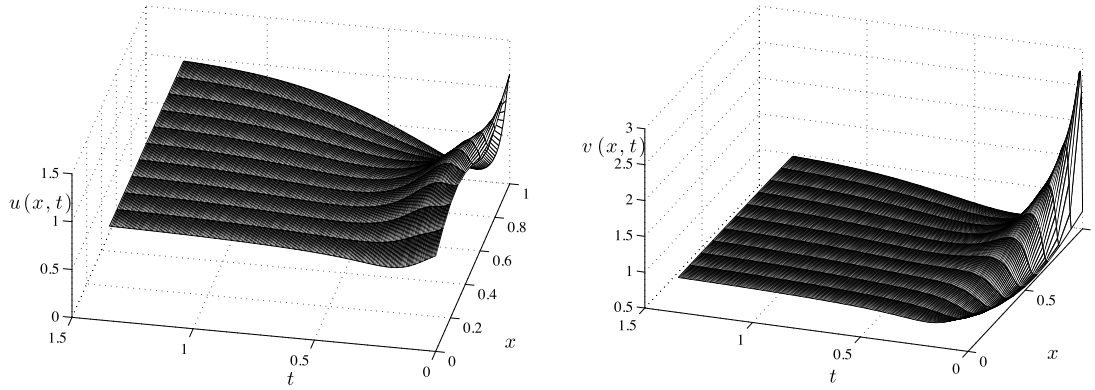**Fig. 2.** Numerical simulation of the open-loop system of the example described in Section 4.1.

Fig. 3. Numerical simulation of the closed-loop system of the example described in Section 4.1, with control input $v(1, t) = U(t)$.
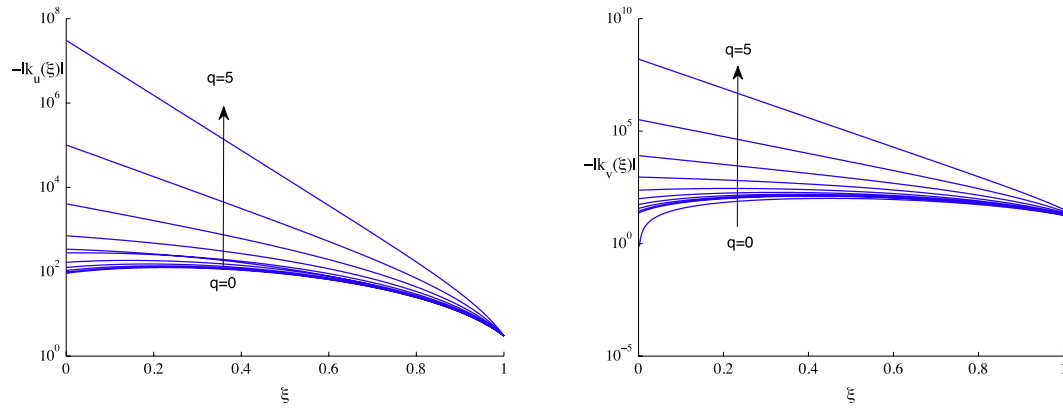

Fig. 4. Control kernels $k_u(\xi)$ and $k_v(\xi)$ for the example of Section 4.2, with fixed $c = 6$ and for different values of q.
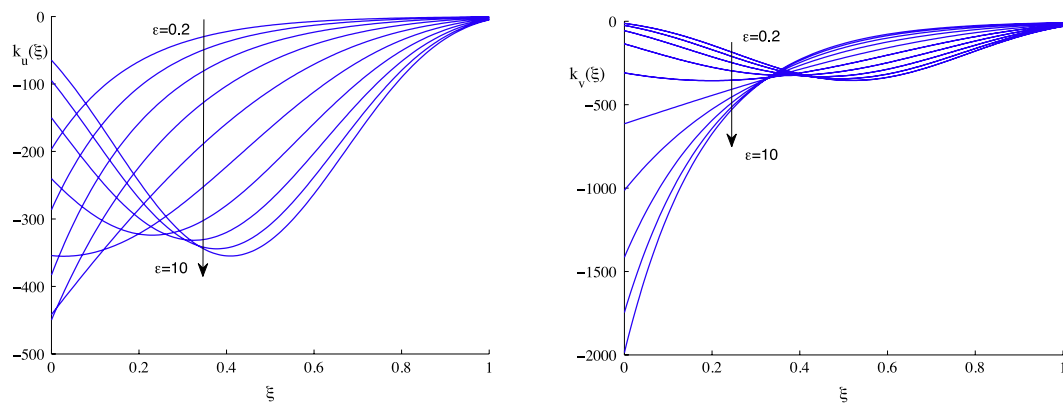

Fig. 5. Control kernels $k_u(\xi)$ and $k_v(\xi)$ for the example of Section 4.3, with fixed $c = 6$ and for different values of ϵ.

Appendix. Some technical results

In the computation of explicit expressions for the kernels, it is necessary to solve a Goursat problem for a function $\Phi(x, \xi)$. The next lemma gives the solution of that problem.

Lemma A.1. *The solution of the following PDE*

$$(\partial_x + \partial_\xi)(\partial_x - \epsilon \partial_\xi)\Phi(x, \xi) = \Phi(x, \xi), \tag{A.1}$$

with boundary conditions

$$\Phi(x, x) = x, \tag{A.2}$$
$$\Phi(x, 0) = 0, \tag{A.3}$$

is given by the smooth function

$$\Phi(x, \xi) = (1 + \epsilon)\xi \frac{I_1\left(\frac{2}{1+\epsilon}\sqrt{(\epsilon x + \xi)(x - \xi)}\right)}{\sqrt{(\epsilon x + \xi)(x - \xi)}}. \tag{A.4}$$

Proof. We present a constructive proof, as the method used in the construction of Φ has some interest; however, we remark that the lemma could be proved just by checking that Φ from (A.4) verifies the equation and by uniqueness of this class of equations (see for instance [17]).

While in principle this PDE could be solved using successive approximation, the resulting integral equation is hard to solve in an explicit fashion. However, we notice that it would be much easier to arrive at a solution if, instead of having boundary data at $\xi = 0$, we had boundary data at the other characteristic $\xi = -\epsilon x$. Following this idea, we substitute boundary condition (A.3) at $\xi = 0$ with a different boundary condition, namely $\Phi(x, -\epsilon x) = f(x)$, and try to find $f(x)$ such that the original PDE is verified. This can be done using a generic Taylor series for f and the finding the values of the coefficients by setting in the resulting solution $\Phi(x, 0) = 0$, which introduces a further complication. Let us guess a simple function $f(x) = \kappa x$. If a value of κ can be found such that the resulting solution verifies that $\Phi(x, 0) = 0$ then we have guessed the correct function. Thus the problem is transformed into

$$(\partial_x + \partial_\xi)(\partial_x - \epsilon \partial_\xi)\Phi(x, \xi) = \Phi(x, \xi), \tag{A.5}$$
$$\Phi(x, x) = x, \tag{A.6}$$
$$\Phi(x, -\epsilon x) = \kappa x. \tag{A.7}$$

Call $\alpha = \epsilon x + \xi$, $\beta = x - \xi$, and $\hat{\Phi}(\alpha, \beta) = \Phi(x, \xi)$. The problem written in α-β variables is:

$$\partial_{\alpha\beta}\hat{\Phi}(\alpha, \beta) = \frac{\hat{\Phi}(\alpha, \beta)}{(1 + \epsilon)^2}, \tag{A.8}$$

$$\hat{\Phi}(\alpha, 0) = \frac{\alpha}{1 + \epsilon}, \tag{A.9}$$

$$\hat{\Phi}(0, \beta) = \frac{\kappa\beta}{1 + \epsilon}. \tag{A.10}$$

Now this is explicitly solvable (as a power series) by using the method of successive approximations, as follows. We integrate $\partial_{\alpha\beta}\hat{\Phi}(\alpha, \beta)$, first in α (from 0 to α), finding

$$\partial_\beta \hat{\Phi}(\alpha, \beta) = \partial_\beta \hat{\Phi}(0, \beta) + \int_0^\alpha \frac{\hat{\Phi}(s, \beta)}{(1 + \epsilon)^2} ds, \tag{A.11}$$

and then we integrate in β (from 0 to β), finding

$$\hat{\Phi}(\alpha, \beta) = \hat{\Phi}(\alpha, 0) + \hat{\Phi}(0, \beta) - \hat{\Phi}(0, 0)$$
$$+ \int_0^\beta \int_0^\alpha \frac{\hat{\Phi}(s, r)}{(1 + \epsilon)^2} ds dr, \tag{A.12}$$

and substituting the boundary conditions

$$\hat{\Phi}(\alpha, \beta) = \frac{\alpha + \kappa\beta}{1 + \epsilon} + \int_0^\beta \int_0^\alpha \frac{\hat{\Phi}(s, r)}{(1 + \epsilon)^2} ds dr. \tag{A.13}$$

Applying to this integral equation the method of successive approximations (see for instance [40]) we directly find a series solution

$$\hat{\Phi}(\alpha, \beta) = \sum_{n=0}^\infty \frac{\left(\frac{\alpha\beta}{(1+\epsilon)^2}\right)^n}{n!(n+1)!} \left(\frac{\kappa\beta + \alpha}{1 + \epsilon}\right), \tag{A.14}$$

which can be easily seen to be convergent and to represent a smooth function (independently of the value of κ). Now, noting that in α-β variables the condition $\xi = 0$ is equivalent to $\alpha = \epsilon\beta$, we make this substitution and find

$$\Phi(x, 0) = \hat{\Phi}(\epsilon\beta, \beta) = \sum_{n=0}^\infty \frac{\left(\frac{\epsilon\beta^2}{(1+\epsilon)^2}\right)^n}{n!(n+1)!} \left(\frac{\kappa\beta + \epsilon\beta}{1 + \epsilon}\right), \tag{A.15}$$

finding that $\Phi(x, 0) = 0$ if and only if $\kappa = -\epsilon$. Thus we used a correct guess for $\Phi(x, -\epsilon x) = -\epsilon x$, and we reach the solution

$$\hat{\Phi}(\alpha, \beta) = \frac{-\epsilon\beta + \alpha}{1 + \epsilon} \sum_{n=0}^\infty \frac{\left(\frac{\alpha\beta}{(1+\epsilon)^2}\right)^n}{n!(n+1)!}$$
$$= (\alpha - \epsilon\beta)\frac{I_1\left(\frac{2}{1+\epsilon}\sqrt{\alpha\beta}\right)}{\sqrt{\alpha\beta}}, \tag{A.16}$$

where Lemma A.2 has been used to express the result as a Bessel function. Substituting Φ in the original variables we reach (A.4). □

Next we present a lemma that allows us to express power series as modified Bessel functions.

Lemma A.2. *The following identity holds.*

$$\sum_{k=0}^\infty \left(\frac{1}{1 + \epsilon}\right)^{2k+n} \frac{(x - \xi)^{k+n}}{(k + n)!} \frac{(\epsilon x + \xi)^k}{k!}$$
$$= \sqrt{\frac{(x - \xi)^n}{(\epsilon x + \xi)^n}} I_n\left(\frac{2}{1 + \epsilon}\sqrt{(x - \xi)(\epsilon x + \xi)}\right). \tag{A.17}$$

Proof. First, using the Taylor series of the corresponding modified Bessel function (see for instance [41, p. 375]), one gets, for integer n,

$$I_n\left(2\sqrt{ab}\right) = \left(\sqrt{ab}\right)^n \sum_{k=0}^\infty \frac{a^k}{(k + n)!} \frac{b^k}{k!} \tag{A.18}$$

and multiplying both sides of the equation by $\sqrt{\frac{a^n}{b^n}}$, we reach

$$\sqrt{\frac{a^n}{b^n}} I_n\left(2\sqrt{ab}\right) = \sum_{k=0}^\infty \frac{a^{k+n}}{(k + n)!} \frac{b^k}{k!}. \tag{A.19}$$

To obtain the result, it is only necessary to substitute $a = \frac{x-\xi}{1+\epsilon}$ and $b = \frac{\epsilon x + \xi}{1+\epsilon}$. □

Finally, we present a result that allows to express a double series as an special function known as the Marcum Q-function (or the generalized Marcum Q-function of order 1, see [34]).

Lemma A.3. *The function $\Pi(x, y)$ defined as*

$$\Pi(x, y) = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{y^n x^{n+m}}{n!(n + m)!} \tag{A.20}$$

can be written as

$$\Pi(x,y) = e^{x+y}Q_1\left(\sqrt{2x},\sqrt{2y}\right),\qquad (A.21)$$

and also it can alternatively be written as

$$\Pi(x,y) = e^{x+y} - y\int_0^1 e^{y(1-s)}I_0\left(2\sqrt{sxy}\right)ds.\qquad (A.22)$$

Additionally, $\Pi\left(\delta x,\frac{y}{\delta}\right)$ can be approximated for small δ as follows

$$\Pi\left(\delta x,\frac{y}{\delta}\right) = I_0\left(2\sqrt{xy}\right) + \delta\sqrt{\frac{x}{y}}I_1\left(2\sqrt{xy}\right)$$
$$+ \delta^2\frac{x}{y}I_2\left(2\sqrt{xy}\right) + O(\delta^3).\qquad (A.23)$$

Proof. Note first that $\Pi(x,y)$ is defined as an (absolutely convergent) double series, so the following manipulations can be rigorously justified. Using Lemma A.2, we can write

$$\Pi(x,y) = \sum_{m=0}^{\infty}\sqrt{\frac{x^m}{y^m}}I_m\left(2\sqrt{xy}\right).\qquad (A.24)$$

On the other hand, the generalized Marcum Q-function of order 1 can be written as [34]

$$Q_1(a,b) = e^{-\frac{a^2+b^2}{2}}\sum_{m=0}^{\infty}\frac{a^m}{b^m}I_m(ab).\qquad (A.25)$$

Substituting $a=\sqrt{2x}$, $b=\sqrt{2x}$, the first part of the lemma follows. To obtain the integral representation (A.22), first we note that it is clear that $\Pi(x,0) = \sum_{m=0}^{\infty}\frac{x^m}{m!} = e^x$. On the other hand

$$\frac{\partial}{\partial y}\Pi(x,y) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{y^{n-1}x^{n+m}}{(n-1)!(n+m)!}$$
$$= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{y^n x^{n+m}}{n!(n+m)!}$$
$$= \Pi(x,y) - \sum_{n=0}^{\infty}\frac{x^n y^n}{n!^2}$$
$$= \Pi(x,y) - I_0\left(2\sqrt{xy}\right),\qquad (A.26)$$

where Lemma A.2 has been used. This expression represents an ordinary differential equation in y, that we can solve

$$\Pi(x,y) = \Pi(x,0)e^y - \int_0^y e^{y-s}I_0\left(2\sqrt{sx}\right)ds,\qquad (A.27)$$

thus:

$$\Pi(x,y) = e^{x+y} - \int_0^y e^{y-s}I_0\left(2\sqrt{sx}\right)ds$$
$$= e^{x+y} - y\int_0^1 e^{y(1-s)}I_0\left(2\sqrt{sxy}\right)ds,\qquad (A.28)$$

obtaining (A.22). Finally (A.23) easily follows from the series representation of $\Pi(x,y)$ and Lemma A.2. □

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