Estimation and Control of the Rijke Tube

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3rd Workshop on Stability and Control of Infinite-Dimensional Systems
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Pieter J. Rijke

(Professor at Leiden University)

"Notice of a new way to set into oscillation the air contained in a tube with both ends open", Annalen der Physik und Chemie, vol. 107, pp. 339-343, 1859. (In German)

ANNALEN

DER

PHYSIK

UND

CHEMIE.

HERAUSGEGEBEN ZU BERLIN

VON

J. C. POGGENDORFF.

HUNDERT UND SIEBENTER BAND.

DER GANZEN FOLGE HUNDERT UND DREI UND ACHTZIGSTER.

NEBST VIER KUPFERTAFELN.

LEIPZIG, 1859.

VERLAG VON JOHANN AMBROSIUS BARTH.

XV. *Notiz über eine neue Art, die in einer an beiden Enden offenen Röhre enthaltene Luft in Schwingungen zu versetzen; von P. L. Rijke.*

1. Meine ersten Versuche wurden mit einer Glasröhre von $0^m,8$ Länge gemacht. Ihr Durchmesser betrug in dem oberen Theil 37^{mm} und in dem unteren 30^{mm}. Im Innern, $0^m,2$ von diesem letzteren Ende ab, hatte ich eine Scheibe von Metallgeflecht, etwa 50^{mm} im Durchmesser haltend, angebracht. Ihre Ränder waren umgebogen, so dafs sie durch den Druck, den diese gegen die Röhrenwandung ausübten, in jeder beliebigen Höhe gehalten werden konnte. Das Metallgeflecht war von $0^{mm},2$ dicken Eisendraht und hielt auf ein Quadrat-Centimeter ungefähr 81 Maschen.

Nachdem der Apparat so vorgerichtet worden, hatte man nur das Metallgeflecht mittelst einer Alkohol- oder Wasserstoff-Lampe in Rothgluth zu versetzen, um einige Augenblicke nach dem Auslöschen oder Fortnehmen der Lampe einen Ton zu vernehmen. Der Ton war beinah der Grundton der Röhre. Er hatte viele Stärke (*éclat*), hielt aber nur einige Sekunden an.

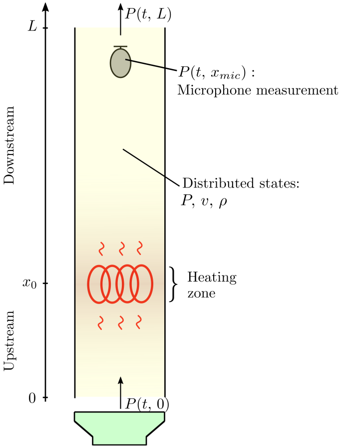
2. Wenn man, statt einer einzigen Scheibe, deren mehre in der Röhre anbringt, so hält der Ton, den man bekommt, länger an.

3. Der Ton hört augenblicklich auf, so wie man die obere Mündung der Röhre verschliefst. Daraus folgt, dafs das Daseyn eines aufsteigenden Luftstromes eine der Bedingungen des Phänomens ausmacht. Auch darf man die Zahl der Scheiben nicht übermäfsig vergröfsern, weil die Verlangsamung des Luftstroms nicht gewisse Gränzen überschreiten darf.

4. Der Versuch gelingt auch, wenn man die Scheibe mittelst einer Kohlenoxyd-Flamme erhitzt. Ich bereitete dieses Gas, indem ich Nordhäuser Schwefelsäure auf Oxal-

The Rijke Tube Experiment

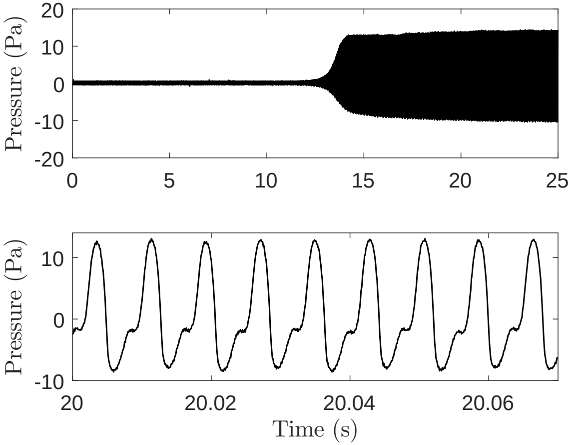
- A vertical tube opened in both ends.
- A heat source is inserted in the lower half of the tube.
- Under the right conditions, the tube begins to hum loudly (thermoacoustic instability).
- A microphone at the top of the tube can be used for measurement of acoustic pressure.
- A speaker at the bottom is used as actuator to stabilize the system.



Click for video

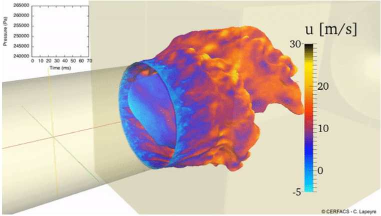
The Rijke Tube Experiment

Microphone signal at the onset of instability showing growth, and then saturation of the limit cycle. A zoomed-in picture shows the periodic, but nonsymetric, limit-cycle behavior.



Motivation

- Thermoacoustic instabilities are often encountered in steam and gas turbines, industrial burners, and jet and ramjet engines.



- At best, they produce vibrations potentially affecting delicate instrumentation and payloads.

  At their worst, the oscillations may increase the average pressure, resulting even in rupture of the system.

  However, these instabilities are notorious difficult to model and study.

  The absence of combustion process in the Rijke tube makes the modeling and analysis more tractable.

  The Rijke tube experiment provides an accessible platform to explore and study stabilization and state estimation of thermoacoustic oscillations.

Previous work

Results on boundary control of thermoacustic instabilities

G. A. de Andrade, R. Vazquez, D. J. Pagano. Backstepping stabilization of a linearized ODE-PDE Rijke tube model. Automatica, v. 96, p. 98 - 109, 2018.

G. A. de Andrade, R. Vazquez, D. J. Pagano. Boundary feedback control of unstable thermoacoustic oscillations in the Rijke tube. In: Proceedings of the 2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations, 2016, v. 48, p. 48 - 53.

G. A. de Andrade, R. Vazquez, D. J. Pagano. Boundary control of a Rijke tube using irrational transfer functions with experimental validation. In: Proceedings of the 20th World Congress of the International Federation of Automatic Control, 2017, v. 50, p. 4528 - 4533.

Previous work

Results on state estimation of thermoacustic instabilities

J. Auriol, G. A. de Andrade, R. Vazquez. A differential-delay estimator for thermoacoustic oscillations in a Rijke tube using in-domain pressure measurements. In: Proceedings of the 59th IEEE Conference on Decision and Control, 2020, p. 4417-4422.

G. A. de Andrade, R. Vazquez. A Backstepping-based observer for estimation of thermoacoustic oscillations in a Rijke tube with in-domain measurements. In: Proceedings of the 21th IFAC World Congress, 2020, v. 53, p. 7521-7526.

G. A. de Andrade, R. Vazquez, D. J. Pagano. Backstepping-based estimation of thermoacoustic oscillations in a Rijke tube with experimental validation. IEEE Transactions on Automatic Control, v. 65, p. 5336 - 5343, 2020.

G. A. de Andrade, R. Vazquez, D. J. Pagano. Backstepping-based linear boundary observer for estimation of thermoacoustic instabilities in a Rijke tube. In: Proceedings of the 57th IEEE Conference on Decision and Control, 2018, p. 2164-2169.

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  - Nonlinear PDE system
  - Simplification assumptions and linearization
- Backstepping for PDEs: a brief introduction
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Thermoacoustic dynamics

Starting from the conservation of mass, momentum, and energy, we arrive at

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \text{mass conservation}$$

$$\partial_t(\rho v) + \partial_x\left(\rho v^2 + P\right) = 0, \qquad \text{momentum balance}$$

$$\partial_t\left(\rho U + \frac{\rho v^2}{2}\right) + \partial_x\left(v\left(\rho U + \frac{\rho v^2}{2}\right) + Pv\right) = q, \qquad \text{energy balance}$$

with boundary conditions

$$P(t,0) = P_0 + g(v(t,\, 0)) + u(t), \qquad \text{open end with speaker}$$

$$P(t,L) = P_0 + f(v(t,\, L)). \qquad \text{open end}$$

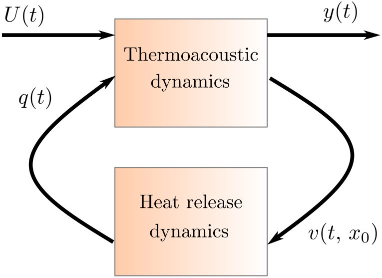
Heat release dynamics

We assume that the heat input is concentrated at a single point x_0:

$$q(x,t) = \frac{1}{A}\delta(x - x_0)Q(t).$$

King's Law describes the dependence of heat transfer on gas velocity:

$$\tau\dot{Q}(t) = -Q(t) + Q_K(t),$$
$$Q_K(t) = l_w(T_w - T)(\kappa + \kappa_v\sqrt{|v(t,x_0)|}).$$



Linearization of thermoacoustic dynamics

Assume constant steady-state solution, $(\rho,\, v,\, P) = (\overline{\rho},\, \overline{v},\, \overline{P}), \forall t \in [0,\, +\infty), \forall x \in [0,\, L]$. Then, we can obtain the following linearized model:

$$
\begin{aligned}
&\partial_t \tilde{\rho} + \overline{v} \partial_x \tilde{\rho} + \overline{\rho} \partial_x \tilde{v} = 0, \\
&\partial_t \tilde{v} + \overline{v} \partial_x \tilde{v} + \frac{1}{\overline{\rho}} \partial_x \tilde{P} = 0, \\
&\partial_t \tilde{P} + \gamma \overline{P} \partial_x \tilde{v} + \overline{v} \partial_x \tilde{P} = \frac{\overline{\gamma}}{A} \delta(x - x_0) \tilde{Q},
\end{aligned}
$$

Taking into account that $\overline{v}$ is very small if compared to the speed of sound, it is easy to see that the contribution of $\overline{v}$ to the gas dynamics is negligible. Therefore, making $\overline{v} = 0$ and noticing that the density $\tilde{\rho}$ is decoupled from the velocity and pressure dynamics, it is obtained that the remaining coupled part of the dynamics is a wave equation!

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\end{aligned}$$

Linearization of heat release dynamics and boundary Conditions

Linearizing King's Law yields

$$\tilde{Q}_K(t) = f(\overline{v})\frac{\overline{T}}{\overline{\rho}}\tilde{\rho} + f'(\overline{v})(T_w - \overline{T})\tilde{v} - f(\overline{v})\frac{\overline{T}}{\overline{P}}\tilde{P}.$$

Comparing the size of the gains of each state in the above equation it is possible to conclude that the velocity fluctuations are the main driver of the heat dynamics, hence it is reasonable to drop out the density and pressure influence of the above equation

$$\tilde{Q}_K(t) \approx f'(\overline{v})(T_w - \overline{T})\tilde{v}(t, x_0).$$

Thus,

$$\tau\dot{\tilde{Q}}(t) = -\tilde{Q}(t) + \tilde{Q}_K(t).$$

Linearization of the boundary conditions yields

$$\tilde{P}(t,0) = -Z_0\tilde{v}(t, L) + U(t),$$
$$\tilde{P}(t,0) = Z_L\tilde{v}(t, L).$$

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Model in Terms of Characteristic Coordinates

Since the system is hyperbolic, there exists an invertible linear transformation such that

$$\begin{pmatrix} \tilde{v} \\ \tilde{P} \end{pmatrix} = \mathbf{T} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{\gamma \overline{P} \overline{\rho}}} & -\frac{1}{2\sqrt{\gamma \overline{P} \overline{\rho}}} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

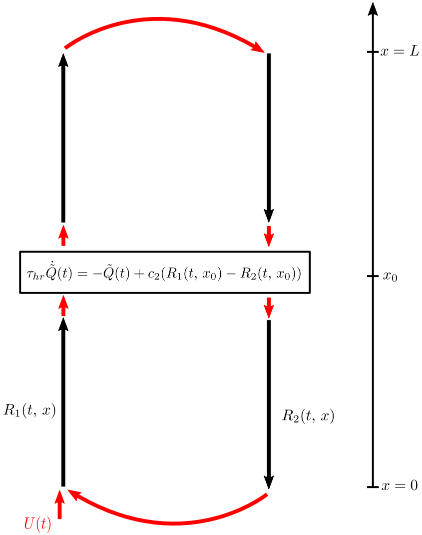
Then, the linearized system is rewritten to

$$\begin{aligned}
&\partial_t R_1 + \lambda \partial_x R_1 = c_1 \delta(x - x_0) \tilde{Q}(t), \\
&\partial_t R_2 - \lambda \partial_x R_2 = c_1 \delta(x - x_0) \tilde{Q}(t), \\
&R_1(t, 0) = k_0 R_2(t,\, 0) + 2U(t), \\
&R_2(t, L) = k_L R_1(t, L). \\
&\tau \dot{\tilde{Q}}(t) = -\tilde{Q}(t) + c_2(R_1(t,\, x_0) - R_2(t,\, x_0)).
\end{aligned}$$

Schematic view of the jumping point at the solution of the PDE system

The following relations are satisfied:

$$R_1(t,\, x_0^+) = R_1(t,\, x_0^-) + c_1\tilde{Q}(t),$$
$$R_2(t,\, x_0^-) = R_2(t,\, x_0^+) + c_1\tilde{Q}(t).$$



Representation in characteristic coordinates

Folding transformation:
Now, we introduce the following state variables

$$R_1(t,x) = \begin{cases} \alpha_1(t,x), & x \in [0, x_0], \\ \beta_2(t,x), & x \in [x_0, L], \end{cases}$$

$$R_2(t,x) = \begin{cases} \beta_1(t,x), & x \in [0, x_0], \\ \alpha_2(t,x), & x \in [x_0, L], \end{cases}$$

and the rescaled spatial variable, so that everything evolves on the same domain:

$$z = \begin{cases} \frac{x}{x_0} & \text{if } x \in [0,\ x_0] \\ \frac{L-x}{L-x_0} & \text{if } x \in [x_0,\ L] \end{cases}$$

Representation in characteristic coordinates

Then, the system linearized system is equivalent to

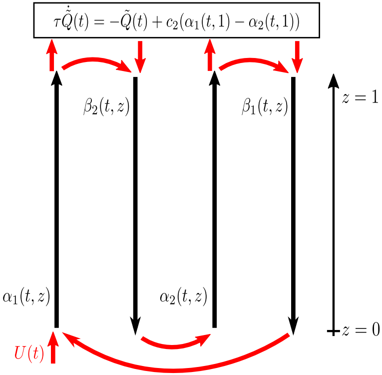
$$\begin{aligned}
\partial_t \alpha_1(t,\, z) + \lambda_1 \partial_z \alpha_1(t,\, z) &= 0,\\
\partial_t \beta_1(t,\, z) - \lambda_1 \partial_z \beta_1(t,\, z) &= 0,\\
\partial_t \beta_2(t,\, z) - \lambda_2 \partial_z \beta_2(t,\, z) &= 0,\\
\partial_t \alpha_2(t,\, z) + \lambda_2 \partial_z \alpha_2(t,\, z) &= 0,
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
\alpha_1(t,0) &= k_0 \beta_1(t,\, 0) + 2U(t),\\
\beta_1(t,1) &= \alpha_2(t,\, 1) + c_1 \tilde{Q}(t),\\
\beta_2(t,1) &= \alpha_1(t,\, 1) + c_1 \tilde{Q}(t),\\
\alpha_2(t,0) &= k_L \beta_2(t,\, 0),\\
\tau \dot{\tilde{Q}}(t) &= -\tilde{Q}(t) + c_2(\alpha_1(t,\, 1) - \alpha_2(t,\, 1)).
\end{aligned}$$

Representation in characteristic coordinates

- The boundary conditions represent two effects: reflection of the acoustic waves; and the feedback coupling between β_2 and α_2, and between α_1 and β_1.
- Under the right conditions the system becomes unstable due to this feedback between the states.
- Our objective is to design an output feedback control law that **exponentially stabilize** the zero equilibrium of the system.



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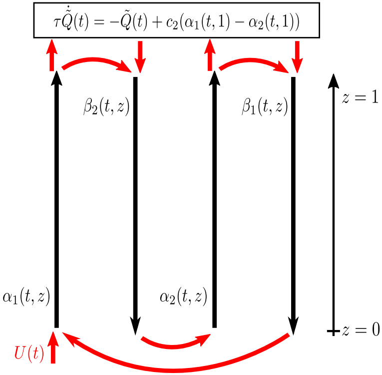

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Backstepping for PDEs: a brief introduction

Roughly speaking, backstepping is a constructive method that achieves Lyapunov stabilization by **transforming** the system into a stable "target system", which is often achieved by collectively shifting all the eigenvalues in a favorable direction in the complex plane, rather than by assigning individual eigenvalues.

Backstepping is not "one-size-fits-all". Requires structure-specific effort by designer.

Reward: elegant controller/observer, (mostly) clear closed-loop behavior.

Backstepping for PDEs: a brief introduction

Basic steps in the backstepping methodology:

1. Identify the undesirable terms in the PDE.

2. Choose a target system in which the undesirable terms are to be eliminated by state transformation and feedback.

3. Find the state transformation.

4. Obtain the boundary feedback/observer gains from the transformation. The transformation alone cannot eliminate the undesirable terms, but the transformation brings them to the boundary, so control can cancel them.

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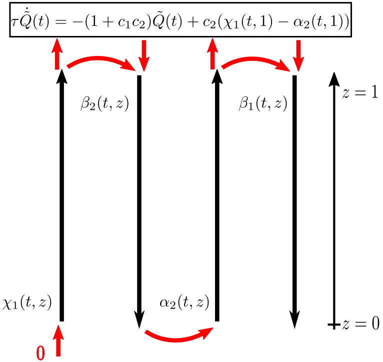
Backstepping-based controller design: target system

We want to map the Rijke tube model into the following target system

$$\begin{aligned}\partial_t \chi_1(t,\, z) + \lambda_1 \partial_z \chi_1(t,\, z) &= 0,\\ \partial_t \beta_1(t,\, z) - \lambda_1 \partial_z \beta_1(t,\, z) &= 0,\\ \partial_t \beta_2(t,\, z) - \lambda_2 \partial_z \beta_2(t,\, z) &= 0,\\ \partial_t \alpha_2(t,\, z) + \lambda_2 \partial_z R_{22}(t,\, z) &= 0,\end{aligned}$$

with the following boundary conditions

$$\begin{aligned}&\chi_1(t,0) = 0,\\ &\beta_1(t,1) = \alpha_2(t,1) + c_1 \tilde{Q}(t),\\ &\beta_2(t,1) = k_L \chi_1(t,1),\\ &\alpha_2(t,0) = \beta_2(t,0),\\ &\quad \tau \dot{\tilde{Q}}(t) = -(1 + c_1 c_2)\tilde{Q}(t) + c_2(\chi_1(t,1) - \alpha_2(t,1)).\end{aligned}$$



Backstepping-based controller design: transformation

To do that, we consider the following backstepping transformation

$$\chi_1(t,\, z) = \alpha_1(t,\, z) - \varphi(z)\tilde{Q}(t) - \int_z^1 \alpha_1(t,\, \xi)K(z,\, \xi)d\xi - \int_0^1 \alpha_2(t, \xi)G(z,\, \xi)d\xi - \int_0^1 \beta_2(t,\, \xi)H(z,\, \xi)d\xi.$$

- Domain of the K kernel:

$$\mathscr{T}_0 = \{(z,\, \xi) \in \mathbb{R}^2 | 0 \le z \le \xi \le 1\},$$

- Domain of G and H kernels:

$$\mathscr{T}_1 = \{(z,\, \xi) \in \mathbb{R}^2 | 0 \le \xi \le 1, 0 \le z \le 1\},$$

- φ is a one-dimensional kernel defined on the interval $z \in [0,\, 1]$.

Backstepping-based controller design: kernel equations

Differentiating the transformation with respect to space and time, integrating by parts, and plugging the target system equation, we obtain that the original system is mapped into the target system if and only if the kernels satisfy the following equations:

$$\partial_\xi K(z,\,\xi) + \partial_z K(z,\,\xi) = 0,$$

$$\partial_\xi G(z,\,\xi) + \frac{\lambda_1}{\lambda_2}\partial_z G(z,\,\xi) = 0,$$

$$\partial_\xi H(z,\,\xi) - \frac{\lambda_1}{\lambda_2}\partial_z H(z,\,\xi) = 0,$$

$$\lambda_1 \varphi'(z) - \frac{1}{\tau}\varphi(z) + \lambda_2 c_1 H(z,\,1) = 0,$$

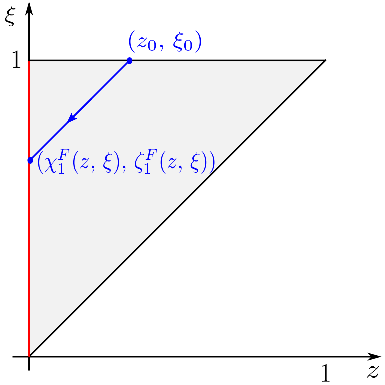
with

$$\lambda_1 K(z,\,1) - \frac{c_2}{\tau}\varphi(z) - \lambda_2 H(z,\,1) = 0, \qquad G(1,\xi) = 0,$$

$$\lambda_2 G(z,\,1) + \frac{c_2}{\tau}\varphi(z) = 0, \qquad H(1,\xi) = 0,$$

$$\alpha G(z,\,0) - H(z,\,0) = 0, \qquad \varphi(1) = -c_1.$$

Backstepping-based controller design: kernel equations



The boundary value problem for the K kernel:

$$\partial_\xi K(z,\, \xi) + \partial_z K(z,\, \xi) = 0,$$

$$\lambda_1 K(z,\, 1) - \frac{c_2}{\tau}\varphi(z) - \lambda_2 H(z,\, 1) = 0.$$

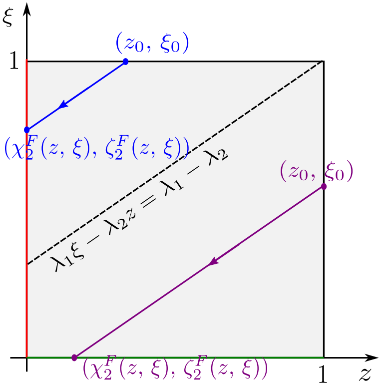
Solution:

$$K(z,\, \xi) = \frac{c_2}{\lambda_1 \tau}\varphi(z - \xi + 1) + \frac{\lambda_2}{\lambda_1} H(z - \xi + 1, 1).$$

Backstepping-based controller design: kernel equations

The boundary value problem for the G kernel:

$$\partial_\xi G(z,\,\xi) + \frac{\lambda_1}{\lambda_2}\partial_z G(z,\,\xi) = 0$$
$$\lambda_2 G(z,\,1) + \frac{c_2}{\tau}\varphi(z) = 0,$$
$$G(1,\xi) = 0.$$



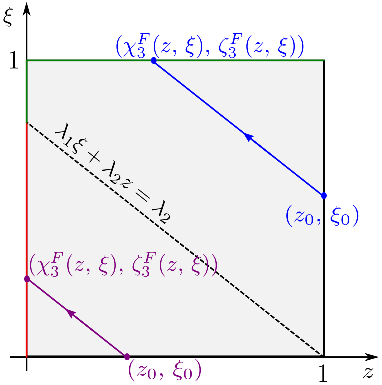
Solution:

$$G(z,\,\xi) = \begin{cases} 0 & \xi - 1 \leq \frac{\lambda_2}{\lambda_1}(z-1), \\ -\frac{c_2}{\lambda_2\tau}\varphi\left(z - \frac{\lambda_1}{\lambda_2}(\xi-1)\right), & \text{otherwise.} \end{cases}$$

Backstepping-based controller design: kernel equations

The boundary value problem for the H kernel:

$$\partial_\xi H(z,\,\xi) - \frac{\lambda_1}{\lambda_2}\partial_z H(z,\,\xi) = 0,$$
$$\alpha G(z,\,0) - H(z,\,0) = 0,$$
$$H(1,\xi) = 0.$$



Solution:

$$H(z,\,\xi) = \begin{cases} 0 & \xi + 1 \geq \frac{\lambda_2}{\lambda_1}(1-z), \\ -\frac{\alpha c_2}{\lambda_2 \tau}\varphi\left(z + \frac{\lambda_1}{\lambda_2}(\xi+1)\right), & \text{otherwise.} \end{cases}$$

Backstepping-based controller design: kernel equations

The boundary value problem for the φ kernel:

$$\lambda_1\varphi'(z) - \frac{1}{\tau}\varphi(z) + \lambda_2 c_1 H(z,\, 1) = 0,$$
$$\varphi(1) = -c_1.$$

φ may have a discontinuity depending on the values of $H(z,\, 1)$.

Case I ($\lambda_1 \geq \lambda_2$): In this case $H(z,\, 1) = 0, \forall z \in [0,\, 1]$. Then,

$$\left.\begin{array}{l} \lambda_1\dot{\varphi}(z) - \frac{1}{\tau}\varphi(z) = 0 \\ \varphi(1) = -c_1 \end{array}\right\} \Rightarrow \varphi(z) = -c_1 \mathrm{e}^{\frac{z-1}{\lambda_1 \tau}}$$

Backstepping-based controller design: kernel equations

Case II ($\lambda_1 < \lambda_2$): In this case, the φ kernel equations can be solved backwards.

Observing the behavior of $H(z, 1)$ backwards, one can note that it is zero for all $z \in [1 - 2\frac{\lambda_1}{\lambda_2}, 1]$. Therefore, the solution φ for $z \in [1 - 2\frac{\lambda_1}{\lambda_2}, 1]$ satisfies

$$\left.\begin{array}{l}\lambda_1 \dot{\varphi}(z) - \frac{1}{\tau}\varphi(z) = 0 \\ \varphi(1) = -c_1\end{array}\right\} \Rightarrow \varphi(z) = -c_1 \mathrm{e}^{\frac{z-1}{\lambda_1 \tau}}$$

Once the solution φ is known on $[1-2\frac{\lambda_1}{\lambda_2}, 1]$, we can repeat the same procedure, starting with the solution on $[1 - 2\frac{\lambda_1}{\lambda_2}, 1]$, to find the solution φ for $z \in [1 - 4\frac{\lambda_1}{\lambda_2}, 1 - 2\frac{\lambda_1}{\lambda_2}]$ by computing the solution of the following boundary value problem:

$$\lambda_1 \dot{\varphi}(z) - \frac{1}{\tau}\varphi(z) + \frac{\alpha c_1^2 c_2}{\tau} \mathrm{e}^{\frac{\lambda_2(z-1)+2\lambda_1}{\lambda_2 \lambda_1 \tau}} = 0,$$

$$\varphi\left(1 - 2\frac{\lambda_1}{\lambda_2}\right) = -c_1 \mathrm{e}^{-\frac{2}{\lambda_2 \tau}}.$$

We can repeat the same procedure, starting with the solution on $[1 - 4\frac{\lambda_1}{\lambda_2}, 1 - 2\frac{\lambda_1}{\lambda_2}]$, to find the solution φ for $z \in [1 - 8\frac{\lambda_1}{\lambda_2}, 1 - 4\frac{\lambda_1}{\lambda_2}]$, and so on.

Backstepping-based controller design: kernel equations

Case II ($\lambda_1 < \lambda_2$): In this case, the φ kernel equations can be solved backwards.

Observing the behavior of $H(z,\,1)$ backwards, one can note that it is zero for all $z \in [1-2\frac{\lambda_1}{\lambda_2},\,1]$. Therefore, the solution φ for $z \in [1-2\frac{\lambda_1}{\lambda_2},\,1]$ satisfies

$$\left.\begin{array}{l}\lambda_1\dot{\varphi}(z) - \frac{1}{\tau}\varphi(z) = 0\\ \varphi(1) = -c_1\end{array}\right\} \Rightarrow \varphi(z) = -c_1 \mathrm{e}^{\frac{z-1}{\lambda_1\tau}}$$

Once the solution φ is known on $[1-2\frac{\lambda_1}{\lambda_2},\,1]$, we can repeat the same procedure, starting with the solution on $[1-2\frac{\lambda_1}{\lambda_2},\,1]$, to find the solution φ for $z \in [1-4\frac{\lambda_1}{\lambda_2},\,1-2\frac{\lambda_1}{\lambda_2}]$ by computing the solution of the following boundary value problem:

$$\begin{aligned}&\lambda_1\dot{\varphi}(z) - \frac{1}{\tau}\varphi(z) + \frac{\alpha c_1^2 c_2}{\tau}\mathrm{e}^{\frac{\lambda_2(z-1)+2\lambda_1}{\lambda_2\lambda_1\tau}} = 0,\\ &\varphi\left(1-2\frac{\lambda_1}{\lambda_2}\right) = -c_1\mathrm{e}^{-\frac{2}{\lambda_2\tau}}.\end{aligned}$$

We can repeat the same procedure, starting with the solution on $[1-4\frac{\lambda_1}{\lambda_2},\,1-2\frac{\lambda_1}{\lambda_2}]$, to find the solution φ for $z \in [1-8\frac{\lambda_1}{\lambda_2},\,1-4\frac{\lambda_1}{\lambda_2}]$, and so on.

Backstepping-based controller design: kernel equations

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Observing the behavior of $H(z, 1)$ backwards, one can note that it is zero for all $z \in [1 - 2\frac{\lambda_1}{\lambda_2}, 1]$. Therefore, the solution φ for $z \in [1 - 2\frac{\lambda_1}{\lambda_2}, 1]$ satisfies

$$\left.\begin{array}{l}\lambda_1 \dot{\varphi}(z) - \frac{1}{\tau}\varphi(z) = 0 \\ \varphi(1) = -c_1\end{array}\right\} \Rightarrow \varphi(z) = -c_1 \mathrm{e}^{\frac{z-1}{\lambda_1 \tau}}$$

Once the solution φ is known on $[1 - 2\frac{\lambda_1}{\lambda_2}, 1]$, we can repeat the same procedure, starting with the solution on $[1 - 2\frac{\lambda_1}{\lambda_2}, 1]$, to find the solution φ for $z \in [1 - 4\frac{\lambda_1}{\lambda_2}, 1 - 2\frac{\lambda_1}{\lambda_2}]$ by computing the solution of the following boundary value problem:

$$\lambda_1 \dot{\varphi}(z) - \frac{1}{\tau}\varphi(z) + \frac{\alpha c_1^2 c_2}{\tau} \mathrm{e}^{\frac{\lambda_2(z-1)+2\lambda_1}{\lambda_2 \lambda_1 \tau}} = 0,$$

$$\varphi\left(1 - 2\frac{\lambda_1}{\lambda_2}\right) = -c_1 \mathrm{e}^{-\frac{2}{\lambda_2 \tau}}.$$

We can repeat the same procedure, starting with the solution on $[1 - 4\frac{\lambda_1}{\lambda_2}, 1 - 2\frac{\lambda_1}{\lambda_2}]$, to find the solution φ for $z \in [1 - 8\frac{\lambda_1}{\lambda_2}, 1 - 4\frac{\lambda_1}{\lambda_2}]$, and so on.

Backstepping-based controller design: kernel equations

Applying this iterative procedure n times, where $n \in \mathbb{N}$ is the largest integer such that $\frac{L-x_0}{x_0} > \frac{1}{2^n}$, yields a unique, globally defined, solution φ when $\lambda_1 < \lambda_2$.

The closer the heat element is to the uncontrolled boundary, the larger the number of pieces of the solution φ.

From a practical point of view, the case $\lambda_1 > \lambda_2$ occurs when the heat release is located in the lower half of the tube. Similarly, $\lambda_1 < \lambda_2$ if the heat release is located in the upper half of the tube.

Backstepping-based controller design: invertibility of the transformation

To ensure that the closed-loop system and the target system have the same stability property, the backstepping transformation has to be invertible.

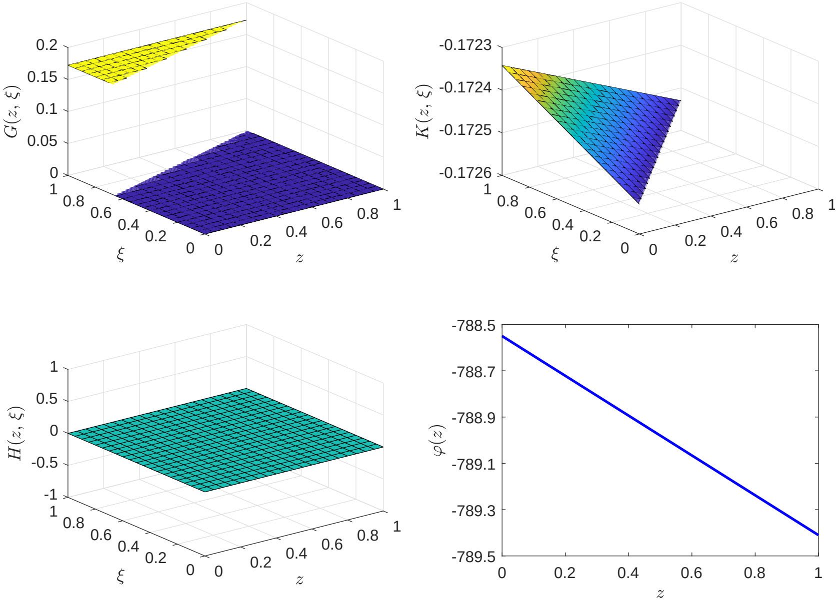
Rewrite the transformation as

$$\alpha_1(t, z) = \int_z^1 \alpha_1(t, \xi) K(z, \xi) d\xi + \psi(t, z).$$

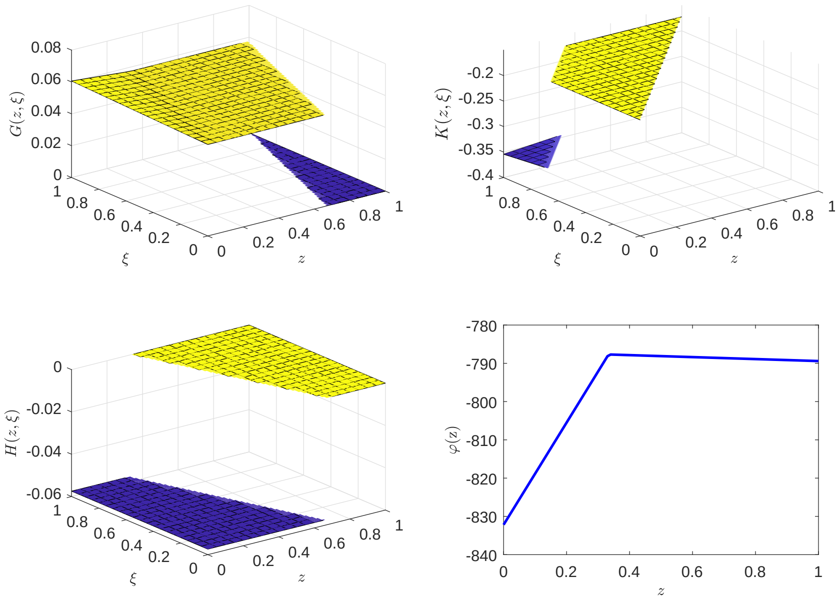
This equation can be seen as a Volterra integral equation of the second kind.

Since $K(z, \xi)$ is bounded, the equation has a unique solution, allowing us to write an inverse transformation, thus proving the invertibility of the transformation.

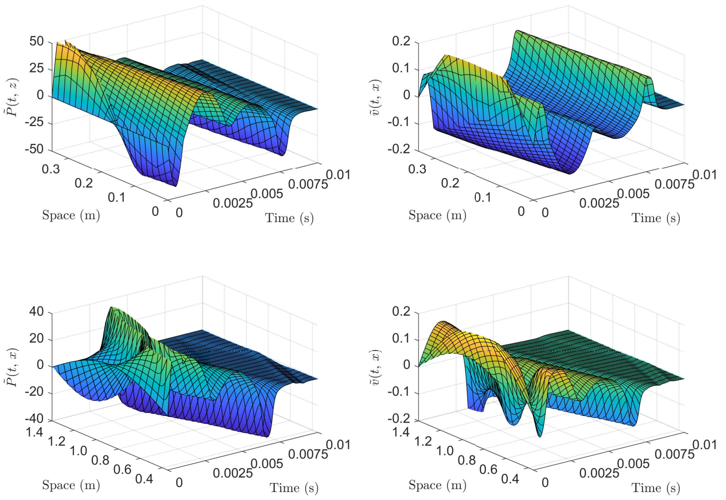
Backstepping-based controller design: kernel visuals (case I)



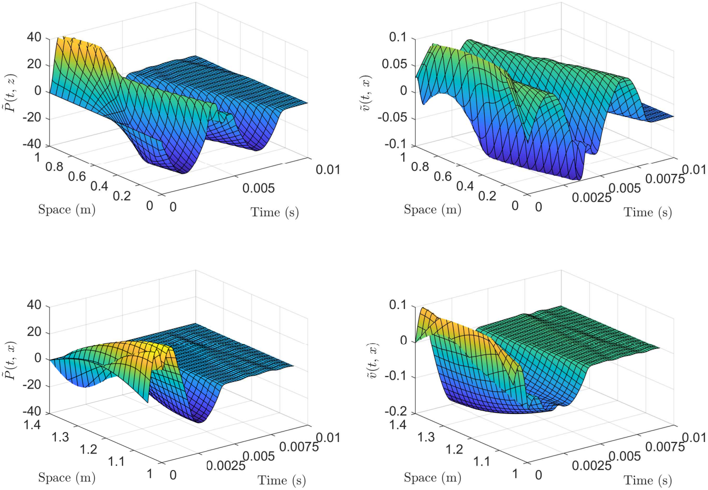
Backstepping-based controller design: kernel visuals (case II)



Backstepping-based controller design: simulation results (case I)



Backstepping-based controller design: simulation results (case II)



Remarks

The backstepping control law requires full-state measurement

$$U(t) = \frac{1}{2}\left(k_0\beta_1(t,0) + \varphi(0)\tilde{Q}(t) + \int_0^1 \alpha_1(t,\,\xi)K(0,\,\xi)d\xi \right.$$
$$\left. + \int_0^1 \alpha_2(t,\,\xi)G(0,\,\xi)d\xi + \int_0^1 \beta_2(t,\,\xi)H(0,\,\xi)d\xi\right).$$

Therefore, the control law must be applied together with a **state-observer** in order to produce experiments.

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Backstepping-based observer design

- We design the observe as a copy of the plant plus output injection terms:

$$\begin{aligned}
&\partial_t \hat{\alpha}_1(t,\, z) + \lambda_1 \partial_z \hat{\alpha}_1(t,\, z) = -p_{11}(z)\tilde{Y}(t),\\
&\partial_t \hat{\beta}_1(t,\, z) - \lambda_1 \partial_z \hat{\beta}_1(t,\, z) = -p_{12}(z)\tilde{Y}(t),\\
&\partial_t \hat{\beta}_2(t,\, z) - \lambda_2 \partial_z \hat{\beta}_2(t,\, z) = -p_{21}(z)\tilde{Y}(t),\\
&\partial_t \hat{\alpha}_{22}(t,\, z) + \lambda_2 \partial_z \hat{\alpha}_2(t,\, z) = -p_{22}(z)\tilde{Y}(t),\\
&\tau \dot{\hat{Q}}(t) = -\hat{Q}(t) + c_2(\hat{\alpha}_1(t,\, 1) - \hat{\alpha}_2(t,\, 1)) - p_Q \tilde{Y}(t),
\end{aligned}$$

  with $\tilde{Y}(t) = \frac{Z_L}{Z_L + \bar{\rho} c} \beta_2(t,\, 0)$.
- The boundary conditions are given by

$$\begin{aligned}
\hat{\alpha}_1(t, 0) &= k_0 \hat{\beta}_1(t,\, 0) + 2U(t),\\
\hat{\beta}_1(t, 1) &= \hat{\alpha}_2(t,\, 1) + c_1 \hat{Q}(t),\\
\hat{\beta}_2(t, 1) &= \hat{\alpha}_1(t,\, 1) + c_1 \hat{Q}(t),\\
\hat{\alpha}_2(t, 0) &= k_L \beta_2(t,\, 0),
\end{aligned}$$

- p_{11}, p_{12}, p_{21}, p_{22}, and p_Q are gains to be found.

Backstepping-based observer design: observed error dynamics

Define the error estimation $\tilde{R}_{ij} = R_{ij} - \hat{R}_{ij}$, $i,\, j = 1,\, 2$, whose dynamics is given by

$$\begin{aligned}
&\partial_t \tilde{\alpha}_1(t,\, z) + \lambda_1 \partial_z \tilde{\alpha}_1(t,\, z) = p_{11}(z)\tilde{Y}(t),\\
&\partial_t \tilde{\beta}_1(t,\, z) - \lambda_1 \partial_z \tilde{\beta}_1(t,\, z) = p_{12}(z)\tilde{Y}(t),\\
&\partial_t \tilde{\beta}_2(t,\, z) - \lambda_2 \partial_z \tilde{\beta}_2(t,\, z) = p_{21}(z)\tilde{Y}(t),\\
&\partial_t \tilde{\alpha}_2(t,\, z) + \lambda_2 \partial_z \tilde{\alpha}_2(t,\, z) = p_{22}(z)\tilde{Y}(t),\\
&\tau \dot{\tilde{Q}}(t) = -\tilde{Q}(t) + c_2(\tilde{\alpha}_1(t,\, 1) - \tilde{\alpha}_2(t,\, 1)) + p_Q \tilde{Y}(t),
\end{aligned}$$

and boundary conditions

$$\begin{aligned}
\tilde{\alpha}_1(t, 0) &= k_0 \tilde{\beta}_1(t,\, 0),\\
\tilde{\beta}_1(t, 1) &= \tilde{\alpha}_2(t,\, 1) + c_1 \tilde{Q}(t),\\
\tilde{\beta}_2(t, 1) &= \tilde{\alpha}_1(t,\, 1) + c_1 \tilde{Q}(t),\\
\tilde{\alpha}_2(t, 0) &= k_L \tilde{\beta}_2(t,\, 0) - p_0 \tilde{Y}.
\end{aligned}$$

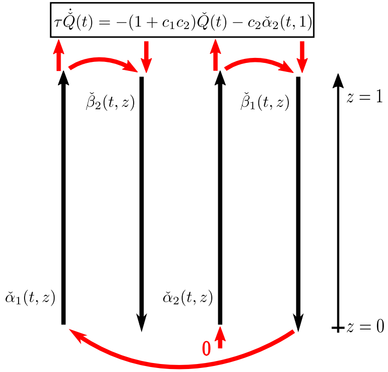
Backstepping-based observer design: target system

To design the observer output injection gains, we map the error estimation dynamics to the following appropriate target system:

$$\partial_t \check{\alpha}_1(t,\, z) + \lambda_1 \partial_z \check{\alpha}_1(t,\, z) = 0,$$
$$\partial_t \check{\beta}_1(t,\, z) - \lambda_1 \partial_z \check{\beta}_1(t,\, z) = 0,$$
$$\partial_t \check{\beta}_2(t,\, z) - \lambda_2 \partial_z \check{\beta}_2(t,\, z) = 0,$$
$$\partial_t \check{\alpha}_2(t,\, z) + \lambda_2 \partial_z \check{\alpha}_2(t,\, z) = 0,$$
$$\tau \dot{\check{Q}}(t) = -(1 + c_1 c_2)\check{Q}(t) - c_2 \check{\alpha}_2(t,\, 1),$$

with boundary conditions

$$\check{\alpha}_1(t, 0) = k_0 \check{\beta}_1(t,\, 0),$$
$$\check{\beta}_1(t, 1) = \check{\alpha}_2(t,\, 1) + c_1 \check{Q}(t),$$
$$\check{\beta}_2(t, 1) = \check{\alpha}_1(t,\, 1) + c_1 \check{Q}(t),$$
$$\check{\alpha}_2(t, 0) = 0.$$



Backstepping-based observer design: transformation

We consider the following backstepping transformation:

$$\tilde{\alpha}_1(t,\, z) = \check{\alpha}_1(t,\, z) - \int_0^1 P_{11}(z,\xi)\check{\beta}_2(t,\xi)d\xi,$$

$$\tilde{\beta}_1(t,\, z) = \check{\beta}_1(t,\, z) - \int_0^1 P_{12}(z,\xi)\check{\beta}_2(t,\xi)d\xi,$$

$$\tilde{\beta}_2(t,\, z) = \check{\beta}_2(t,\, z) - \int_0^z P_{21}(z,\xi)\check{\beta}_2(t,\xi)d\xi,$$

$$\tilde{Q}(t) = \check{Q}(t) - \int_0^1 P_Q(\xi)\check{\beta}_2(t,\xi)d\xi,$$

- Domain of the P_{21} kernel:

$$\mathscr{T}_0 = \{(z,\, \xi) \in \mathbb{R}^2 | 0 \leq z \leq \xi \leq 1\},$$

- Domain of P_{11} and P_{12} kernels:

$$\mathscr{T}_1 = \{(z,\, \xi) \in \mathbb{R}^2 | 0 \leq \xi \leq 1, 0 \leq z \leq 1\},$$

- P_Q is a one-dimensional kernel defined on the interval $\xi \in [0,\, 1]$.

Backstepping-based observer design: kernel equations

Differentiating the transformation with respect to space and time, plugging the target system equation and integrating by parts, we obtain that the error system dynamics is mapped into the target system if and only if the kernels satisfy the following equations:

$$
\begin{aligned}
&\lambda_2 \partial_\xi P_{11}(z,\xi) - \lambda_1 \partial_z P_{11}(z,\xi) = 0,\\
&\lambda_2 \partial_\xi P_{12}(z,\xi) + \lambda_1 \partial_z P_{12}(z,\xi) = 0,\\
&\partial_\xi P_{21}(z,\xi) + \partial_z P_{21}(z,\xi) = 0,\\
&\tau \lambda_2 P_Q'(\xi) = P_Q(\xi) - c_2 P_{11}(1,\xi),
\end{aligned}
$$

and

$$
\begin{aligned}
P_{21}(1,\xi) &= P_{11}(1,\xi) + c_1 P_Q(\xi), & P_{11}(z,1) &= 0,\\
P_Q(1) &= -\frac{c_2}{\tau\lambda_2}, & P_{12}(z,1) &= 0,\\
P_{11}(0,\xi) &= -P_{12}(0,\xi), & P_{12}(1,\xi) &= c_1 P_Q(\xi).
\end{aligned}
$$

The existence and uniqueness of the solution of the kernel equations can be proved in a similar way to the backstepping control design.

Backstepping-based observer design: kernel equations

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\end{aligned}
$$

and

$$
\begin{aligned}
P_{21}(1,\xi) &= P_{11}(1,\xi) + c_1 P_Q(\xi), & P_{11}(z,1) &= 0, \\
P_Q(1) &= -\frac{c_2}{\tau \lambda_2}, & P_{12}(z,1) &= 0, \\
P_{11}(0,\xi) &= -P_{12}(0,\xi), & P_{12}(1,\xi) &= c_1 P_Q(\xi).
\end{aligned}
$$

The existence and uniqueness of the solution of the kernel equations can be proved in a similar way to the backstepping control design.

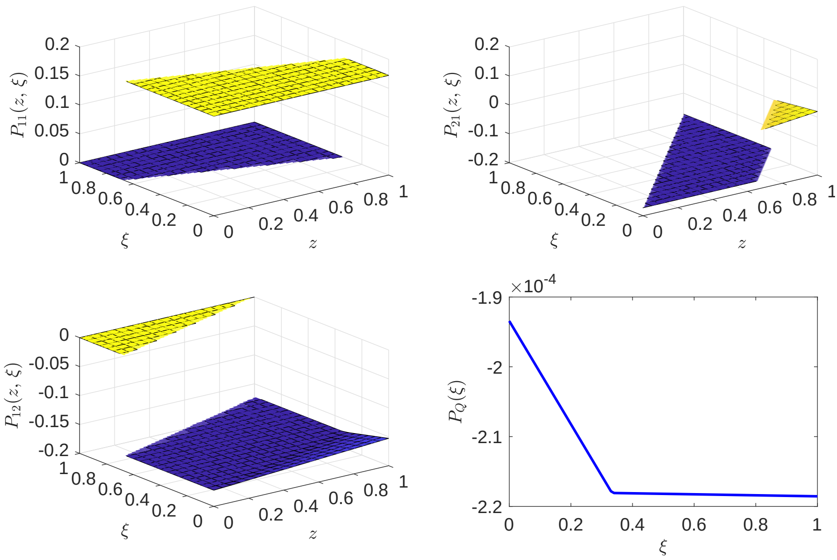
Backstepping-based observer design: observer gains

The closer the heat element is to the unmeasured boundary, the larger the number of pieces of the solution φ and thus, the higher the complexity of the observer.

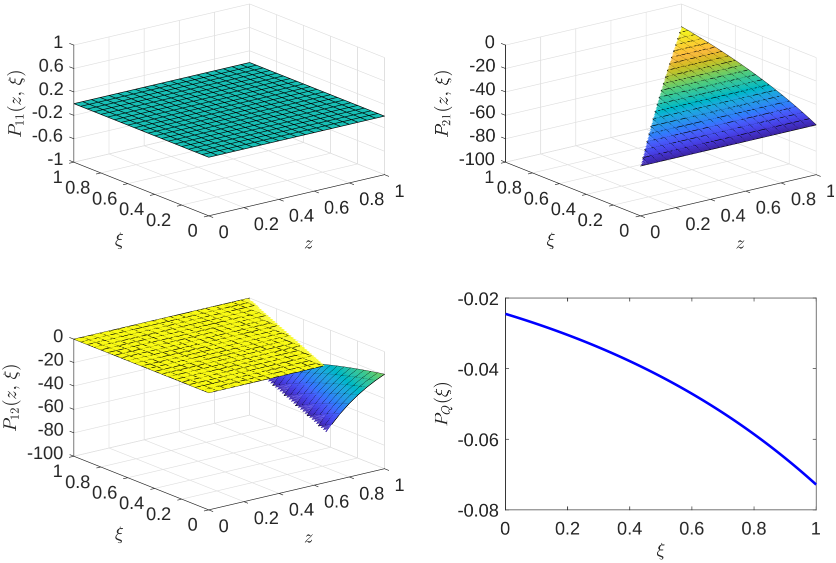
Besides, the observer gains are given by

$$
\begin{aligned}
p_{11}(z) &= \lambda_2 P_{11}(z,0), \\
p_{12}(z) &= \lambda_2 P_{12}(z,0), \\
p_{21}(z) &= \lambda_2 P_{21}(z,0), \\
p_Q &= \tau \lambda_2 P_Q(0).
\end{aligned}
$$

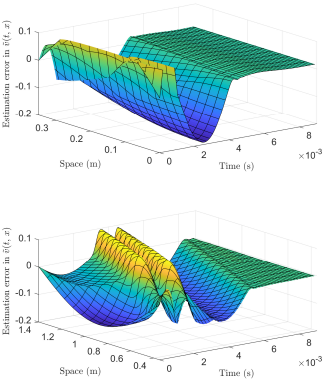
Backstepping-based observer design: kernel visuals (case $\lambda_2 > \lambda_1$)



Backstepping-based observer design: kernel visuals (case $\lambda_2 < \lambda_1$)



Simulation results: Observer error dynamics



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Backstepping-based observer design: Case II

Assume that the system's outputs are

$$y_1(t) = \tilde{P}(t,\, x_m),$$
$$y_2(t) = \tilde{v}(t,\, x_m),$$

with $x_m \in (0, L)$.

The in-domain observer problem is the problem to design an observer that provides accurate online estimates of $\tilde{Q}(t)$, $\tilde{P}(t,\, x)$ and $\tilde{v}(t,\, x)$. The observer must only make use of the system input U and outputs $y_1(t)$ and $y_2(t)$.

Applying a **double folding** transformation into the Riemann coordinates representation of the system we can directly extend the previous backstepping design to solve this problem.

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Applying a **double folding** transformation into the Riemann coordinates representation of the system we can directly extend the previous backstepping design to solve this problem.

Double folding transformation

Define x_0 and x_m as the points to fold the system. We then consider the following piecewise definition of R_1 and R_2:

$$R_1(t,x) = \begin{cases} \alpha_1(t,x), & x \in [0, x_0], \\ \beta_2(t,x), & x \in [x_0, x_m], \\ \alpha_3(t,x), & x \in [x_m, L], \end{cases}$$

$$R_2(t,x) = \begin{cases} \beta_1(t,x), & x \in [0, x_0], \\ \alpha_2(t,x), & x \in [x_0, x_m], \\ \beta_3(t,x), & x \in [x_m, L], \end{cases}$$

and define the following piecewise spatial transformation in z:

$$z = \begin{cases} \frac{x}{x_0}, & x \in [0, x_0], \\ \frac{L-x}{L-x_0}, & x \in [x_0, x_m], \\ \frac{x-x_m}{L-x_m}, & x \in [x_m, L]. \end{cases}$$

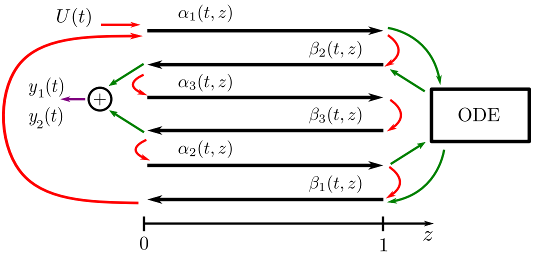
Double folding transformation

This set of scaling and folding transformations allows us to map the system into the following matrix system:

$$\partial_t \alpha(t,z) + \Lambda \partial_z \alpha(t,z) = 0,$$
$$\partial_t \beta(t,z) - \Lambda \partial_z \beta(t,z) = 0,$$
$$\tau \dot{Q}(t) + Q(t) = q(\alpha_1(t,1) - \alpha_2(t,1)),$$

The measurements verify

$$y_1(t) = \frac{1}{2}(\beta_2(t,\,0) + \beta_3(t,\,0)),$$
$$y_2(t) = \frac{1}{2\sqrt{\gamma \overline{P}\overline{\rho}}}\left(\beta_2(t,\,0) - \beta_3(t,\,0)\right).$$



Backstepping-based observer design

We design the observe as a copy of the plant plus output injection terms:

$$\partial_t \hat{\alpha}(t,z) + \Lambda \partial_z \hat{\alpha}(t,z) = -p^+(z)\tilde{y}_{\beta_2}(t),$$
$$\partial_t \hat{\beta}(t,z) - \Lambda \partial_z \hat{\beta}(t,z) = -p^-(z)\tilde{y}_{\beta_2}(t),$$
$$\tau \dot{\hat{Q}}(t) + \hat{Q}(t) = q(\hat{\alpha}_1(t,\,1) - \hat{\alpha}_2(t,\,1)) - p_Q \tilde{y}_{\beta_2}(t),$$

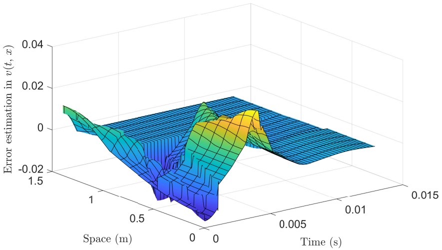
with boundary conditions

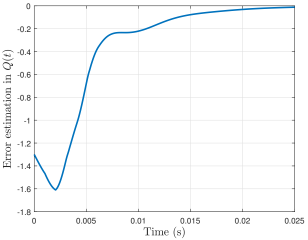
$$\hat{\alpha}(t,0) = N_i \hat{\beta}(t,0) + N_u U(t) + p_{bc}\tilde{y}(t),$$
$$\hat{\beta}(t,\,1) = N_f \hat{\alpha}(t,1), + N_q \hat{Q}(t),$$

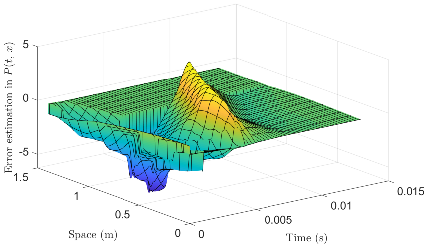
where p^+, p^-, p_Q and p_{bc} are the gains to be found.

The value of the gains can be found using a **similar** target system and backstepping transformation as the previous case!

Simulation results: Observer error dynamics







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  - Transfer function representation of the system
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- Final remarks

Differential-delay observer

Our objective now is to design a state observer that only relies on the measurement of the pressure fluctuations at some arbitrary location x_m:

$$y = \tilde{P}(t, x_m).$$

The challenge of this observer design is that the available measurement is a linear combination of two states in Riemann coordinates.

The backstepping design remains an open problem for this case.

Physical properties of the system

We assume that the physical parameters of the system satisfies the two following conditions:

Condition 1 *(robustness to small measurements delays):* The coefficients k_0 and k_L verify $|k_0| < 1$, $|k_L| < 1$.

Condition 2 *(heat addition to the system is sufficiently small):* The coefficients of the system verify the following inequalities:

$$c_2 c_1 k_0 < -1,$$

$$\frac{2}{\lambda_1} < \tau \frac{\arccos(\frac{1}{c_1 c_2 k_0})}{\sqrt{(c_1 c_2 k_0)^2 - 1}}.$$

Observer design

Copy of the original dynamics (considering x_0 and x_m as the points to fold) with output injection gains

$$\partial_t \hat{\alpha}(t,z) + \Lambda \partial_z \hat{\alpha}(t,z) = 0,$$
$$\partial_t \hat{\beta}(t,z) - \Lambda \partial_z \hat{\beta}(t,z) = 0,$$
$$\tau \dot{\hat{Q}}(t) + \hat{Q}(t) = c_2(\hat{\alpha}_1(t,1) - \hat{\alpha}_2(t,1)) - P_0(2(\tilde{y}(t))),$$

with boundary conditions

$$\hat{\alpha}(t,0) = N_i \hat{\beta}(t,0) + N_u U(t),$$
$$\hat{\beta}(t,1) = N_f \hat{\alpha}(t,1) + N_q \hat{Q}(t),$$

where P_0 is a **Linear operator** to be found.

Error dynamics

Define the estimation error as $\tilde{\alpha} = \alpha - \hat{\alpha}$, $\tilde{\beta} = \beta - \hat{\beta}$ and $\tilde{Q} = Q - \hat{Q}$

$$\partial_t \tilde{\alpha}(t,z) + \Lambda \partial_z \tilde{\alpha}(t,z) = 0,$$
$$\partial_t \tilde{\beta}(t,z) - \Lambda \partial_z \tilde{\beta}(t,z) = 0,$$
$$\tau \dot{\tilde{Q}}(t) + \tilde{Q}(t) = c_2(\tilde{\alpha}_1(t,1) - \tilde{\alpha}_2(t,1)) + 2P_0(\tilde{y}(t)),$$

where the boundary conditions are given by

$$\tilde{\alpha}(t,0) = N_i \tilde{\beta}(t,0),$$
$$\tilde{\beta}(t,1) = N_f \tilde{\alpha}(t,1) + N_q \tilde{Q}(t),$$

Estabilization of the error system: The exponential convergence of $\tilde{Q}$ to zero implies the exponential stabilization of the error system.

Time-delay representation

Objective: Express $\tilde{Q}$ as the solution of a neutral equation and stabilize it.

$$
\begin{aligned}
2\tilde{y}(t) =& \tilde{\beta}_2(t,0) + \tilde{\beta}_3(t,0) = \tilde{\beta}_2(t,0) + k_L \tilde{\beta}_2\left(t - \frac{2}{\lambda_3}, 0\right), \\
\tilde{\beta}_2(t,0) =& k_0 k_L \tilde{\beta}_2(t-\tau,0) + c_1\left(\tilde{Q}\left(t - \frac{1}{\lambda_2}\right) + k_0 \tilde{Q}\left(t - \frac{1}{\lambda_2} - \frac{2}{\lambda_1}\right)\right) \\
\tilde{\alpha}_1(t,1) =& k_0 k_L \tilde{\alpha}_1(t-\tau,1) + k_0 c_1\left(\tilde{Q}\left(t - \frac{2}{\lambda_1}\right) + k_L \tilde{Q}(t-\tau)\right), \\
\tilde{\alpha}_2(t,1) =& k_0 k_L \tilde{\alpha}_2(t-\tau,1) + k_L c_1\left(k_0 \tilde{Q}(t-\tau) + \tilde{Q}\left(t - \frac{2}{\lambda_2} - \frac{2}{\lambda_3}\right)\right) \\
\tau \dot{\tilde{Q}}(t) =& -\tilde{Q}(t) + c_2(\tilde{\alpha}_1(t,1) - \tilde{\alpha}_2(t,1)) + 2P_0(\tilde{y}(t))
\end{aligned}
$$

Laplace transform

We denote s the Laplace variable

$$\begin{aligned} 2\tilde{y}(s) =& (1 + k_L \mathrm{e}^{-\frac{2}{\lambda_3}s})\tilde{\beta}_2(s,0), \\ (1 - k_0 k_L \mathrm{e}^{-\tau s})\tilde{\beta}_2(s,0) =& c_1 \mathrm{e}^{-\frac{1}{\lambda_2}s}(1 + k_0 \mathrm{e}^{-\frac{2}{\lambda_1}s})\tilde{Q}(s), \\ (1 - k_0 k_L \mathrm{e}^{-\tau s})\tilde{\alpha}_1(s,1) =& k_0 c_1 (\mathrm{e}^{-\frac{2}{\lambda_1}s} + k_L \mathrm{e}^{-\tau s})\tilde{Q}(s), \\ (1 - k_0 k_L \mathrm{e}^{-\tau s})\tilde{\alpha}_2(s,1) =& k_L c_1 (k_0 \mathrm{e}^{-\tau s} + \mathrm{e}^{-\frac{2}{\lambda_2}s - \frac{2}{\lambda_3}s})\tilde{Q}(s), \\ (s\tau + 1)\tilde{Q}(s) =& c_2(\tilde{\alpha}_1(s,1) - \tilde{\alpha}_2(s,1)) + 2P_0(\tilde{y}(s)). \end{aligned}$$

The operator $(1 - k_0 k_L \mathrm{e}^{-\tau s})$ does not vanish on the complex Right Half Plane. Multiplying the first and last equation by it, we obtain

$$\begin{aligned} (1 - k_0 k_L \mathrm{e}^{-\tau s})(s\tau + 1)\tilde{Q}(s) &= c_2 c_1 (k_0 \mathrm{e}^{-\frac{2}{\lambda_1}s} - k_L \mathrm{e}^{-\frac{2}{\lambda_2}s - \frac{2}{\lambda_3}s})\tilde{Q}(s) \\ &\quad + (1 - k_0 k_L \mathrm{e}^{-\tau s}) 2P_0(\tilde{y}(s)). \\ 2(1 - k_0 k_L \mathrm{e}^{-\tau s})\tilde{y}(s) &= c_1 \mathrm{e}^{-\frac{1}{\lambda_2}s}(1 + k_L \mathrm{e}^{-\frac{2}{\lambda_3}s})(1 + k_0 \mathrm{e}^{-\frac{2}{\lambda_1}s})\tilde{Q}(s) \end{aligned}$$

Design of the operator P_0

Let us consider the operator P_0 defined by

$$
\begin{aligned}
P_0(\tilde{y}(t)) = & - k_0 P_0 \left(\tilde{y}(t - \frac{2}{\lambda_1})\right) - k_L P_0 \left(\tilde{y}(t - \frac{2}{\lambda_3})\right) - k_0 k_L P_0 \left(\tilde{y}(t - \frac{2}{\lambda_1} - \frac{2}{\lambda_3})\right) \\
& + 2 c_2 k_L \tilde{y} \left(t - \frac{2}{\lambda_3} - \frac{1}{\lambda_2}\right) - 2 c_2 k_L k_0^2 \tilde{y} \left(t - \frac{4}{\lambda_1} - \frac{1}{\lambda_2} - \frac{2}{\lambda_3}\right).
\end{aligned}
$$

Then, the state $\tilde{Q}$ exponentially converges to zero.

<u>Proof:</u> Laplace transform:

$$
\begin{aligned}
& 2(1 + k_L \mathrm{e}^{-\frac{2}{\lambda_3} s})(1 + k_0 \mathrm{e}^{-\frac{2}{\lambda_1} s}) P_0(\tilde{y}(s)) = \\
& \qquad (c_2 k_L \mathrm{e}^{-\frac{1}{\lambda_2} s - \frac{2}{\lambda_3} s} - c_2 k_0^2 k_L \mathrm{e}^{-\frac{2}{\lambda_1} s} \mathrm{e}^{\frac{1}{\lambda_2} s} \mathrm{e}^{-\tau s}) 2 \tilde{y}(s).
\end{aligned}
$$

P_0 is **properly defined**: $(1 + k_L \mathrm{e}^{-\frac{2}{\lambda_3} s})(1 + k_0 \mathrm{e}^{-\frac{2}{\lambda_1} s})$ is strictly positive in the complex RHP.

Design of the operator P_0

Let us consider the operator P_0 defined by

$$P_0(\tilde{y}(t)) = -k_0 P_0\left(\tilde{y}(t-\frac{2}{\lambda_1})\right) - k_L P_0\left(\tilde{y}(t-\frac{2}{\lambda_3})\right) - k_0 k_L P_0\left(\tilde{y}(t-\frac{2}{\lambda_1}-\frac{2}{\lambda_3})\right)$$
$$+ 2c_2 k_L \tilde{y}\left(t-\frac{2}{\lambda_3}-\frac{1}{\lambda_2}\right) - 2c_2 k_L k_0^2 \tilde{y}\left(t-\frac{4}{\lambda_1}-\frac{1}{\lambda_2}-\frac{2}{\lambda_3}\right).$$

Then, the state $\tilde{Q}$ exponentially converges to zero.

Proof: Laplace transform:

$$2(1+k_L e^{-\frac{2}{\lambda_3}s})(1+k_0 e^{-\frac{2}{\lambda_1}s})P_0(\tilde{y}(s)) =$$
$$(c_2 k_L e^{-\frac{1}{\lambda_2}s-\frac{2}{\lambda_3}s} - c_2 k_0^2 k_L e^{-\frac{2}{\lambda_1}s} e^{\frac{1}{\lambda_2}s} e^{-\tau s}) 2\tilde{y}(s).$$

P_0 is **properly defined**: $(1+k_L e^{-\frac{2}{\lambda_3}s})(1+k_0 e^{-\frac{2}{\lambda_1}s})$ is strictly positive in the complex RHP.
We obtain

$$2(1-k_0 k_L e^{-\tau s})P_0(\tilde{y}(s)) = c_1(c_2 k_L e^{-\frac{2}{\lambda_2}s-\frac{2}{\lambda_3}s} - c_2 k_0^2 k_L e^{-\frac{2}{\lambda_1}s} e^{-\tau s})\tilde{Q}(s),$$

Convergence of $\tilde{Q}$ to zero

$$(1 - k_0 k_L \mathrm{e}^{-\tau s})(s\tau + 1)\tilde{Q}(s) = c_2 c_1 (k_0 \mathrm{e}^{-\frac{2}{\lambda_1} s} - k_L \mathrm{e}^{-\frac{2}{\lambda_2} s - \frac{2}{\lambda_3} s})\tilde{Q}(s) + (1 - k_0 k_L \mathrm{e}^{-\tau s}) 2 P_0(\tilde{y}(s)).$$

$$2(1 - k_0 k_L \mathrm{e}^{-\tau s}) P_0(\tilde{y}(s)) = c_1 (c_2 k_L \mathrm{e}^{-\frac{2}{\lambda_2} s - \frac{2}{\lambda_3} s} - c_2 k_0^2 k_L \mathrm{e}^{-\frac{2}{\lambda_1} s} \mathrm{e}^{-\tau s})\tilde{Q}(s),$$

It gives the neutral equation

$$(1 - k_0 k_L \mathrm{e}^{-\tau s})(s\tau + 1)\tilde{Q}(s) = c_2 c_1 (1 - k_0 k_L \mathrm{e}^{-\tau s}) k_0 \mathrm{e}^{-\frac{2}{\lambda_1} s} \tilde{Q}(s).$$

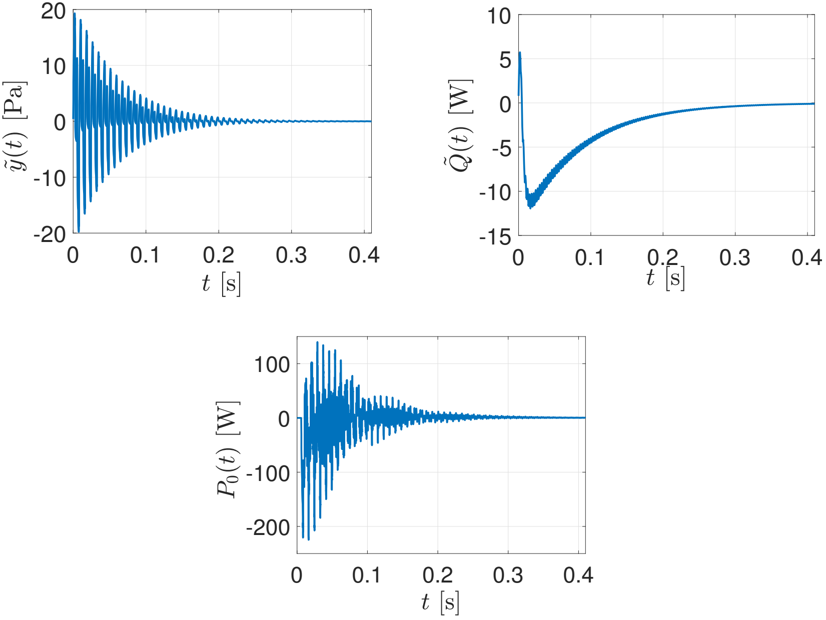
$$\Rightarrow \qquad (s\tau + 1)\tilde{Q}_1(s) = c_2 c_1 k_0 \mathrm{e}^{-\frac{2}{\lambda_1} s} \tilde{Q}_1(s)$$

Detectability of $\tilde{Q}$ from $\tilde{Q}_1 = (1 - k_0 k_L \mathrm{e}^{-\tau s})\tilde{Q}$.

$$\dot{\tilde{Q}}_1(t) = -\frac{1}{\tau}\tilde{Q}_1(t) + \frac{c_2 c_1 k_0}{\tau}\tilde{Q}_1(t - \frac{2}{\lambda_1}),$$

which goes asymptotically to zero since $-\frac{c_2 c_1 k_0}{\tau} > \frac{1}{\tau}$ (Condition 2, applying a Proposition from Nicolescu's book on DDEs).

Simulation results: Observer error dynamics



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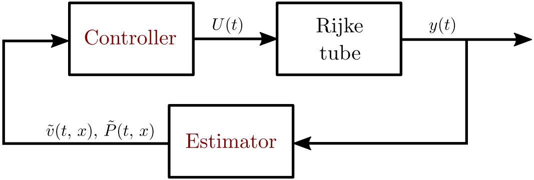
Final remarks

An output feedback control law can be designed using the reconstruction of the estimated states profile through the exponential convergent observer with the measurements.



Since our design is based on the linear system, the separation principle holds; i.e., the combination of a separately designed state feedback controller and observer results in a stabilizing output-feedback controller.

Thanks for your attention