Rapid Stabilization of Timoshenko Beam by PDE Backstepping

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***Abstract*—In this paper, we present rapid boundary stabilization of a Timoshenko beam with anti-damping and anti-stiffness at the uncontrolled boundary, by using infinite-dimensional backstepping. We introduce a Riemann transformation to map the Timoshenko beam states into a set of coordinates that verify a 1-D hyperbolic PIDE-ODE system. Then backstepping is applied to obtain a control law guaranteeing closed-loop stability of the origin in the L^2 sense. Arbitrarily rapid stabilization can be achieved by adjusting control parameters, and has not been achieved in previous results. Finally, a numerical simulation shows the effectiveness of the proposed controller. This result extends a previous work which considered a slender Timoshenko beam with Kelvin-Voigt damping, by allowing destabilizing boundary conditions at the uncontrolled boundary and attaining an arbitrarily rapid convergence rate.**

***Index Terms*—Timoshenko beam, PDE backstepping, hyperbolic systems, boundary control, distributed parameter systems.**

I. Introduction

Flexible beams are widely used in many applications ranging from aerospace to civil structures. Correspondingly, beam stabilization has become an important research topic. Among all the beam models, Timoshenko model, as the most realistic of the 1D distributed parameter models, takes into account both the rotatory inertia of the beam cross-sections and the deflection due to shear effect. In this paper, we focus on the control of such a model by applying the backstepping method.

Extensive literature exists on the control of beams and particularly on Timoshenko beams. We next give an overview of past results. More than three decades ago, Kim and Renardy [18] used a classical boundary damper feedback which required both the space and time derivatives at the tip of the beam. Later, considering a clamped-free Timoshenko beam, Morgul [27] proposed a more general dynamic boundary feedback. Balakrishnan [5], [6] considered boundary conditions leading to superstability (vanishing of the beam states in finite time), for clamped boundary conditions on the uncontrolled end. Taylor and Yau [33] studied a rotating Timoshenko beam that can be stabilized by both applying a force at the free end and a torque at the pivoted end. Xu et al. investigated the use of pointwise feedback controls based on asymptotic analysis of eigenvalues and the eigenfunctions [36]. Soufyane et al. achieved uniform stabilization by using a locally distributed damping; in this case, stability can be guaranteed if and only if the two wave equations have the same speeds [32]. Alabau firstly established the polynomial stabilization for Timoshenko beam with a single damping including the case of different speeds of propagation in the two coupled wave equations [1]–[3]. Macchelli et al. used a distributed port Hamiltonian (dpH) approach to describe Timoshenko beams and proposed a finite dimensional passive controller that shapes the beam's total energy [24]. This approach has also been followed by other authors; for instance, Siuka et al. also adopted a dpH model and proposed an invariant-based method to achieve stabilization [28] and Wu et al. used a passive LQG control design method [35]. Xu presented a boundary feedback design for the exponential stabilization of a Timoshenko beam with both ends free, and gave an explicit asymptotic formula of eigenvalues of the closed loop system [37]. Considering a Timoshenko beam with local Kelvin-Voigt damping, Zhao et al. obtained exponential stability under some additional hypotheses [39]. Krstic et al. extended the backstepping method, by using a singular perturbation approach, to controller and observer design for a slender Timoshenko beam, with actuation only at the beam base and sensing only at the beam tip [19], [29]. For a nonuniform Timoshenko beam with spatial-varying parameters, Ammar-Khodja et al. [4] studied the stabilization for both internal and boundary cases with one control force. He et al. designed an output-feedback control law using a Lyapunov-based approach with a disturbance observers [15]; the Lyapunov approach is a powerful tool in design of control laws for beams, not only in the Timoshenko model (see e.g. [10]). Extending the approach, He et al. proposed an adaptive integral-Barrier Lyapunov function boundary control for inhomogeneous Timoshenko beams with constraints [16]. Considering both system uncertainties and uncertain input backlash non-linearity, He et al. gave vibration boundary control law using a disturbance observer [14]. Allowing for hysteresis of the boundary control input, Liu and Xu proposed a dynamic feedback control law that exponentially stabilized the beam with distributed delay [22]. Yildirim et al. proposed a novel optimal piezoelectric control approach for suppressing vibrations [38]. Finally, to cite several very recent contributions, Ma et al. introduced a prescribed performance function restricting within an arbitrarily small residual set [23], Guo and Meng consider a two-dimensional robust output tracking for a Timoshenko beam equation by using an observer-based error feedback control approach [12], and Mattioni et al. address

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a beam clamped on a moving inertia actuated by an external torque and force with the dpH method using strong dissipation feedback [25], and also in the case of having a mass at the controlled end [26].

In recent years, the backstepping method has proven itself as a powerful design method for control of infinite dimensional systems. However, beyond the results in [19], [29] more than a decade ago, backstepping has not been fully exploited for Timoshenko beam control, even though it has produced results for the shear beam model [20] and the Euler-Bernoulli model [30], [34]. Previous works consider only stable versions of the system and provide controllers which render them exponentially stable, but do not provide arbitrarily fast decay. In the present paper, we achieve rapid stabilization of a Timoshenko beam with anti-damping and anti-stiffness at the uncontrolled boundary. The decay rate can be prescribed arbitrarily by setting the controller parameters. Specifically, we propose an initial Riemann transformations of the Timoshenko beam states to a new set of variables verifying a system of $(2+2)\times(2+2)$ hyperbolic PIDEs[1], coupled with two ODEs. Then the backstepping method is directly applied to controller design of the new system, by extending previously-developed tools [11].

Thus, the main contribution of this paper with respect to previous results is allowing destabilizing boundary conditions at the uncontrolled boundary (numerous works consider simple clamped conditions) of Timoshenko beams and attaining an arbitrarily rapid convergence rate.

The paper is organized as follows: Section II presents the Timoshenko beam model. Section III gives the design of the boundary controller and the main result. Then, Section IV analyzes the resulting controller. Section V studies the closed-loop stability. Finally, Section VI validates the effectiveness of the proposed controllers by numerical simulation and Section VII closes the paper with some concluding remarks.

II. Problem Statement

The goal of this work is to design an exponentially stabilizing feedback control law (with arbitrary convergence rate) for the equilibrium at the origin of the following Timoshenko beam model, which is given by the following system of PDEs

$$\varepsilon u_{tt} = u_{xx} - \alpha_x, \tag{1}$$

$$\mu\alpha_{tt} = \alpha_{xx} + \frac{a}{\varepsilon}\left(u_x - \alpha\right). \tag{2}$$

where $u(x,t)$ denotes the displacement, $\alpha(x,t)$ denotes the angle of rotation, for $x \in (0,1)$, $t > 0$. The positive coefficients ε and μ and the real coefficient a are non-dimensional physical parameters defined in [13]. The boundary conditions of (1)–(2) are

$$u_x(0,t) = \alpha(0,t) - \theta u_t(0,t) - \xi u(0,t), \tag{3}$$

$$u_x(1,t) = V_1(t), \tag{4}$$

$$\alpha_x(0,t) = 0, \tag{5}$$

$$\alpha_x(1,t) = V_2(t). \tag{6}$$

with $\theta, \xi \in \mathbb{R}$ (which represent, respectively, anti-damping and anti-stiffness), and $V_1(t)$ and $V_2(t)$ the actuation variables that have to be designed. The initial conditions for the system (1)–(6) are denoted by $u_0(x) = u(x,0)$, $\alpha_0(x) = \alpha(x,0)$, $u_{0t}(x) = u_t(x,0)$, $\alpha_{0t}(x) = \alpha_t(x,0)$.

Assumption 1: The anti-damping coefficient θ appearing in (3) verifies $\theta \neq \sqrt{\varepsilon}$.

To understand this assumption, consider just a simple wave equation $\varepsilon u_{tt} = u_{xx}$; then, a boundary condition of the type $u_x(0,t) = -\sqrt{\varepsilon}u_t(0,t)$ is ill-posed, since a solution by the method of characteristics (or alternatively, d'Alambert's solution) will end up with one over-determined characteristic and one under-determined characteristic; depending on the other boundary condition, this leads to only trivial or constant functions solving the equation, which in general will not agree with the initial conditions.

III. Controller design and main result

This section presents the design of our boundary control law, starting in Section III-A with a transformation of the Timoshenko beam states to a new set of variables verifying a hyperbolic-ODE coupled system. Next, in Section III-B the control law is introduced and the main result is stated.

A. Transformation of the wave PDE representation of the beam to a system of hyperbolic PDEs coupled with ODEs

As a first step, following the classical Riemann transformation, the Timoshenko beam is mapped into a first-order hyperbolic integro-differential system coupled with ODEs; while Riemann transformations have been used before in the context of Timoshenko beams with easier boundary conditions (see, e.g. [4]), mapping this plant into a PIDE-ODE system is a novelty in controller design for beams. This represents an alternative idea to design a controller for this plant, since it opens the door to apply 1-D hyperbolic backstepping control designs. The system becomes a $(2+2)\times(2+2)$ system of 1-D hyperbolic PIDEs, coupled with two ODEs, by using the following transformations

$$p = u_x + \sqrt{\varepsilon}u_t, \tag{7}$$

$$q = u_x - \sqrt{\varepsilon}u_t, \tag{8}$$

$$r = \alpha_x + \sqrt{\mu}\alpha_t, \tag{9}$$

$$s = \alpha_x - \sqrt{\mu}\alpha_t, \tag{10}$$

$$x_1 = u(0,t), \tag{11}$$

$$x_2 = \alpha(0,t). \tag{12}$$

Then (1)–(6) is equivalent to the PDE-ODE system

$$p_t = \frac{1}{\sqrt{\varepsilon}}p_x - \frac{1}{2\sqrt{\varepsilon}}\left(r+s\right), \tag{13}$$

$$r_t = \frac{1}{\sqrt{\mu}}r_x + \frac{a}{2\varepsilon\sqrt{\mu}}\left(p+q\right) - \frac{a}{2\varepsilon\sqrt{\mu}}\left[\int_0^x \left(r(y,t)+s(y,t)\right)dy + 2x_2\right], \tag{14}$$

$$q_t = -\frac{1}{\sqrt{\varepsilon}}q_x + \frac{1}{2\sqrt{\varepsilon}}\left(r+s\right), \tag{15}$$

[1]The notation $(n+m)\times(n+m)$ is usual in the backstepping literature and it refers to a hyperbolic 1-D system having n convecting (typically uncontrolled) and m counterconvecting (typically controlled) states.

$$s_t = -\frac{1}{\sqrt{\mu}}s_x - \frac{a}{2\varepsilon\sqrt{\mu}}(p+q) + \frac{a}{2\varepsilon\sqrt{\mu}}\left[\int_0^x (r(y,t)+s(y,t))\,dy + 2x_2\right], \quad (16)$$

$$\dot{x}_1 = \frac{1}{\sqrt{\varepsilon}-\theta}\left[\xi x_1 - x_2 + p(0,t)\right], \quad (17)$$

$$\dot{x}_2 = \frac{1}{\sqrt{\mu}}r(0,t). \quad (18)$$

with boundary conditions

$$q(0,t) = -\frac{(\sqrt{\varepsilon}+\theta)}{\sqrt{\varepsilon}-\theta}p(0,t) - \frac{2\sqrt{\varepsilon}}{\sqrt{\varepsilon}-\theta}(\xi x_1 - x_2), \quad (19)$$

$$s(0,t) = -r(0,t), \quad (20)$$

$$p(1,t) = V_p(t), \quad (21)$$

$$r(1,t) = V_r(t). \quad (22)$$

where $V_p(t) = V_1(t) + \sqrt{\varepsilon}u_t(1,t)$ and $V_r(t) = V_2(t) + \sqrt{\mu}\alpha_t(1,t)$ are the redefined control variables for this plant.

It must be noted that (13)–(16) is a system of $(2+2)\times(2+2)$ 1-D hyperbolic PIDEs coupled with two ODEs (17)–(18) that has not been explored before, but facilitates analysis and design of backstepping controllers, as 1-D hyperbolic systems have been widely explored [7]. For instance, the "superstability" (convergence in finite time) result stated in [5], [6], using static output feedback, is only possible with clamped boundary conditions $u(0,t) = \alpha(0,t) = 0$, as they automatically impose that the finite-dimensional states x_1 and x_2 are zero. Note that in that case a straightforward application of the standard backstepping method for hyperbolic systems [17] can achieve finite-time stabilization in the minimum possible time without the need of the additional conditions in [5].

The system (13)–(22) is similar to the one stabilized with backstepping in [11]. Thus, the method therein can be easily adapted. However, as a first step, the procedure requires ordering the states p and r as function of their transport speeds, namely $\frac{1}{\sqrt{\varepsilon}}$ and $\frac{1}{\sqrt{\mu}}$. There are three possible cases: $\frac{1}{\sqrt{\varepsilon}} > \frac{1}{\sqrt{\mu}}$, $\frac{1}{\sqrt{\varepsilon}} < \frac{1}{\sqrt{\mu}}$, and $\frac{1}{\sqrt{\varepsilon}} = \frac{1}{\sqrt{\mu}}$. In what follows, we assume $\frac{1}{\sqrt{\varepsilon}} > \frac{1}{\sqrt{\mu}}$; the case $\frac{1}{\sqrt{\varepsilon}} < \frac{1}{\sqrt{\mu}}$ can be treated analogously by switching the order of the states p and r in all subsequent steps, and the equality case becomes a straightforward extension of the 2×2 backstepping design. Define

$$Z = \begin{bmatrix} p \\ r \end{bmatrix}, Y = \begin{bmatrix} q \\ s \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, V = \begin{bmatrix} V_p \\ V_r \end{bmatrix} \quad (23)$$

Then, (13)–(22) can be written in the following simplified matrix form

$$Z_t = \Sigma Z_x + \Lambda_1(Z+Y) + \Lambda_2 X + \int_0^x F[Z(y,t)+Y(y,t)]dy \quad (24)$$

$$Y_t = -\Sigma Y_x - \Lambda_1(Y+Z) - \Lambda_2 X - \int_0^x F[Z(y,t)+Y(y,t)]dy \quad (25)$$

$$\dot{X} = AX + (B_1 + B_2 C)Z(0,t) \quad (26)$$

with boundary conditions

$$Z(1,t) = V \quad (27)$$

$$Y(0,t) = CZ(0,t) + DX \quad (28)$$

where

$$\Sigma = \begin{bmatrix} \frac{1}{\sqrt{\varepsilon}} & 0 \\ 0 & \frac{1}{\sqrt{\mu}} \end{bmatrix}, \ \Lambda_1 = \begin{bmatrix} 0 & -\frac{1}{2\sqrt{\varepsilon}} \\ \frac{a}{2\varepsilon\sqrt{\mu}} & 0 \end{bmatrix}, \quad (29)$$

$$\Lambda_2 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{a}{\varepsilon\sqrt{\mu}} \end{bmatrix}, \ F = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{a}{2\varepsilon\sqrt{\mu}} \end{bmatrix}, \quad (30)$$

$$A = \begin{bmatrix} \frac{\xi}{\sqrt{\varepsilon}-\theta} & -\frac{1}{\sqrt{\varepsilon}-\theta} \\ 0 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} \frac{1}{\sqrt{\varepsilon}-\theta} & 0 \\ 0 & 0 \end{bmatrix}, \quad (31)$$

$$B_2 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{\sqrt{\mu}} \end{bmatrix}, \ C = \begin{bmatrix} -\frac{\sqrt{\varepsilon}+\theta}{\sqrt{\varepsilon}-\theta} & 0 \\ 0 & -1 \end{bmatrix}, \quad (32)$$

$$D = \begin{bmatrix} -\frac{2\sqrt{\varepsilon}\xi}{\sqrt{\varepsilon}-\theta} & \frac{2\sqrt{\varepsilon}}{\sqrt{\varepsilon}-\theta} \\ 0 & 0 \end{bmatrix}. \quad (33)$$

The system (24)–(28), differently from [11], contains integral coupling terms, and the states of ODEs appearing inside the domain of the PDEs.

B. Stabilizing control law and main result

For system (24)–(28), the following control law is obtained in Section IV.

$$V = \int_0^1 K(1,y)Z(y,t)dy + \int_0^1 L(1,y)Y(y,t)dy + \Phi(1)X \quad (34)$$

whose gain kernels are the particular values of the 2×2 matrices

$$K(x,y) = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, L(x,y) = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}, \quad (35)$$

$$\Phi(x) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}. \quad (36)$$

evaluated at $x = 1$. $K(x,y)$ and $L(x,y)$ are both defined in the triangle $\{(x,y)\in\mathbb{R}^2 | 0\le y\le x\le 1\}$, and $\Phi(x)$ is defined in $[0,1]$. These matrices verify the following (well-posed) kernel equations

$$\Sigma K_x + K_y\Sigma = (K-L)\Lambda_1 - \Omega(x)K - F + \int_y^x [K(x,s)-L(x,s)]Fds, \quad (37)$$

$$\Sigma L_x - L_y\Sigma = (K-L)\Lambda_1 - \Omega(x)L - F + \int_y^x [K(x,s)-L(x,s)]Fds. \quad (38)$$

$$\Phi_x = \Sigma^{-1}\Phi A - \Sigma^{-1}\Lambda_2 - \Sigma^{-1}\Omega(x)\Phi + \int_0^x \Sigma^{-1}(K-L)\Lambda_2 dy + \Sigma^{-1}L(x,0)\Sigma D \quad (39)$$

with boundary conditions for K and L

$$\Sigma L(x,x) + L(x,x)\Sigma = -\Lambda_1, \quad (40)$$

$$\Sigma K(x,x) - K(x,x)\Sigma = -\Lambda_1 + \Omega(x), \quad (41)$$

$$K(x,0)\Sigma - L(x,0)\Sigma C = \Phi B_1 + \Phi B_2 C. \quad (42)$$

with

$$\Omega(x) = \begin{bmatrix} 0 & 0 \\ \omega_{21} & 0 \end{bmatrix} \tag{43}$$

where $\omega_{21}(x,t) = (\frac{1}{\sqrt{\mu}} - \frac{1}{\sqrt{\varepsilon}})k_{21}(x,x) + \frac{a}{2\varepsilon}$ and boundary conditions for $\Phi(x)$ in (39) as follows

$$\Phi(0) = \begin{bmatrix} -\xi - \frac{\delta_1}{\kappa} & 1 \\ 0 & -\delta_2\sqrt{\mu} \end{bmatrix} \tag{44}$$

where $\kappa = 1/(\sqrt{\varepsilon} - \theta)$. The parameters δ_1, δ_2 are arbitrary values which directly determine the decay rate of the closed-loop controlled Timoshenko beam (see Section V). The well-posedness of the kernel equations for $K(x,y), L(x,y), \Phi(x)$ is stated in Theorem 2 in Section IV-B.

Expressing (34) in terms of the Timoshenko beam variables:

$$\begin{aligned} V_1 = &-\int_0^1 \left(k_{11,y}(1,y) + l_{11,y}(1,y)\right) u(y,t)dy \\ &+ \int_0^1 \sqrt{\varepsilon}\left(k_{11}(1,y) + l_{11}(1,y)\right) u_t(y,t)dy \\ &- \int_0^1 \left(k_{12,y}(1,y) + l_{12,y}(1,y)\right) \alpha(y,t)dy \\ &+ \int_0^1 \sqrt{\mu}\left(k_{12}(1,y) + l_{12}(1,y)\right) \alpha_t(y,t)dy \\ &+ \left(k_{11}(1,1) + l_{11}(1,1)\right) u(1,t) \\ &- \left(k_{11}(1,0) + l_{11}(1,0) - \phi_{11}(1)\right) u(0,t) \\ &- \left(k_{12}(1,0) + l_{12}(1,0) - \phi_{12}(1)\right)\alpha(0,t) \\ &+ \left(k_{12}(1,1) + l_{12}(1,1)\right)\alpha(1,t) - \sqrt{\varepsilon}u_t(1,t), \end{aligned} \tag{45}$$

$$\begin{aligned} V_2 = &-\int_0^1 (k_{21,y}(1,y) + l_{21,y}(1,y))u(y,t)dy \\ &+ \int_0^1 \sqrt{\varepsilon}(k_{21}(1,y) + l_{21}(1,y))u_t(y,t)dy \\ &- \int_0^1 (k_{22,y}(1,y) + l_{22,y}(1,y))\alpha(y,t)dy \\ &+ \int_0^1 \sqrt{\mu}(k_{22}(1,y) + l_{22}(1,y))\alpha_t(y,t)dy \\ &+ \left(k_{21}(1,1) + l_{21}(1,1)\right) u(1,t) \\ &- \left(k_{21}(1,0) + l_{21}(1,0) - \phi_{21}(1)\right) u(0,t) \\ &- \left(k_{22}(1,0) + l_{22}(1,0) - \phi_{22}(1)\right) \alpha(0,t) \\ &+ \left(k_{22}(1,1) + l_{22}(1,1)\right) \alpha(1,t) - \sqrt{\mu}\alpha_t(1,t). \end{aligned} \tag{46}$$

The main result is stated next.

***Theorem** 1:* Consider system (1)–(6), with initial conditions $u_0, \alpha_0 \in H^1(0,1), u_{0t}, \alpha_{0t} \in L^2$, under the control law (45)–(46). If the values of δ_1, δ_2 (the controller parameters appearing in (44)) are set large enough so that the constant

$$C_2 = 2\min\{\delta_1, \delta_2\} - 1 \tag{47}$$

is positive, there exists a solution $u(\cdot,t), \alpha(\cdot,t) \in H^1(0,1)$, $u_t(\cdot,t), \alpha_t(\cdot,t) \in L^2(0,1)$ for $t > 0$, and the following inequality is verified, guaranteeing the exponential stability of the equilibrium $u \equiv \alpha \equiv u_t \equiv \alpha_t \equiv 0$:

$$\begin{aligned} &\|u(\cdot,t)\|^2_{H^1} + \|\alpha(\cdot,t)\|^2_{H^1} + \|u_t(\cdot,t)\|^2_{L^2} + \|\alpha_t(\cdot,t)\|^2_{L^2} \\ &\leq C_1 \mathrm{e}^{-C_2 t}\Big(\|u_0\|^2_{H^1} + \|\alpha_0\|^2_{H^1} + \|u_{0t}\|^2_{L^2} + \|\alpha_{0t}\|^2_{L^2}\Big). \end{aligned} \tag{48}$$

The proof of Theorem 1 is given in Section V.

Note that the constant C_2 of Theorem 1 only depends on the controller parameters δ_1 and δ_2. Under Assumption 1 it is always possible to set C_2 as large as desired, thus achieving arbitrary convergence rate.

IV. Controller Analysis

This section presents the steps leading to (34). The backstepping method is used: first, the target system is presented in Section IV-A; next, the backstepping transformation (of Volterra type) is introduced in Section IV-B. The well-posedness of the kernel equations is stated in Theorem 2.

A. Target system

We design a target system as follows

$$\sigma_t = \Sigma\sigma_x + \Omega(x)\sigma, \tag{49}$$

$$\begin{aligned} \psi_t = &-\Sigma\psi_x - \Lambda_1(\psi + \sigma) - \int_0^x \Xi_2(x,y)\sigma(y,t)dy \\ &- \int_0^x \Xi_3(x,y)\psi(y,t)dy - \Xi_1(x)X, \end{aligned} \tag{50}$$

$$\dot{X} = E_1X + E_2\sigma(0,t). \tag{51}$$

with boundary conditions

$$\sigma(1,t) = 0,\ \psi(0,t) = E_3X + C\sigma(0,t) \tag{52}$$

where

$$\sigma = \begin{bmatrix} \eta \\ \beta \end{bmatrix},\ E_1 = (B_1 + B_2C)\Phi(0) + A, \tag{53}$$

$$E_2 = B_1 + B_2C,\ E_3 = C\Phi(0) + D. \tag{54}$$

and where the values of $\Xi_1(x,)$, $\Xi_2(x,y)$, and $\Xi_3(x,y)$ are obtained in terms of the inverse backstepping transformation, in Section IV-B. The stability of this target system is shown in Section V.

B. Backstepping transformation

Firstly, inspired by [21], we introduce the following backstepping transformation, of Volterra type

$$\begin{aligned} \sigma = &Z - \int_0^x K(x,y)\,Z(y,t)dy \\ &- \int_0^x L(x,y)\,Y(y,t)dy - \Phi(x)X(t), \end{aligned} \tag{55}$$

$$\psi = Y. \tag{56}$$

The kernel equations are deduced as usual, by a tedious but straightforward procedure of taking derivatives in the transformation, replacing the original and target equations, and integrating by parts. The details are skipped for brevity. Regarding their well-posedness, the following result holds.

***Theorem** 2:* There exists a unique bounded solution $k_{ij}(x,y), l_{ij}(x,y), i = 1,2; j = 1,2$ to the kernel equations (37)–(42); in particular, there exists a positive number M such that for $i,j = 1,2$

$$|k_{ij}(x,y)|, |l_{ij}(x,y)| \leq M\mathrm{e}^{Mx} \tag{57}$$

The proof follows along the lines of [11] and is skipped; it is based on using the method of characteristics to write (37)–(42) in the form of integral equations and then posing a solution in terms of a successive approximation series, whose convergence is proven recursively. It is evident that the derivations of [11] can be easily adapted to the presence of integral terms and the differences in the boundary conditions without much effort.

Since the kernels appearing in (55) are bounded, the transformation is invertible from the theory of Volterra integral equation. Thus one can define

$$\begin{aligned} Z =&\sigma + \int_0^x \breve{K}(x,y)\,\sigma(y,t)dy \\ &+ \int_0^x \breve{L}(x,y)\,\psi(y,t)dy + \breve{\Phi}(x)\,X, \end{aligned} \tag{58}$$

$$Y = \psi. \tag{59}$$

with bounded kernels. Both the transformation and its inverse map L^2 functions into L^2 functions (see e.g. [17]).

From the inverse transformation, the kernels $\Xi_1(x)$, $\Xi_2(x,y)$, $\Xi_3(x,y)$ appearing in (50) are

$$\Xi_1(x) = \Lambda_1\breve{\Phi}(x) + \Lambda_2 + \int_0^x F\breve{\Phi}(y)dy, \tag{60}$$

$$\Xi_2(x,y) = \Lambda_1\breve{K}(x,y) + F + \int_0^x F\breve{K}(s,y)ds, \tag{61}$$

$$\Xi_3(x,y) = \Lambda_1\breve{L}(x,y) + F + \int_0^x F\breve{L}(s,y)ds. \tag{62}$$

from which it can be deduced that they are bounded kernels. $\Xi_2(x,y)$ and $\Xi_3(x,y)$ are both defined in the triangle $\{(x,y) \in \mathbb{R}^2 | 0 \le y \le x \le 1\}$, and $\Xi_1(x)$ is defined in $[0,1]$.

V. Stability of Closed Loop

This section proves Theorem 1. First, in Section V-A, the solution of (49)–(52) is studied with the method of characteristics, to motivate our choice of target system. This helps to find stability conditions in Section V-B. Then, a Lyapunov analysis in Section V-C shows exponential stability for the full target system. Section V-D finishes the proof of Theorem 1.

A. A semi-explicit solution for the target system

We start solving (49)–(52) with the method of characteristics (see e.g. [17]). Writing down the solution for $\sigma(x,t) = [\eta \;\; \beta]^T$:

$$\eta = \begin{cases} \eta(x + \frac{t}{\sqrt{\varepsilon}}, 0), & 0 \le t \le t_1, \\ 0, & t > t_1 \end{cases}$$

$$\beta = \begin{cases} \beta(x + \dfrac{t}{\sqrt{\mu}}, 0) + \displaystyle\int_0^t \omega_{21}\left(x + \dfrac{t-s}{\sqrt{\mu}}\right) \\ \times\, \eta\left(x + \dfrac{t-s}{\sqrt{\mu}} + \dfrac{s}{\sqrt{\varepsilon}}, 0\right) ds, 0 \le t \le t_1, \\ \beta(x + \dfrac{t}{\sqrt{\mu}}, 0) + \displaystyle\int_0^{t^*(t,x)} \omega_{21}\left(x + \dfrac{t-s}{\sqrt{\mu}}\right) \\ \times\, \eta\left(x + \dfrac{t-s}{\sqrt{\mu}} + \dfrac{s}{\sqrt{\varepsilon}}, 0\right) ds, t_1 \le t \le t_2, \\ 0, \qquad t > t_2. \end{cases} \tag{63}$$

where

$$t_1 = \sqrt{\varepsilon}(1-x), t_2 = \sqrt{\mu}(1-x), t^*(t,x) = \frac{1 - x - \frac{t}{\sqrt{\mu}}}{\frac{1}{\sqrt{\varepsilon}} - \frac{1}{\sqrt{\mu}}}.$$

Thus, $\sigma(x,t)$ converges to zero in finite time $\sqrt{\mu}$. For $t > \sqrt{\mu}$,

$$\begin{aligned} \psi_t(x,t) =& -\Sigma\psi_x(x,t) - \Lambda_1\psi(x,t) - \Xi_1(x)X \\ &- \int_0^x \Xi_3(x,y)\psi(y,t)dy, \\ \dot{X} =& E_1 X, \\ \psi(0,t) =& E_3 X. \end{aligned} \tag{64}$$

Solving for X we get $X(t) = X(0)\mathrm{e}^{E_1 t}$, where we have used the matrix exponential. Then

$$\begin{aligned} \psi_t(x,t) =& -\Sigma\psi_x(x,t) - \Lambda_1\psi(x,t) - \Xi_1(x)X(0)\mathrm{e}^{E_1 t} \\ &- \int_0^x \Xi_3(x,y)\psi(y,t)dy, \\ \psi(0,t) =& E_3 X(0)\mathrm{e}^{E_1 t}. \end{aligned} \tag{65}$$

Applying the method of characteristics, two Volterra-type integral equations can be found for the components of ψ. The details are skipped, but one can always find a unique L^2 solution for ψ.

B. Stability conditions

The only requirement for stability is that E_1 is Hurwitz as then the origin of the state is exponentially stable for (64), see e.g. [11]. Nevertheless, for rapid arbitrary stabilization, the eigenvalues of E_1 need to be set. According to (53), E_1 has the following expression:

$$E_1 = A + (B_1 + B_2 C)\Phi(0) \tag{66}$$

Which, remembering the definition $\kappa = 1/(\sqrt{\varepsilon} - \theta)$, results in

$$E_1 = \begin{bmatrix} \kappa\xi + \kappa\phi_{11}(0) & -\kappa + \kappa\phi_{12}(0) \\ \frac{\phi_{21}(0)}{\sqrt{\mu}} & \frac{\phi_{22}(0)}{\sqrt{\mu}} \end{bmatrix} \tag{67}$$

If we choose the boundary conditions $\Phi(0)$ as follows:

$$\phi_{11}(0) = -\xi - \frac{\delta_1}{\kappa}, \;\; \phi_{12}(0) = 1, \tag{68}$$

$$\phi_{21}(0) = 0, \;\; \phi_{22}(0) = -\delta_2\sqrt{\mu}. \tag{69}$$

with $\delta_1, \delta_2 > 0$, then E_1 is a diagonal matrix with entries $-\delta_1$ and $-\delta_2$, which become its eigenvalues. The rate of convergence of X can be arbitrarily set by adjusting the value δ_1, δ_2 and will be equal to $c = \min\{\delta_1, \delta_2\}$.

C. Lyapunov-based stability analysis of target system

Next, we use a Lyapunov functional for the stability analysis of target system, to show exponential stability of the origin with a fixed convergence rate. Define

$$\begin{aligned} V =& \zeta_1 X^T X + \zeta_2 \int_0^1 \mathrm{e}^{\delta x}\sigma^T(x,t)\Sigma^{-1}\sigma(x,t)dx \\ &+ \int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)\Sigma^{-1}\psi(x,t)dx \end{aligned} \tag{70}$$

Differentiating (70) with respect to t, we have

$$\dot{V} = 2\zeta_1 X^T X_t + 2\zeta_2 \int_0^1 \mathrm{e}^{\delta x} \sigma^T(x,t)\Sigma^{-1}\sigma_t(x,t)dx + 2\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)\Sigma^{-1}\psi_t(x,t)dx \tag{71}$$

Substituting (49)–(51) into (71) and then using integration by parts and the fact that $X^T E_1 X \leq -cX^T X$, we have

$$\begin{aligned}\dot{V} \leq &-2\zeta_1 c X^T X + 2\zeta_1 X^T E_2 \sigma(0,t) - \zeta_2 \sigma^T(0,t)\sigma(0,t)\\ &-\zeta_2 \int_0^1 \mathrm{e}^{\delta x}\sigma^T(x,t)(\delta I - 2\Sigma^{-1}\Omega(x))\sigma(x,t)dx\\ &-\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)(\delta I + 2\Sigma^{-1}\Lambda_1)\psi(x,t)dx\\ &-2\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)\Sigma^{-1}(\Lambda_1\sigma(x,t) + \Xi_1(x)X)dx\\ &-2\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)\Sigma^{-1}\int_0^x \Xi_2(x,y)\sigma(y,t)dydx\\ &-2\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)(\Sigma)^{-1}\int_0^x \Xi_3(x,y)\psi(y,t)dydx\\ &+\psi^T(0,t)\psi(0,t)\end{aligned} \tag{72}$$

Regarding the last line of (72), using $\psi(0,t) = E_3 X + C\sigma(0,t)$, we have $\psi^T(0,t)\psi(0,t) = X^T E_3^T E_3 X + 2X^T E_3^T C\sigma(0,t) + \sigma^T(0,t)C^T C\sigma(0,t)$. Then, the first line and last line of (72) become

$$\begin{aligned}&-X^T(2\zeta_1 cI - E_3^T E_3)X + 2X^T(\zeta_1 E_2 + E_3^T C)\sigma(0,t)\\ &-\sigma^T(0,t)(\zeta_2 I - C^T C)\sigma(0,t)\\ \leq &-(2\zeta_1 c - M_1)X^T X - (\zeta_2 - M_2)\sigma^T(0,t)\sigma(0,t)\end{aligned} \tag{73}$$

with $M_1 = \|E_3\|^2 + 1$, $M_2 = \|\zeta_1 E_2 + E_3^T C\|^2 + \|C\|^2$. The fourth line of (72) is bounded as follows

$$\begin{aligned}&-2\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)\Sigma^{-1}(\Lambda_1\sigma(x,t) + \Xi_1(x)X)dx\\ \leq &(1+M_4)\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)\psi(x,t)dx\\ &+ M_3\int_0^1 \mathrm{e}^{\delta x}\sigma^T(x,t)\sigma(x,t)dx + X^T X\end{aligned} \tag{74}$$

with $M_3 = \|\Sigma^{-1}\Lambda_1\|^2$ and $M_4 = \max_{x\in[0,1]}\|\Sigma^{-1}\Xi_1(x)\|^2$. The fifth line of (72) can be bounded as follows

$$\begin{aligned}&-2\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)(\Sigma)^{-1}\int_0^x \Xi_2(x,y)\sigma(y,t)dydx\\ \leq &2\int_0^1\int_0^1 \mathrm{e}^{-\delta x}|\psi^T(x,t)|\Sigma^{-1}|\Xi_2(x,y)||\sigma(y,t)|dydx\\ \leq &\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)\psi(x,t)dx\\ &+ M_5\int_0^1 \mathrm{e}^{\delta x}\sigma^T(x,t)\sigma(x,t)dx\end{aligned} \tag{75}$$

where $M_5 = \max_{x,y\in[0,1]}\|\Sigma^{-1}\Xi_2(x,y)\|^2$. Finally, the sixth line of (72) is also bounded

$$\begin{aligned}&-2\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)(\Sigma)^{-1}\int_0^x \Xi_3(x,y)\psi(y,t)dydx\\ \leq &2\int_0^1 \mathrm{e}^{-\delta x/2}|\psi^T(x,t)|\Sigma^{-1}\\ &\times\int_0^x \mathrm{e}^{-\delta x/2}|\Xi_3(x,y)||\psi(y,t)|dydx\\ \leq &2\int_0^1\int_0^1 \mathrm{e}^{-\delta x/2}|\psi^T(x,t)|\Sigma^{-1}\\ &\times \mathrm{e}^{-\delta y/2}|\Xi_3(x,y)||\psi(y,t)|dydx\\ \leq &M_6\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)\psi(x,t)dx\end{aligned} \tag{76}$$

with $M_6 = 1 + \max_{x,y\in[0,1]}\|\Sigma^{-1}\Xi_3(x,y)\|^2$. Thus,

$$\begin{aligned}\dot{V} \leq &-(2\zeta_1 c - M_1 - 1)X^T X - (\zeta_2 - M_2)\sigma^T(0,t)\sigma(0,t)\\ &-\int_0^1 \mathrm{e}^{\delta x}\sigma^T(x,t)(\zeta_2\delta I - 2\zeta_2\Sigma^{-1}\Omega(x))\sigma(x,t)dx\\ &+\int_0^1 \mathrm{e}^{\delta x}\sigma^T(x,t)(M_3 + M_5)\sigma(x,t)dx\\ &+\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)(M_6 - 2\Sigma^{-1}\Lambda_1)\psi(x,t)dx\\ &+\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)(2 + M_4 - \delta I)\psi(x,t)dx\end{aligned} \tag{77}$$

Choosing $c = (c'+1)/2$ with $c' > 0$, $\zeta_1 = M_1 + 1$, $\delta > \max\{(2\|\Lambda_1\| + c')\|\Sigma^{-1}\| + M_6 + 2 + M_4, (c' + 2\max_{x\in[0,1]}\|\Omega(x)\|)\|\Sigma^{-1}\| + 1\}$, and $\zeta_2 > \max\{M_3 + M_5, M_2\}$, we obtain:

$$\begin{aligned}\dot{V} \leq &-c'\zeta_1 X^T X - c'\zeta_2\int_0^1 \mathrm{e}^{\delta x}\sigma^T(x,t)\Sigma^{-1}\sigma(x,t)dx\\ &-c'\int_0^1 \mathrm{e}^{-\delta x}\psi^T(x,t)\Sigma^{-1}\psi(x,t)dx \leq -c'V\end{aligned} \tag{78}$$

with $c' = 2\min\{\delta_1, \delta_2\} - 1$. Thus setting the controller parameters δ_1 and δ_2 sufficiently large, an arbitrary convergence rate $c' > 0$ is achieved for V.

From the Lyapunov inequality just obtained and using norm equivalences, and the boundedness of the kernels of both direct (55) and inverse (59) transformations, one obtains

$$\begin{aligned}&\|p(\cdot,t)\|_{L^2}^2 + \|q(\cdot,t)\|_{L^2}^2 + \|r(\cdot,t)\|_{L^2}^2 + \|s(\cdot,t)\|_{L^2}^2\\ &+x_1^2(t) + x_2^2(t)\\ \leq &K_1\mathrm{e}^{-c't}\Big(\|p_0\|_{L^2}^2 + \|q_0\|_{L^2}^2 + \|r_0\|_{L^2}^2 + \|s_0\|_{L^2}^2\\ &+x_1^2(0) + x_2^2(0)\Big)\end{aligned} \tag{79}$$

for some $K_1 > 0$. When rewritten in terms of the physical Timoshenko beam states, the exponential stability bound of Theorem 1 follows, since

$$u(t,x) = x_1(t) + \frac{1}{2}\int_0^x (p(t,y) + q(t,y))dy, \tag{80}$$

$$\alpha(t,x) = x_2(t) + \frac{1}{2}\int_0^x (r(t,y) + s(t,y))dy, \tag{81}$$

$$u_t(t,x) = \frac{p(t,x) - q(t,x)}{2\sqrt{\varepsilon}}, \tag{82}$$

$$\alpha_t(t,x) = \frac{r(t,x) - s(t,x)}{2\sqrt{\mu}} \tag{83}$$

D. Properties of the closed-loop system

Under the assumptions of Theorem 1, we have that $u_0 \in H^1, \alpha_0 \in H^1, u_{0t} \in L^2, \alpha_{0t} \in L^2$, with $u_0(0), \alpha_0(0) \in \mathbb{R}$ and therefore the initial conditions of p, q, r, s belong to L^2. Therefore the initial conditions of the transformed states are also L^2. The target system has an L^2 solution (see Section IV–B); thus the original system has as well, since the inverse transformation maps L^2 into L^2. This finally proves Theorem 1, by (80)–(83).

VI. Numerical Simulation

To verify the effectiveness of the proposed boundary controller, (1)–(6) is simulated with $\varepsilon = 1$, $\mu = 2$, $a = 1$, $\theta = -1$, $\xi = 1$. The initial values are set to $u_0 = 2.8 - 2.8x - 1.8x^2, u_{t0} = 0, \alpha_0 = x^2, \alpha_{t0} = 0$. We use the HPDE tool in MATLAB, in which the four equivalent one-order hyperbolic PDEs (13)–(16) and the ODEs (17)–(18) are solved, and the evolution of $u(x,t), \alpha(x,t)$ is obtained by using (80)–(82). We first show in Fig. 1 the unstable response of the open-loop system, which diverges due to anti-damping. Next, we apply the proposed controller (45)–(46) to the Timoshenko beam. The controller parameters are chosen as $\delta_1 = 5, \delta_2 = 2$. The feedback gains $K(1,y)$, $L(x,y)$ and $\Phi(x)$ are shown in Fig. 2 and were computed using a power series approach as in [8]. There is a discontinuity in the kernel function $k_{12}(1,y)$, which is typically present when applying the backstepping method to a $(2+2)\times(2+2)$ system and does not impact the result [17]. The variables $u(x,t)$, $u_t(x,t)$, $\alpha(x,t)$ and $\alpha_t(x,t)$ evolve as shown in Fig. 3, converging to zero exponentially, as expected from Theorem 1.

VII. Concluding remarks

This work considered boundary control of a Timoshenko beam with anti-damping and anti-stiffness at the uncontrolled boundary; firstly, we transform the Timoshenko beam states into a hyperbolic PIDE-ODE system. Then, backstepping is applied, obtaining arbitrarily fast decay. Simulations show the effectiveness of the controller in setting the convergence rate. A (dual) observer could be designed with the same methodology; such an observer would, in principle, require two measurements at either boundary, resulting in a control law using only boundary information; the analysis of such an observer is left as future work. In addition, the possibility of achieving *finite-time* convergence (in the spirit of the superstability results of [5]), by using time-varying backstepping [31], will be investigated.

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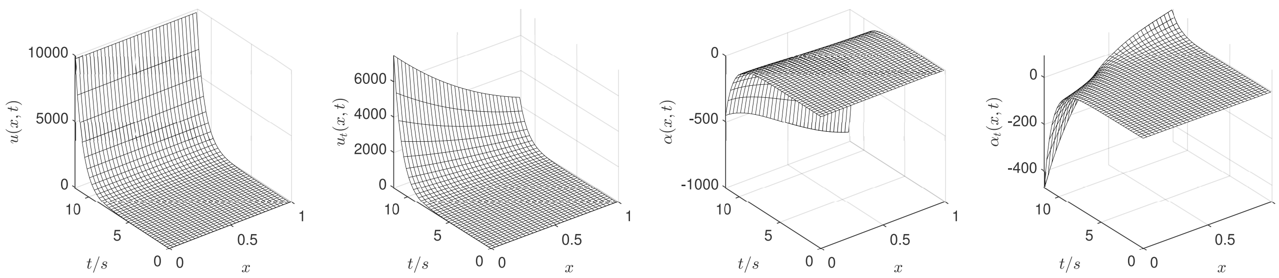

Fig. 1: Open-loop evolution of Timoshenko beam states u, u_t, α, α_t over time.

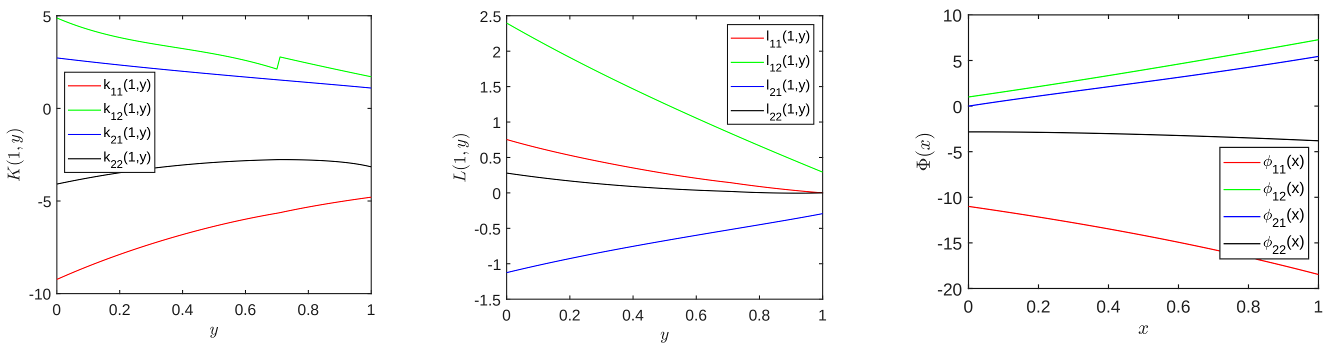

Fig. 2: Feedback gains $K(1,y)$, $L(1,y)$, $\Phi(x)$. Note the discontinuity in the kernel function $k_{12}(1,y)$.

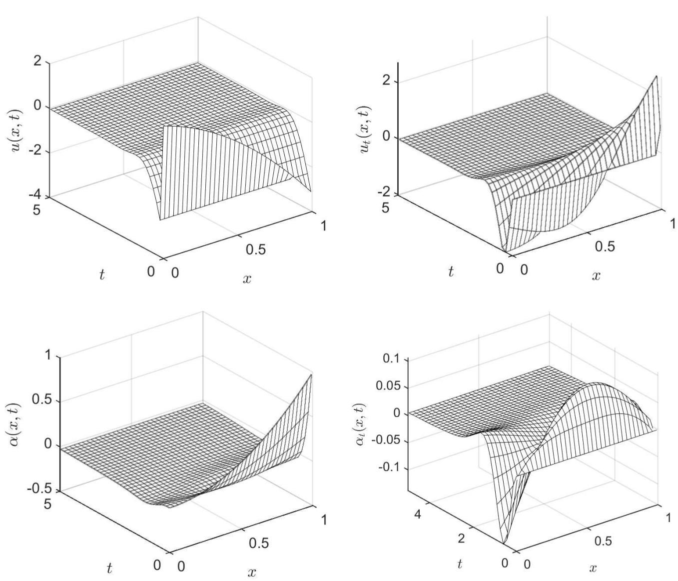

Fig. 3: Closed-loop evolution of Timoshenko beam states u, u_t, α, α_t over time.

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