Backstepping boundary control and state estimation for reaction-diffusion PDEs on arbitrary-dimensional balls

Rafael Vazquez (Univ. Seville, Spain)

Miroslav Krstic (Univ. California San Diego, USA)

Jie Qi (Donghua Univ., Shanghai, China)

UPMC, Paris, 18 September 2015

Outline

- Reaction-diffusion equation on an n-dimensional ball
- Control design: Spherical harmonics & backstepping
- Stability (sketch of proof)
- Observer design
- Extensions & open problems: non-constant coefficients
- Application to motion planning problems

Reaction-diffusion equation on an n-dimensional ball

Let the state $u = u(t,\vec{x})$, with $\vec{x} = [x_1, x_2, \ldots, x_n]^T$, verify

$$\frac{\partial u}{\partial t} \quad = \quad \varepsilon \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \right) + \lambda u = \varepsilon \triangle_n u + \lambda u,$$

for constant $\varepsilon > 0$, $\lambda(r,\vec{\theta})$, and for $t > 0$, in the n-ball $B^n(R)$ defined as

$$B^n(R) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| < R\},$$

with b.c. on the boundary of $B^n(R)$, the $(n-1)$-sphere $S^{n-1}(R)$:

$$S^{n-1}(R) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = R\}.$$

Reaction-diffusion equation on an n-dimensional ball

Let the state $u = u(t, \vec{x})$, with $\vec{x} = [x_1, x_2, \ldots, x_n]^T$, verify

$$\frac{\partial u}{\partial t} = \varepsilon \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \right) + \lambda u = \varepsilon \triangle_n u + \lambda u,$$

for constant $\varepsilon > 0$, $\lambda(r, \vec{\theta})$, and for $t > 0$, in the n-ball $B^n(R)$ defined as

$$B^n(R) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| < R\},$$

with b.c. on the boundary of $B^n(R)$, the $(n-1)$-sphere $S^{n-1}(R)$:

$$S^{n-1}(R) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = R\}.$$

The b.c. is of Dirichlet type:

$$u(t, \vec{x})\Big|_{\vec{x} \in S^{n-1}(R)} = U(t, \vec{x})$$

where $U(t, \vec{x})$ is the actuation variable.

Reaction-diffusion equation on an n-dimensional ball

Ball geometry is the simplest possible n-dimensional geometry, appears in applications (typically $n = 2, 3$).

Unstable system for large values of $\frac{\lambda}{\varepsilon}$

Objective: find an explicit stabilizing feedback law

Reaction-diffusion equation on an n-dimensional ball

Ball geometry is the simplest possible n-dimensional geometry, appears in applications (typically $n = 2, 3$).

Unstable system for large values of $\frac{\lambda}{\varepsilon}$

Objective: find an explicit stabilizing feedback law

Inspiration: The backstepping method stabilizes the 1-D problem

$$u_t = \varepsilon u_{xx} + \lambda u,\ x \in [0, L],\ u(t,0) = 0,\ u(t,L) = U(t)$$

with feedback law (Smyshlyaev&Krstic 2002, published in IEEE TAC 2004)

$$U(t,x) = \int_0^L -\xi \frac{\lambda}{\varepsilon} \frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(L^2 - \xi^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(L^2 - \xi^2)}} u(t,\xi) d\xi$$

Can we obtain a similar result?

Can we obtain an explicit feedback law?

Utility of an explicit control law:

- Understanding the structure of the control law
- Understanding the dependence with respect to parameters of the plant
- Very easy and precise to implement (rare commodity in PDEs)
- Adaptive control!

Explicit solutions are possible for this case (constant coefficients ε and λ, arbitrary dimension)!

Ultraspherical coordinates

The n-ball domain is well described in n-dimensional spherical coordinates, also known as ultraspherical coordinates:

- one radial coordinate r, $r \in [0,R)$.
- $n-1$ angular coordinates: $\vec{\theta} = [\theta_1, \theta_2, \ldots, \theta_{n-1}]^T$, with $\theta_1 \in [0, 2\pi)$ and $\theta_i \in [0,\pi]$ for $2 \leq i \leq n-1$.

Definition:

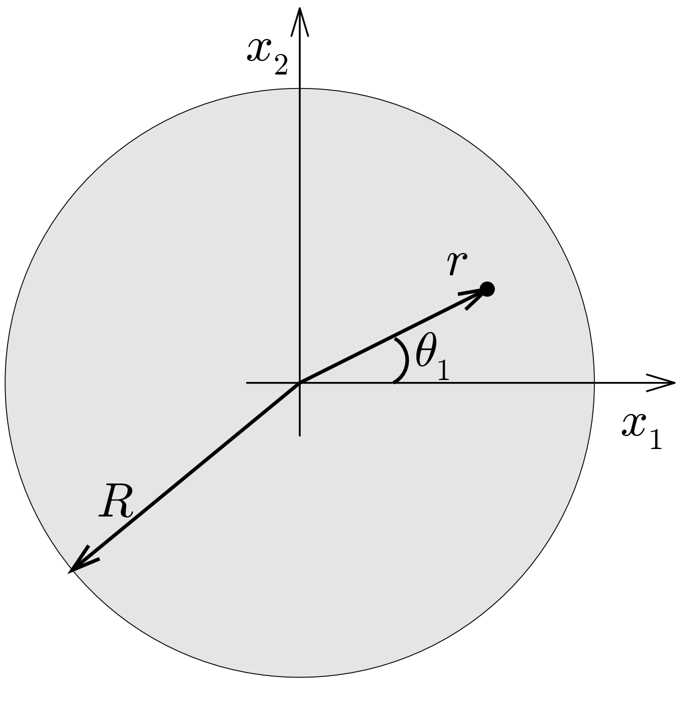
$$
\begin{aligned}
x_1 &= r\cos\theta_1 \sin\theta_2 \sin\theta_3 \ldots \sin\theta_{n-1}, \\
x_2 &= r\sin\theta_1 \sin\theta_2 \sin\theta_3 \ldots \sin\theta_{n-1}, \\
x_3 &= r\cos\theta_2 \sin\theta_3 \ldots \sin\theta_{n-1}, \\
&\vdots \\
x_{n-1} &= r\cos\theta_{n-2} \sin\theta_{n-1}, \\
x_n &= r\cos\theta_{n-1}.
\end{aligned}
$$

Ultraspherical coordinates: Examples

n=2

Polar coordinates: $r \in [0,R)$, $\theta_1 \in [0,2\pi)$.

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1 \end{aligned}$$

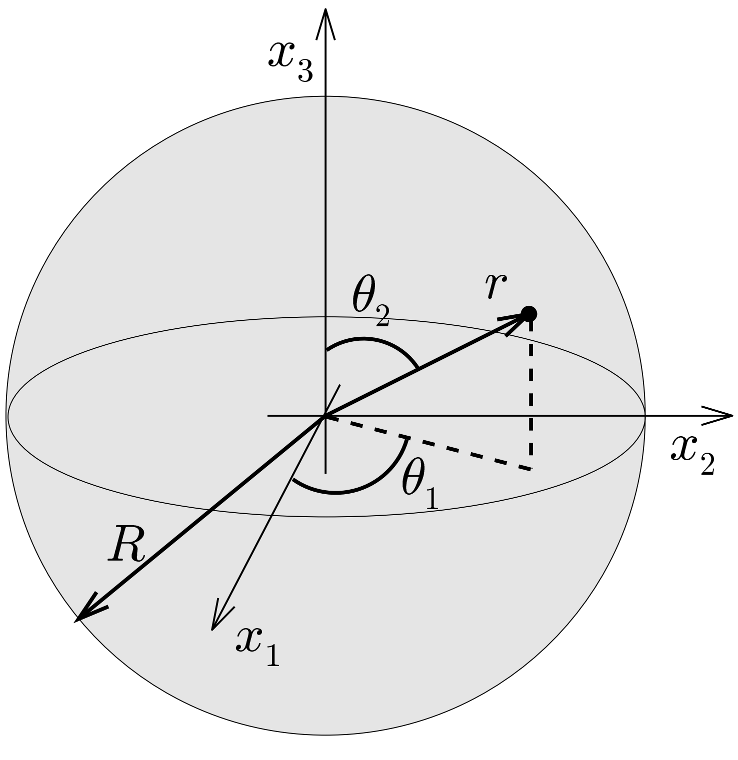


Ultraspherical coordinates: Examples

n=3

Spherical coordinates: $r \in [0,R)$, $\theta_1 \in [0,2\pi)$, $\theta_2 \in [0,\pi]$

$$\begin{aligned} x_1 &= r\cos\theta_1\cos\theta_2 \\ x_2 &= r\sin\theta_1\cos\theta_2 \\ x_3 &= r\sin\theta_2 \end{aligned}$$



Laplacian in ultraspherical coordinates

Writing the reaction diffusion equation in ultraspherical coordinates

$$
\begin{aligned}
u_t &= \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u\right)+\frac{1}{r^2}\triangle^*_{n-1}u+\lambda u,\\
u(t,R,\vec{\theta}\,) &= U(t,\vec{\theta}\,),
\end{aligned}
$$

where $\triangle^*_{n-1}$ is called the Laplace-Beltrami operator and represents the Laplacian over the $(n-1)$-sphere.

It is defined recursively as

$$
\begin{aligned}
\triangle^*_1 &= \frac{\partial^2}{\partial\theta_1^2},\\
\triangle^*_n &= \frac{1}{\sin^{n-1}\theta_n}\frac{\partial}{\partial\theta_n}\left(\sin^{n-1}\theta_n\frac{\partial}{\partial\theta_n}\right)+\frac{\triangle^*_{n-1}}{\sin^2\theta_n},
\end{aligned}
$$

Example:

$$
\triangle^*_2 = \frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial}{\partial\theta_2}\right)+\frac{1}{\sin^2\theta_2}\frac{\partial^2}{\partial\theta_1^2}.
$$

Designing a boundary feedback law

- Exploit periodicity in $\vec{\theta}$ by using Spherical Harmonics
- Apply the backstepping method to each harmonic coefficient
- Solve the backstepping kernel equations to find a feedback law for each harmonic
- Re-assemble the feedback law in Spherical Harmonics back to physical space

Spherical Harmonics

Develop u and U in term of Spherical Harmonics coefficients u_l^m and U_l^m:

$$u(t,r,\vec{\theta}) = \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=N(l,n)-1} u_l^m(r,t) Y_{lm}^n(\vec{\theta}), \quad U(t,\vec{\theta}) = \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=N(l,n)-1} U_l^m(t) Y_{lm}^n(\vec{\theta}),$$

$N(l,n)$: number of (linearly independent) n-dimensional spherical harmonics of degree l

$$N(l,n) = \frac{2l+n-2}{l} \binom{l+n-3}{l-1}, \quad l>0; \qquad N(0,n)=1$$

$Y_{lm}^n(\vec{\theta})$: m-th order n-dimensional spherical harmonic of degree l

Coefficients are defined as:

$$u_l^m(r,t) = \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} u(t,r,\vec{\theta}) \bar{Y}_{lm}^n(\vec{\theta}) \sin^{n-2}\theta_{n-1} \sin^{n-3}\theta_{n-2} \dots \sin\theta_2 d\vec{\theta},$$

$$U_l^m(t) = \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} U(t,\vec{\theta}) \bar{Y}_{lm}^n(\vec{\theta}) \sin^{n-2}\theta_{n-1} \sin^{n-3}\theta_{n-2} \dots \sin\theta_2 d\vec{\theta},$$

($d\vec{\theta} = d\theta_{n-1} d\theta_{n-2} \dots d\theta_2 d\theta_1$, $\bar{Y}_{lm}^n$ is the complex conjugate of Y_{lm}^n)

Spherical Harmonics

The n-dimensional spherical harmonics are **eigenfunctions** for the Laplacian $\triangle^*_{n-1}$:

$$\triangle^*_{n-1} Y^n_{lm} = -l(l+n-2)Y^n_{lm}.$$

Thus, each harmonic coefficient $u^m_l(t,r)$ for $l \in \mathbb{N}$ and $0 \leq m \leq N(l,n)$, verifies

$$\partial_t u^m_l = \frac{\varepsilon}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r u^m_l \right) - l(l+n-2) \frac{\varepsilon}{r^2} u^m_l + \lambda u^m_l,$$

evolving in $r \in [0,R]$, $t > 0$, with boundary conditions

$$u^m_l(t,R) \quad = \quad U^m_l(t),$$

The PDEs for the harmonics are not coupled: we can independently design each U^m_l and later assemble all of the them to find an expression for U.

Backstepping control of Spherical Harmonics coefficients

To design $U_l^m(t)$ seek transformation of

$$\partial_t u_l^m = \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}u_l^m + \lambda u_l^m$$

into the (stable) target system

$$\partial_t w_l^m = \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r w_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}w_l^m$$

with boundary conditions

$$w_l^m(t,R) = 0$$

Backstepping control of Spherical Harmonics coefficients

To design $U_l^m(t)$ seek transformation of

$$\partial_t u_l^m = \frac{\varepsilon}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r u_l^m \right) - l(l+n-2) \frac{\varepsilon}{r^2} u_l^m + \lambda u_l^m$$

into the (stable) target system

$$\partial_t w_l^m = \frac{\varepsilon}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r w_l^m \right) - l(l+n-2) \frac{\varepsilon}{r^2} w_l^m$$

with boundary conditions

$$w_l^m(t,R) = 0$$

The transformation is

$$w_l^m(t,r) = u_l^m(t,r) - \int_0^r K_{lm}^n(r,\rho) u_l^m(t,\rho) d\rho$$

with kernels K_{lm}^n to be found.

Backstepping control of Spherical Harmonics coefficients

To design $U_l^m(t)(t)$ seek transformation of

$$\partial_t u_l^m = \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}u_l^m + \lambda u_l^m$$

into the (stable) target system

$$\partial_t w_l^m = \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r w_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}w_l^m$$

with boundary conditions

$$w_l^m(t,R) = 0$$

The transformation is

$$w_l^m(t,r) = u_l^m(t,r) - \int_0^r K_{lm}^n(r,\rho)u_l^m(t,\rho)d\rho$$

with kernels K_{lm}^n to be found.

Substituting at $r=R$ we find $U_l^m(t)$ as

$$U_l^m(t)(t) = \int_0^R K_{lm}^n(R,\rho)u_l^m(t,\rho)d\rho$$

Kernel equation

The control kernels $K^n_{lm}(r,\rho)$ are found, for a given $n \geq 2$ and each l,m, from

$$\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r K^n_{lm}\right)-\partial_\rho\left(\rho^{n-1}\partial_\rho\left(\frac{K^n_{lm}}{\rho^{n-1}}\right)\right)-l(l+n-2)\left(\frac{1}{r^2}-\frac{1}{\rho^2}\right)K^n_{lm}=\frac{\lambda}{\varepsilon}K^n_{lm}.$$

with BC

$$\begin{aligned}
\lambda+2\varepsilon\frac{d}{dr}\left(K^n_{lm}(r,r)\right) &= 0\\
K^n_{lm}(r,0) &= 0\\
(n-2)\partial_\rho K^n_{lm}(r,\rho)|_{\rho=0} &= 0
\end{aligned}$$

Kernel equation

The control kernels $K_{lm}^n(r,\rho)$ are found, for a given $n \geq 2$ and each l,m, from

$$\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r K_{lm}^n\right) - \partial_\rho\left(\rho^{n-1}\partial_\rho\left(\frac{K_{lm}^n}{\rho^{n-1}}\right)\right) - l(l+n-2)\left(\frac{1}{r^2}-\frac{1}{\rho^2}\right)K_{lm}^n = \frac{\lambda}{\varepsilon}K_{lm}^n.$$

with BC

$$\begin{aligned} \lambda + 2\varepsilon\frac{d}{dr}\left(K_{lm}^n(r,r)\right) &= 0 \\ K_{lm}^n(r,0) &= 0 \\ (n-2)\partial_\rho K_{lm}^n(r,\rho)|_{\rho=0} &= 0 \end{aligned}$$

The first BC integrates (using $K_{lm}^n(0,0)=0$) to

$$K_{lm}^n(r,r) = -\int_0^r \frac{\lambda}{2\varepsilon}d\rho = -\frac{\lambda r}{2\varepsilon}$$

Solving the kernel equation

To solve

$$
\begin{aligned}
\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r K^n_{lm}\right)-\partial_\rho\left(\rho^{n-1}\partial_\rho\left(\frac{K^n_{lm}}{\rho^{n-1}}\right)\right)-l(l+n-2)\left(\frac{1}{r^2}-\frac{1}{\rho^2}\right)K^n_{lm} &= \frac{\lambda}{\varepsilon}K^n_{lm}\\
K^n_{lm}(r,r) &= -\frac{\lambda r}{2\varepsilon}\\
K^n_{lm}(r,0) &= 0\\
(n-2)\partial_\rho K^n_{lm}(r,\rho)|_{\rho=0} &= 0
\end{aligned}
$$

define $K^n_{lm}(r,\rho)=G^n_{lm}(r,\rho)\rho\left(\frac{\rho}{r}\right)^{l+n-2}$. The two last BCs are automatically verified, and writing the kernel equation in terms of G^n_{lm}

$$
\begin{aligned}
\partial_{rr}G^n_{lm}+(3-n-2l)\frac{\partial_r G^n_{lm}}{r}-\partial_{\rho\rho}G^n_{lm}+(1-n-2l)\frac{\partial_\rho G^n_{lm}}{\rho} &= \frac{\lambda}{\varepsilon}G^n_{lm}\\
G^n_{lm}(r,r) &= -\frac{\lambda}{2\varepsilon}
\end{aligned}
$$

Solving the kernel equation

To solve

$$
\begin{aligned}
\partial_{rr}G^n_{lm} + (3-n-2l)\frac{\partial_r G^n_{lm}}{r} - \partial_{\rho\rho}G^n_{lm} + (1-n-2l)\frac{\partial_\rho G^n_{lm}}{\rho} &= \frac{\lambda}{\varepsilon}G^n_{lm} \\
G^n_{lm}(r,r) &= -\frac{\lambda}{2\varepsilon}
\end{aligned}
$$

assume a solution of the form $G^n_{lm}(r,\rho) = \Phi\left(\left(\frac{\lambda}{\varepsilon}(r^2-\rho^2)\right)^{1/2}\right)$, where $\Phi(s)$ is to be found (independent of n, l and m!).

We find, calling $x = \left(\frac{\lambda}{\varepsilon}(r^2-\rho^2)\right)^{1/2}$,

$$
\begin{aligned}
\Phi''(x) + \frac{3}{x}\Phi'(x) - \Phi(x) &= 0 \\
\Phi(0) &= -\frac{\lambda}{2\varepsilon}
\end{aligned}
$$

Note that n, l and m do not appear in the equation.

Note that we have gone from a PDE to an ODE.

Solving the kernel equation

To solve

$$\begin{aligned} \Phi''(x) + \frac{3}{x}\Phi'(x) - \Phi(x) &= 0 \\ \Phi(0) &= -\frac{\lambda}{2\varepsilon} \end{aligned}$$

call $\Psi(x) = x\Phi(x)$:

$$\left(\frac{\Psi''}{x} - 2\frac{\Psi'}{x^2} + 2\frac{\Psi}{x^3}\right) + \frac{3}{x}\left(\frac{\Psi'}{x} - \frac{\Psi}{x^2}\right) - \frac{\Psi}{x} = 0$$

which cross-multiplied by x^3 gives

$$x^2\Psi'' + x\Psi' - (1 + x^2)\Psi = 0$$

.

Solving the kernel equation

To solve

$$\begin{aligned}\Phi''(x) + \frac{3}{x}\Phi'(x) - \Phi(x) &= 0 \\ \Phi(0) &= -\frac{\lambda}{2\varepsilon}\end{aligned}$$

call $\Psi(x) = x\Phi(x)$:

$$\left(\frac{\Psi''}{x} - 2\frac{\Psi'}{x^2} + 2\frac{\Psi}{x^3}\right) + \frac{3}{x}\left(\frac{\Psi'}{x} - \frac{\Psi}{x^2}\right) - \frac{\Psi}{x} = 0$$

which cross-multiplied by x^3 gives

$$x^2\Psi'' + x\Psi' - (1 + x^2)\Psi = 0$$

Bessel's modified differential equation of order 1, whose bounded solution is

$$\Psi(x) = C_1 \mathrm{I}_1(x)$$

where I_1 is the first-order modified Bessel function of the first kind.

Solving the kernel equation

Undoing all the transformations:

$$\Phi(x) = C_1 \frac{\mathrm{I}_1(x)}{x}$$

since $\Phi(0) = -\frac{\lambda}{2\varepsilon}$ and $\lim_{x\to 0} \frac{\mathrm{I}_1(x)}{x} = 1/2$ we obtain $C_1 = -\frac{\lambda}{\varepsilon}$

Solving the kernel equation

Undoing all the transformations:

$$\Phi(x) = C_1 \frac{\mathrm{I}_1(x)}{x}$$

since $\Phi(0) = -\frac{\lambda}{2\varepsilon}$ and $\lim_{x\to 0} \frac{\mathrm{I}_1(x)}{x} = 1/2$ we obtain $C_1 = -\frac{\lambda}{\varepsilon}$

Thus

$$\Phi(x) = -\frac{\lambda}{\varepsilon} \frac{\mathrm{I}_1(x)}{x}$$

Solving the kernel equation

Undoing all the transformations:

$$\Phi(x) = C_1 \frac{\mathrm{I}_1(x)}{x}$$

since $\Phi(0) = -\frac{\lambda}{2\varepsilon}$ and $\lim_{x\to 0} \frac{\mathrm{I}_1(x)}{x} = 1/2$ we obtain $C_1 = -\frac{\lambda}{\varepsilon}$

Thus

$$\Phi(x) = -\frac{\lambda}{\varepsilon} \frac{\mathrm{I}_1(x)}{x}$$

therefore

$$G^n_{lm}(r,\rho) = -\frac{\lambda}{\varepsilon} \frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}}$$

Solving the kernel equation

Undoing all the transformations:

$$\Phi(x) = C_1 \frac{\mathrm{I}_1(x)}{x}$$

since $\Phi(0) = -\frac{\lambda}{2\varepsilon}$ and $\lim_{x\to 0} \frac{\mathrm{I}_1(x)}{x} = 1/2$ we obtain $C_1 = -\frac{\lambda}{\varepsilon}$

Thus

$$\Phi(x) = -\frac{\lambda}{\varepsilon}\frac{\mathrm{I}_1(x)}{x}$$

therefore

$$G_n(r,\rho) = -\frac{\lambda}{\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}}$$

and finally

$$K_{lm}^{n}(r,\rho) = -\rho\left(\frac{\rho}{r}\right)^{l+n-2}\frac{\lambda}{\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}}$$

Explicit feedback law

The feedback law for each spherical harmonic is

$$U_l^m(t) = \int_0^R K_{lm}^n(R,\rho)u_l^m(t,\rho)d\rho = \int_0^R -\rho\left(\frac{\rho}{R}\right)^{l+n-2}\frac{\lambda}{\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}}u_l^m(t,\rho)d\rho$$

Summing to obtain the physical-space feedback law

$$\begin{aligned}
U(t,\vec{\theta}) &= \sum_{l=0}^{l=\infty}\sum_{m=0}^{m=N(l,n)-1} U_l^m(t)Y_{lm}^n(\vec{\theta}) \\
&= \sum_{l=0}^{l=\infty}\sum_{m=0}^{m=N(l,n)-1} \int_0^R -\rho\left(\frac{\rho}{R}\right)^{l+n-2}\frac{\lambda}{\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}}u_l^m(t,\rho)d\rho Y_{lm}^n(\vec{\theta})
\end{aligned}$$

Formally exchanging the integral with the infinite sum (it can be proved correct)

$$U(t,\theta) = \int_0^R -\rho\frac{\lambda}{\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}}\left[\sum_{l=0}^{l=\infty}\sum_{m=0}^{m=N(l,n)-1}\left(\frac{\rho}{R}\right)^{l+n-2}u_l^m(t,\rho)Y_{lm}^n(\vec{\theta})\right]d\rho$$

Explicit feedback law

In the term in brackets, inserting the definition of u_l^m in terms of u

$$\sum_{l=0}^{l=\infty} \sum_{m=0}^{m=N(l,n)-1} \left(\frac{\rho}{R}\right)^{l+n-2} u_l^m(t,\rho) Y_{lm}^n(\vec{\theta})$$

$$= \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=N(l,n)-1} \left(\frac{\rho}{R}\right)^{l+n-2} \int_0^{\pi} \ldots \int_0^{\pi} \int_0^{2\pi} u(t,r,\vec{\phi}) \bar{Y}_{lm}^n(\vec{\phi}) \sin^{n-2}\phi_{n-1} \sin^{n-3}\phi_{n-2} \ldots \sin\phi_2 d\vec{\phi} Y_{lm}^n(\vec{\theta})$$

Explicit feedback law

In the term in brackets, inserting the definition of u_l^m in terms of u

$$
\begin{aligned}
&\sum_{l=0}^{l=\infty}\sum_{m=0}^{m=N(l,n)-1}\left(\frac{\rho}{R}\right)^{l+n-2} u_l^m(t,\rho)Y_{lm}^n(\vec{\theta}) \\
=&\sum_{l=0}^{l=\infty}\sum_{m=0}^{m=N(l,n)-1}\left(\frac{\rho}{R}\right)^{l+n-2}\int_0^{\pi}\dots\int_0^{\pi}\int_0^{2\pi} u(t,r,\vec{\phi})\bar{Y}_{lm}^n(\vec{\phi})\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\dots\sin\phi_2 d\vec{\phi}Y_{lm}^n(\vec{\theta})
\end{aligned}
$$

The Addition Theorem for Spherical Harmonics states:

$$
\sum_{m=0}^{N(l,n)-1} Y_{lm}^n(\vec{\theta})\bar{Y}_{lm}^n(\vec{\phi}) = \frac{N(l,n)}{\mathrm{Area}(S^{n-1})}P_{l,n}(\cos\omega)
$$

where $P_{l,n}$ is the Legendre polynomial of degree l in n dimensions, $\mathrm{Area}(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the surface area of the unit $(n-1)$-sphere, and ω is the **geodesic distance** between the points given by $\vec{\theta}$ and $\vec{\phi}$ on the unit $(n-1)$-sphere:

$$
\begin{aligned}
\omega \;=\; & \cos^{-1}\{\cos\phi_{n-1}\cos\theta_{n-1}+\sin\phi_{n-1}\sin\theta_{n-1}\times[\cos\phi_{n-2}\cos\theta_{n-2}+\sin\phi_{n-2}\sin\theta_{n-2} \\
& \times[\dots[\cos\phi_2\cos\theta_2+\sin\phi_2\sin\theta_2\cos(\theta_1-\phi_1)]\dots]]\}.
\end{aligned}
$$

Explicit feedback law

Thus, the term in brackets is

$$\sum_{l=0}^{l=\infty} \sum_{m=0}^{m=N(l,n)-1} \left(\frac{\rho}{R}\right)^{l+n-2} u_l^m(t,\rho) Y_{lm}^n(\vec{\theta})$$

$$= \sum_{l=0}^{l=\infty} \left(\frac{\rho}{R}\right)^{l+n-2} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} u(t,r,\vec{\phi}) \frac{N(l,n) P_{l,n}(\cos\omega)}{\mathrm{Area}(S^{n-1})} \sin^{n-2}\phi_{n-1} \sin^{n-3}\phi_{n-2} \ldots \sin\phi_2 \, d\vec{\phi}$$

Explicit feedback law

Thus, the term in brackets is

$$\sum_{l=0}^{l=\infty}\sum_{m=0}^{m=N(l,n)-1}\left(\frac{\rho}{R}\right)^{l+n-2}u_l^m(t,\rho)Y_{lm}^n(\vec{\theta})$$

$$=\sum_{l=0}^{l=\infty}\left(\frac{\rho}{R}\right)^{l+n-2}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}u(t,r,\vec{\phi})\frac{N(l,n)P_{l,n}(\cos\omega)}{\mathrm{Area}(S^{n-1})}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\ldots\sin\phi_2 d\vec{\phi}$$

On the other hand, the Poisson identity states

$$\sum_{l=0}^{\infty}N(l,n)s^lP_{l,n}(t)=\frac{1-s^2}{\left(1+s^2-2st\right)^{n/2}}$$

thus

$$\sum_{l=0}^{l=\infty}\sum_{m=0}^{m=N(l,n)-1}\left(\frac{\rho}{R}\right)^{l+n-2}u_l^m(t,\rho)Y_{lm}^n(\vec{\theta})$$

$$=\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}u(t,r,\vec{\phi})\frac{\rho^{n-2}}{\mathrm{Area}(S^{n-1})}\frac{R^2-\rho^2}{\left(R^2+\rho^2-2\rho R\cos\omega\right)^{n/2}}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\ldots\sin\phi_2 d\vec{\phi}$$

Explicit feedback law

The function

$$P(R,\rho,\vec{\theta},\vec{\phi}) = \frac{1}{\mathrm{Area}(S^{n-1})}\frac{R^2-\rho^2}{\left(R^2+\rho^2-2\rho R\cos\omega\right)^{n/2}}$$

is the **Poisson kernel for an n-ball**

- Used to express the solution for the Laplace problem in an n-ball as an integral:

$$\Delta v(r,\vec{\theta}) = 0,\ v(R,\vec{\theta}) = F(\vec{\theta})$$

 can be explicitly solved as

$$v(r,\vec{\theta}) = \int_0^{\pi}\!\!\ldots\int_0^{\pi}\int_0^{2\pi} P(R,r,\vec{\theta},\vec{\phi})F(\vec{\phi})R^{n-2}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\ldots\sin\phi_2 d\vec{\phi}$$

- Tends to a Dirac delta $\delta(\vec{\theta}-\vec{\psi})$ when r goes to ρ

Explicit feedback law

Thus we obtain finally our explicit feedback law

$$
\begin{aligned}
U(t,\theta) &= -\int_0^R \int_0^{\pi} \dots \int_0^{\pi} \int_0^{2\pi} \frac{\lambda}{\varepsilon} \frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}} P(R,\rho,\vec{\theta},\vec{\phi}) \\
&\quad \times u(t,\rho,\vec{\phi})\rho^{n-1}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\dots\sin\phi_2\, d\vec{\phi}\, d\rho \\
&= -\frac{\lambda}{\varepsilon}\int_0^R \rho \frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}} \\
&\quad \times \left[\int_0^{\pi} \dots \int_0^{\pi} \int_0^{2\pi} P(R,\rho,\vec{\theta},\vec{\phi}) u(t,\rho,\vec{\phi})\rho^{n-1}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\dots\sin\phi_2\, d\vec{\phi}\right] d\rho
\end{aligned}
$$

Explicit feedback law

Thus we obtain finally our explicit feedback law

$$
\begin{aligned}
U(t,\theta) &= -\int_0^R \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \frac{\lambda}{\varepsilon} \frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}} P(R,\rho,\vec{\theta},\vec{\phi}) \\
&\quad \times u(t,\rho,\vec{\phi})\rho^{n-1}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\ldots\sin\phi_2 d\vec{\phi}d\rho \\
&= -\frac{\lambda}{\varepsilon}\int_0^R \rho \frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}} \\
&\quad \times \left[\int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} P(R,\rho,\vec{\theta},\vec{\phi})u(t,\rho,\vec{\phi})\rho^{n-1}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\ldots\sin\phi_2 d\vec{\phi}\right] d\rho
\end{aligned}
$$

Compare with the explicit backstepping controller for 1-D reaction-diffusion equation:

$$
U(t,x) = -\frac{\lambda}{\varepsilon}\int_0^L \rho \frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(L^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(L^2-\rho^2)}} u(t,\rho)d\rho
$$

Explicit feedback law in rectangular coordinates

Noticing that

$$\rho^{n-1} \sin^{n-2}\phi_{n-1} \sin^{n-3}\phi_{n-2} \ldots \sin\phi_2$$

is the "volume" element for an n-ball, we can write the control law back in rectangular coordinates

$$U(t,\vec{x}) \;=\; -\frac{1}{\mathrm{Area}(S^{n-1})}\sqrt{\frac{\lambda}{\varepsilon}} \int_{B^n(R)} \mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2 - \|\vec{\xi}\|^2)}\right] \frac{\sqrt{R^2 - \|\vec{\xi}\|^2}}{\|\vec{x} - \vec{\xi}\|^n} u(t,\vec{\xi})\, d\vec{\xi},$$

where the integral is extended to the complete n-ball $B^n(R)$ and $\vec{x} \in S^{n-1}(R)$.

The transformation in physical coordinates

To get additional insight the backstepping transformation can be expressed in physical coordinates.

We have found a transformation from

$$u_t = \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}u_r\right)_r + \frac{1}{r^2}\triangle^*_{n-1}u + \lambda u,$$

The transformation in physical coordinates

To get additional insight the backstepping transformation can be expressed in physical coordinates.

We have found a transformation from

$$u_t = \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}u_r\right)_r + \frac{1}{r^2}\triangle^*_{n-1}u + \lambda u,$$

into

$$w_t = \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}w_r\right)_r + \frac{1}{r^2}\triangle^*_{n-1}w$$

The transformation in physical coordinates

To get additional insight the backstepping transformation can be expressed in physical coordinates.

We have found a transformation from

$$u_t = \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}u_r\right)_r + \frac{1}{r^2}\triangle^*_{n-1}u + \lambda u,$$

into

$$w_t = \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}w_r\right)_r + \frac{1}{r^2}\triangle^*_{n-1}w$$

as follows:

$$\begin{aligned} w(t,r,\vec{\theta}) &= u(t,r,\vec{\theta}) - \int_0^r \int_0^\pi \ldots \int_0^\pi \int_0^{2\pi} K(r,\rho,\vec{\theta},\vec{\phi}) \\ &\quad \times u(t,\rho,\vec{\phi}) \sin^{n-2}\phi_{n-1} \sin^{n-3}\phi_{n-2} \ldots \sin\phi_2 d\phi_1 d\phi_2 \ldots d\phi_{n-1}, \end{aligned}$$

The transformation in physical coordinates

To get additional insight the backstepping transformation can be expressed in physical coordinates.

We have found a transformation from

$$u_t = \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}u_r\right)_r + \frac{1}{r^2}\triangle^*_{n-1}u + \lambda u,$$

into

$$w_t = \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}w_r\right)_r + \frac{1}{r^2}\triangle^*_{n-1}w$$

as follows:

$$\begin{aligned} w(t,r,\vec{\theta}) &= u(t,r,\vec{\theta}) - \int_0^r \int_0^\pi \ldots \int_0^\pi \int_0^{2\pi} K(r,\rho,\vec{\theta},\vec{\phi}) \\ &\quad \times u(t,\rho,\vec{\phi}) \sin^{n-2}\phi_{n-1} \sin^{n-3}\phi_{n-2} \ldots \sin\phi_2 d\phi_1 d\phi_2 \ldots d\phi_{n-1}, \end{aligned}$$

where

$$K(r,\rho,\vec{\theta},\vec{\phi}) = -\frac{\rho^{n-1}}{\mathrm{Area}(S^{n-1})}\sqrt{\frac{\lambda}{\varepsilon}}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{r^2-\rho^2}}P(r,\rho,\vec{\theta},\vec{\phi}),$$

Stability result (L^2)

Theorem

Consider the following PDE on the n-ball $B^n(R)$

$$\begin{aligned} \frac{\partial u(t,\vec{x})}{\partial t} &= \varepsilon \triangle_n u(t,\vec{x}) + \lambda u(t,\vec{x}) \\ u(t,\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} &= U(t,\vec{x}), \end{aligned}$$

with initial conditions $u_0(\vec{x})$ and

$$U(t,\vec{x}) = -\frac{1}{\mathrm{Area}(S^{n-1})}\sqrt{\frac{\lambda}{\varepsilon}}\int_{B^n(R)} \mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\|\vec{\xi}\|^2)}\right]\frac{\sqrt{R^2-\|\vec{\xi}\|^2}}{\|\vec{x}-\vec{\xi}\|^n}u(t,\vec{\xi})d\vec{\xi},$$

Assume in addition that $u_0 \in L^2(B^n(R))$.

Then the closed-loop system has a unique $C([0,\infty);L^2(B^n(R)))$ solution, and the equilibrium profile $u \equiv 0$ is exponentially stable in the $L^2(B^n(R))$ norm, i.e., there exists $c_1, c_2 > 0$ such that

$$\|u(t,\cdot)\|_{L^2(B^n(R))} \leq c_1 \mathrm{e}^{-c_2 t}\|u_0\|_{L^2(B^n(R))}.$$

Stability result (H^1)

Theorem

For the previous PDE, assume in addition that $u_0 \in H^1(B^n(R))$ and the compatibility condition

$$u_0(\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} = -\frac{1}{\mathrm{Area}(S^{n-1})}\sqrt{\frac{\lambda}{\varepsilon}}\int_{B^n(R)} \mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\|\vec{\xi}\|^2)}\right]\frac{\sqrt{R^2-\|\vec{\xi}\|^2}}{\|\vec{x}-\vec{\xi}\|^n}u_0(\vec{\xi})d\vec{\xi},$$

Then the closed-loop system has a unique $C([0,\infty);H^1(B^n(R)))$ solution, and the equilibrium profile $u \equiv 0$ is exponentially stable in the $H^1(B^n(R))$ norm, i.e., there exists $c_1, c_2 > 0$ such that

$$\|u(t,\cdot)\|_{H^1(B^n(R))} \leq c_1 \mathrm{e}^{-c_2 t}\|u_0\|_{H^1(B^n(R))}.$$

Sketch of proof

The strategy of the proof (for both L^2 and H^1 norms) is as follows.

1. We start from a well-known well-posedness and stability open-loop result on the n-ball for a Sobolev space $W(B^n(R))$, apply to the target system.

$$\begin{aligned} w_t(t,\vec{x}) &= \varepsilon \triangle_n w(t,\vec{x}), \qquad t>0, \vec{x} \in B^n(R) \\ w(t,\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} &= 0, \\ w(0,\vec{x}) &= w_0(\vec{x}), \quad w_0 \in W(B^n(R)). \end{aligned}$$

(this might require compatibility conditions)

and deduce the stability result for the target system (using e.g. known energy estimates or Lyapunov analysis)

$$\|w(t,\cdot)\|_W \leq b_1 \mathrm{e}^{-b_2 t} \|w_0\|_W.$$

with $b_1, b_2 > 0$.

Sketch of proof

2. We then show that the backstepping transformation is a map from $W(B^n(R))$ to $W(B^n(R))$:

$$w(t,\vec{x}) = u(t,\vec{x}) - \int_{B^n(\|\vec{x}\|)} K(\vec{x},\vec{\xi})u(t,\vec{\xi})d\xi = \mathcal{K}[u(t,\vec{x})](\vec{x}).$$

In particular we need to show $\|w(t,\cdot)\|_W \leq K\|u(t,\cdot)\|_W$ for $K > 0$

Thus, if the initial conditions in u coordinates (u_0) are in $W(B^n(R))$, then the corresponding $w_0 = \mathcal{K}[u_0]$ are in $W(B^n(R))$ as well, and $\|w_0(\cdot)\|_W \leq K\|u_0(\cdot)\|_W$

Sketch of proof

3. We show that the backstepping transformation is invertible:

$$u(t,\vec{x}) = w(t,\vec{x}) + \int_{B^n(\|\vec{x}\|)} L(\vec{x},\vec{\xi})w(t,\vec{\xi})d\xi = \mathcal{L}[w(t,\vec{x})](\vec{x}).$$

and the inverse transformation is again a map from $W(B^n(R))$ to $W(B^n(R))$, i.e. $\|u(t,\cdot)\|_W \leq L\|w(t,\cdot)\|_W$ for $L > 0$.

Therefore the u system inherits the well-posedness properties of the target system.

Sketch of proof

4. Having shown well-posedness of the closed-loop system, we can now finally state the desired results, namely, well-posedness and stability properties which are expressed as exponential decay with time of the Sobolev norm of the state.

In particular:

$$\begin{aligned} \|u(t,\cdot)\|_W &\leq L\|w(t,\cdot)\|_W \\ &\leq Lb_1 \mathrm{e}^{-b_2 t}\|w_0\|_W \\ &\leq Lb_1 K \mathrm{e}^{-b_2 t}\|u_0\|_W \end{aligned}$$

Sketch of proof

Example for L^2:

1. For the PDE

$$\begin{aligned} w_t(t,\vec{x}) &= \varepsilon \triangle_n w(t,\vec{x}), \qquad t>0, \vec{x}\in B^n(R) \\ w(t,\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} &= 0, \\ w(0,\vec{x}) &= w_0(\vec{x}), \quad w_0 \in W(B^n(R)). \end{aligned}$$

we have (see any standard textbook such as Brezis, "Functional Analysis, Sobolev Spaces, and Partial Differential Equations") that $u \in C^1((0,\infty);L^2(B^n(R)))$.

Sketch of proof

Example for L^2:

1. For the PDE

$$\begin{aligned} w_t(t,\vec{x}) &= \varepsilon \triangle_n w(t,\vec{x}), \qquad t>0, \vec{x}\in B^n(R) \\ w(t,\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} &= 0, \\ w(0,\vec{x}) &= w_0(\vec{x}), \quad w_0 \in W(B^n(R)). \end{aligned}$$

we have (see any standard textbook such as Brezis, ”Functional Analysis, Sobolev Spaces, and Partial Differential Equations”) that $u \in C^1((0,\infty);L^2(B^n(R)))$.

The stability result can be found, as usual, from using the definition $\|w(t,\cdot)\|^2_{L^2(B^n(R))} = \int_{B^n(R)} w^2(t,\vec{x})d\vec{x}$, and then

$$\frac{d}{dt}\frac{1}{2}\|w(t,\cdot)\|^2_{L^2(B^n(R))} = \varepsilon \int_{B^n(R)} w(t,\vec{x}) \triangle_n w(t,\vec{x})d\vec{x} = -\varepsilon \int_{B^n(R)} (\nabla_n w(t,\vec{x}))^2 d\vec{x} \le -c_0\varepsilon \|w(t,\cdot)\|^2_{L^2(B^n(R))}$$

therefore finding

$$\|w(t,\cdot)\|_{L^2(B^n(R))} \le \mathrm{e}^{-b_2 t}\|w_0\|_{L^2(B^n(R))}$$

Sketch of proof

Example for L^2:

2. We next need to show $\|w(t,\cdot)\|_W \leq K\|u(t,\cdot)\|_W$

$$
\begin{aligned}
|w_{lm}|^2 &= \left| u_{lm}(r) - \int_0^r K_{lm}^n(r,\rho)u_{lm}(\rho)d\rho \right|^2 \\
&\leq 2|u_{lm}|^2 + 2\left| \int_0^r K_{lm}^n(r,\rho)u_{lm}(\rho)d\rho \right|^2 \\
&\leq 2|u_{lm}|^2 + 2C_1^2 \left(\int_0^r \rho \left(\frac{\rho}{r}\right)^{l+n-2} d\rho \right) \left(\int_0^r \rho \left(\frac{\rho}{r}\right)^{l+n-2} |u_{lm}(\rho)|^2 d\rho \right) \\
&\leq 2|u_{lm}|^2 + C_1^2 r^{4-n} \left(\int_0^r \rho^{n-1}|u_{lm}(\rho)|^2 d\rho \right),
\end{aligned}
$$

Sketch of proof

Example for L^2:

2. We next need to show $\|w(t,\cdot)\|_W \leq K\|u(t,\cdot)\|_W$

$$
\begin{aligned}
|w_{lm}|^2 &= \left| u_{lm}(r) - \int_0^r K^n_{lm}(r,\rho)u_{lm}(\rho)d\rho \right|^2 \\
&\leq 2|u_{lm}|^2 + 2\left| \int_0^r K^n_{lm}(r,\rho)u_{lm}(\rho)d\rho \right|^2 \\
&\leq 2|u_{lm}|^2 + 2C_1^2 \left(\int_0^r \rho \left(\frac{\rho}{r}\right)^{l+n-2} d\rho \right) \left(\int_0^r \rho \left(\frac{\rho}{r}\right)^{l+n-2} |u_{lm}(\rho)|^2 d\rho \right) \\
&\leq 2|u_{lm}|^2 + C_1^2 r^{4-n} \left(\int_0^r \rho^{n-1} |u_{lm}(\rho)|^2 d\rho \right),
\end{aligned}
$$

and therefore

$$
\|w_{lm}\|^2_{L^2} = \int_0^R r^{n-1}|w_{lm}(r)|^2 dr \;\leq\; \left(2 + \frac{R^4 C_1^2}{4}\right) \|u^2_{lm}\|_{L^2} = K\|u^2_{lm}\|_{L^2}
$$

Sketch of proof

Example for L^2:

2. We next need to show $\|w(t,\cdot)\|_W \leq K\|u(t,\cdot)\|_W$

$$
\begin{aligned}
|w_{lm}|^2 &= \left| u_{lm}(r) - \int_0^r K^n_{lm}(r,\rho) u_{lm}(\rho) d\rho \right|^2 \\
&\leq 2|u_{lm}|^2 + 2\left| \int_0^r K^n_{lm}(r,\rho) u_{lm}(\rho) d\rho \right|^2 \\
&\leq 2|u_{lm}|^2 + 2C_1^2 \left(\int_0^r \rho \left(\frac{\rho}{r}\right)^{l+n-2} d\rho \right) \left(\int_0^r \rho \left(\frac{\rho}{r}\right)^{l+n-2} |u_{lm}(\rho)|^2 d\rho \right) \\
&\leq 2|u_{lm}|^2 + C_1^2 r^{4-n} \left(\int_0^r \rho^{n-1} |u_{lm}(\rho)|^2 d\rho \right),
\end{aligned}
$$

and therefore

$$
\|w_{lm}\|^2_{L^2} = \int_0^R r^{n-1} |w_{lm}(r)|^2 dr \quad \leq \quad \left(2 + \frac{R^4 C_1^2}{4}\right) \|u^2_{lm}\|_{L^2} = K\|u^2_{lm}\|_{L^2}
$$

so finally

$$
\|w\|^2_{L^2(B^n(R))} = \sum_{l=0}^{\infty} \sum_{m=0}^{N(l,n)-1} \|w_{lm}\|^2_{L^2} \leq K \sum_{l=0}^{\infty} \sum_{m=0}^{N(l,n)-1} \|u_{lm}\|^2_{L^2} = K\|u\|^2_{L^2(B^n(R))}
$$

Sketch of proof

Example for L^2:

3. We show that the backstepping transformation is invertible. Pose an inverse transform:

$$u_{lm}(t,r) = w_{lm}(t,r) + \int_0^r L_{lm}^n(r,\rho) w_{lm}(t,\rho) d\rho,$$

Sketch of proof

Example for L^2:

3. We show that the backstepping transformation is invertible. Pose an inverse transform:

$$u_{lm}(t,r) = w_{lm}(t,r) + \int_0^r L_{lm}^n(r,\rho) w_{lm}(t,\rho) d\rho,$$

and as before we find the following kernel equations for L_{lm}^n:

$$L_{lmrr}^n + (n-1)\frac{L_{lmr}^n}{r} - L_{lm\rho\rho}^n + (n-1)\frac{L_{lm\rho}^n}{\rho} - (n-1)\frac{L_{lm}^n}{\rho^2} - l(l+n-2)\left(\frac{1}{r^2} - \frac{1}{\rho^2}\right) L_{lm}^n = -\frac{\lambda}{\varepsilon} L_{lm}^n$$
$$L_{lm}^n(r,0) = (n-2) L_{lm\rho}^n(r,0) = 0,$$
$$L_{lm}^n(r,r) = -\frac{\lambda r}{2\varepsilon},$$

Sketch of proof

Example for L^2:

3. We show that the backstepping transformation is invertible. Pose an inverse transform:

$$u_{lm}(t,r) = w_{lm}(t,r) + \int_0^r L^n_{lm}(r,\rho) w_{lm}(t,\rho) d\rho,$$

and as before we find the following kernel equations for L^n_{lm}:

$$L^n_{lmrr} + (n-1)\frac{L^n_{lmr}}{r} - L^n_{lm\rho\rho} + (n-1)\frac{L^n_{lm\rho}}{\rho} - (n-1)\frac{L^n_{lm}}{\rho^2} - l(l+n-2)\left(\frac{1}{r^2} - \frac{1}{\rho^2}\right) L^n_{lm} = -\frac{\lambda}{\varepsilon} L^n_{lm}$$
$$L^n_{lm}(r,0) = (n-2) L^n_{lm\rho}(r,0) = 0,$$
$$L^n_{lm}(r,r) = -\frac{\lambda r}{2\varepsilon},$$

same as for K^n_{lm} but substituting λ by $-\lambda$ and changing sign!We thus find:

$$L^n_{lm}(r,\rho) = -\rho\left(\frac{\rho}{r}\right)^{l+n-2} \frac{\lambda}{\varepsilon} \frac{\mathrm{J}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}}.$$

Sketch of proof

Example for L^2:

3. We show that the backstepping transformation is invertible. Pose an inverse transform:

$$u_{lm}(t,r) = w_{lm}(t,r) + \int_0^r L_{lm}^n(r,\rho) w_{lm}(t,\rho) d\rho,$$

and as before we find the following kernel equations for L_{lm}^n:

$$L_{lmrr}^n + (n-1)\frac{L_{lmr}^n}{r} - L_{lm\rho\rho}^n + (n-1)\frac{L_{lm\rho}^n}{\rho} - (n-1)\frac{L_{lm}^n}{\rho^2} - l(l+n-2)\left(\frac{1}{r^2} - \frac{1}{\rho^2}\right) L_{lm}^n = -\frac{\lambda}{\varepsilon} L_{lm}^n$$
$$L_{lm}^n(r,0) = (n-2) L_{lm\rho}^n(r,0) = 0,$$
$$L_{lm}^n(r,r) = -\frac{\lambda r}{2\varepsilon},$$

same as for K_{lm}^n but substituting λ by $-\lambda$ and changing sign. We thus find:

$$L_{lm}^n(r,\rho) = -\rho \left(\frac{\rho}{r}\right)^{l+n-2} \frac{\lambda}{\varepsilon} \frac{\mathrm{J}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}}.$$

and as before $\|u\|_{L^2(B^n(R))}^2 \leq \left(2 + \frac{R^4 C_2^2}{4}\right) \|w\|_{L^2(B^n(R))}^2 = L \|w\|_{L^2(B^n(R))}^2$

Sketch of proof

4. We finish finally:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2(B^n(R))} &\leq L\|L^2(B^n(R))(t,\cdot)\|_{L^2(B^n(R))} \\
&\leq L\mathrm{e}^{-b_2 t}\|L^2(B^n(R))_0\|_{L^2(B^n(R))} \\
&\leq LK\mathrm{e}^{-b_2 t}\|u_0\|_{L^2(B^n(R))}
\end{aligned}$$

Further remarks about stability

- For $n = 2$ it is possible to prove exponential stability in the $H^p(B^n(R))$ space, for any positive integer p, under suitable compatibility conditions. Thus any degree of smoothness is possible (even C^∞!).

- The critical step is proving $\|w(t,\cdot)\|_{H^p(B^n(R))} \leq K_p \|u(t,\cdot)\|_{H^p(B^n(R))}$.

- The main idea of the proof is taking derivatives of the backstepping transformation and then integrating by parts to pass the derivatives in the kernel to derivatives in the state.

- This idea does not seem to generalize for $n > 2$. So far, no more than $H^1(B^n(R))$ has been proved for $n > 2$.

Observer design

Consider now the same equation

$$
\begin{aligned}
u_t &= \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u\right) + \frac{1}{r^2}\triangle^*_{n-1} u + \lambda u, \\
u(t,R,\vec{\theta}\,) &= U(t,\vec{\theta}\,),
\end{aligned}
$$

but now our objective is to estimate $u(r,\vec{\theta})$ from measurements at the boundary. In particular, $u_r(t,R,\vec{\theta}\,)$ is measured.

Observer design

Consider now the same equation

$$
\begin{aligned}
u_t &= \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u\right) + \frac{1}{r^2}\triangle^*_{n-1}\, u + \lambda u, \\
u(t,R,\vec{\theta}\,) &= U(t,\vec{\theta}\,),
\end{aligned}
$$

but now our objective is to estimate $u(r,\vec{\theta})$ from measurements at the boundary. In particular, $u_r(t,R,\vec{\theta}\,)$ is measured.

The following observer produces a convergent estimate $\hat{u}(r,\vec{\theta})$:

$$
\begin{aligned}
\hat{u}_t &= \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}\hat{u}_r\right)_r + \frac{1}{r^2}\triangle^*_{n-1}\,\hat{u} + \lambda\hat{u} + \mathcal{P}\left[u_r(t,R,\vec{\theta}) - \hat{u}_r(t,R,\vec{\theta}\right](r,\vec{\theta}) \\
\hat{u}(t,R,\vec{\theta}\,) &= U(t,\vec{\theta}\,).
\end{aligned}
$$

where $\mathcal{P}$ is defined:

$$
\begin{aligned}
\mathcal{P}[\Psi(\vec{\theta})](r,\vec{\theta}) &= -\frac{R^{n-1}\sqrt{\lambda\varepsilon}}{\mathrm{Area}(S^{n-1})}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-r^2)}\right]}{\sqrt{R^2-r^2}}\int_0^{\pi}\!\!\ldots\int_0^{\pi}\int_0^{2\pi}\Psi(\vec{\phi})P(r,\rho,\vec{\theta},\vec{\phi}) \\
&\quad\times\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\ldots\sin\phi_2 d\phi_1 d\phi_2\ldots d\phi_{n-1}d\rho
\end{aligned}
$$

Observer design

Expressing the observer equationin rectangular coordinates, we obtain

$$\hat{u}_t = \varepsilon \triangle_n \hat{u} + \lambda \hat{u} - \frac{\sqrt{\lambda\varepsilon}}{\mathrm{Area}(S^{n-1})} \mathrm{I}_1 \left[\sqrt{\frac{\lambda}{\varepsilon}(R^2 - \|\vec{x}\|^2)} \right] \sqrt{R^2 - \|\vec{x}\|^2} \int_{S^{n-1}(R)} \frac{u_r(t,\vec{\xi}) - \hat{u}_r(t,\vec{\xi})}{\|\vec{x} - \vec{\xi}\|^n} d\vec{\xi}$$

with BC

$$\hat{u}(t,\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} = U(t,\vec{x})$$

We can show $\|u(t,\cdot) - \hat{u}(t,\cdot)\|$ goes to zero as $t \to \infty$ exponentially, in both L^2 and H^1 norms.

Observer design

The idea is the same as for the controller. Starting with the plant expressed in spherical harmonics:

$$\begin{aligned} u_{lmt} &= \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}u_{lmr}\right)_r - l(l+n-2)\frac{\varepsilon}{r^2}u_{lm} + \lambda u_{lm}, \\ u_{lm}(t,R) &= U_{lm}(t), \end{aligned}$$

We assume we measure $u_{lmr}(t,R)$ and wish to estimate the state u_{lm} inside the domain.

Observer design

The idea is the same as for the controller. Starting with the plant expressed in spherical harmonics:

$$
\begin{aligned}
u_{lmt} &= \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}u_{lmr}\right)_r - l(l+n-2)\frac{\varepsilon}{r^2}u_{lm} + \lambda u_{lm}, \\
u_{lm}(t,R) &= U_{lm}(t),
\end{aligned}
$$

We assume we measure $u_{lmr}(t,R)$ and wish to estimate the state u_{lm} inside the domain.

Construct our observer as a copy of the plant plus output injection terms:

$$
\begin{aligned}
\hat{u}_{lmt} &= \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}\hat{u}_{lmr}\right)_r - l(l+n-2)\frac{\varepsilon}{r^2}\hat{u}_{lm} + \lambda \hat{u}_{lm} + p^n_{lm}(u_{lmr}(t,R) - \hat{u}_{lmr}(t,R)), \\
\hat{u}_{lm}(t,R) &= \hat{U}_{lm}(t).
\end{aligned}
$$

We need to design $p(r)$.

Observer design

The idea is the same as for the controller. Starting with the plant expressed in spherical harmonics:

$$
\begin{aligned}
u_{lmt} &= \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}u_{lmr}\right)_r - l(l+n-2)\frac{\varepsilon}{r^2}u_{lm} + \lambda u_{lm}, \\
u_{lm}(t,R) &= U_{lm}(t),
\end{aligned}
$$

We assume we measure $u_{lmr}(t,R)$ and wish to estimate the state u_{lm} inside the domain.

Construct our observer as a copy of the plant plus output injection terms:

$$
\begin{aligned}
\hat{u}_{lmt} &= \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}\hat{u}_{lmr}\right)_r - l(l+n-2)\frac{\varepsilon}{r^2}\hat{u}_{lm} + \lambda\hat{u}_{lm} + p^n_{lm}(u_{lmr}(t,R) - \hat{u}_{lmr}(t,R)), \\
\hat{u}_{lm}(t,R) &= \hat{U}_{lm}(t).
\end{aligned}
$$

We need to design $p(r)$.

Define the observer error as $\tilde{u} = u - \hat{u}$. The observer error dynamics are given by

$$
\begin{aligned}
\tilde{u}_{lmt} &= \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}\tilde{u}_{lmr}\right)_r - l(l+n-2)\frac{\varepsilon}{r^2}\tilde{u}_{lm} + \lambda\tilde{u}_{lm} - p^n_{lm}(r)\tilde{u}_{lmr}(t,R), \\
\tilde{u}_{lm}(t,R) &= 0.
\end{aligned}
$$

Observer design

Need to make the dynamics of $\tilde{u}$ stable with $p_{lm}^n(r)$. Our approach to design $p_{lm}^n(r)$ is to seek a mapping that transforms $\tilde{u}$ into the following target system

$$
\begin{aligned}
\tilde{w}_{lmt} &= \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}\tilde{w}_{lmr}\right)_r - l(l+n-2)\frac{\varepsilon}{r^2}\tilde{w}_{lm}, \\
\tilde{w}_{lm}(t,R) &= 0.
\end{aligned}
$$

Observer design

Need to make the dynamics of $\tilde{u}$ stable with $p_{lm}^n(r)$. Our approach to design $p_{lm}^n(r)$ is to seek a mapping that transforms $\tilde{u}$ into the following target system

$$\begin{aligned}
\tilde{w}_{lmt} &= \frac{\varepsilon}{r^{n-1}}\left(r^{n-1}\tilde{w}_{lmr}\right)_r - l(l+n-2)\frac{\varepsilon}{r^2}\tilde{w}_{lm}, \\
\tilde{w}_{lm}(t,R) &= 0.
\end{aligned}$$

The transformation is defined as follows:

$$\tilde{u}_{lm}(t,r) = \tilde{w}_{lm}(t,r) - \int_r^R P_{lm}^n(r,\rho)\tilde{w}_{lm}(t,\rho)d\rho$$

and then $p_{lm}^n(r)$ will be found from the transformation kernel P_{lm}^n.

Observer design

The following kernel equation is found:

$$P^n_{lmrr} + (n-1)\frac{P^n_{lmr}}{r} - P^n_{lm\rho\rho} + (n-1)\frac{P^n_{lm\rho}}{\rho} - (n-1)\frac{P^n_{lm}}{\rho^2} - l(l+n-2)\left(\frac{1}{r^2} - \frac{1}{\rho^2}\right)P^n_{lm} = -\frac{\lambda}{\varepsilon}P^n_{lm}$$

$$P^n_{lm}(0,\rho) = P^n_{lm\rho}(0,\rho) = 0,$$

$$P^n_{lm}(r,r) = -\frac{\lambda r}{2\varepsilon},$$

and once the kernel is found $p^n_{lm}(r) = \varepsilon P^n_{lm}(r,R)$

Observer design

The following kernel equation is found:

$$P^n_{lmrr} + (n-1)\frac{P^n_{lmr}}{r} - P^n_{lm\rho\rho} + (n-1)\frac{P^n_{lm\rho}}{\rho} - (n-1)\frac{P^n_{lm}}{\rho^2} - l(l+n-2)\left(\frac{1}{r^2} - \frac{1}{\rho^2}\right)P^n_{lm} = -\frac{\lambda}{\varepsilon}P^n_{lm}$$

$$P^n_{lm}(0,\rho) = P^n_{lm\rho}(0,\rho) = 0,$$

$$P^n_{lm}(r,r) = -\frac{\lambda r}{2\varepsilon},$$

and once the kernel is found $p^n_{lm}(r) = \varepsilon P^n_{lm}(r,R)$

It turns out this equation can be solved by the control kernel found previously , by defining

$$P^n_{lm}(r,\rho) = \frac{\rho^{n-1}}{r^{n-1}} K^n_{lm}(\rho, r)$$

Then, by summing the spherical harmonics we reach again a Poisson kernel-like function times a Bessel function.

Output feedback design

Consider now the output feedback problem. For

$$
\begin{aligned}
u_t &= \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u\right)+\frac{1}{r^2}\triangle^*_{n-1}u+\lambda u,\\
u(t,R,\vec{\theta}\,) &= U(t,\vec{\theta}\,),
\end{aligned}
$$

design U to stabilize $u(r,\vec{\theta})$, but only using measurement $u_r(t,R,\vec{\theta}\,)$.

Output feedback design

Consider now the output feedback problem. For

$$
\begin{aligned}
u_t &= \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u\right)+\frac{1}{r^2}\triangle^*_{n-1}u+\lambda u,\\
u(t,R,\vec{\theta}\,) &= U(t,\vec{\theta}\,),
\end{aligned}
$$

design U to stabilize $u(r,\vec{\theta})$, but only using measurement $u_r(t,R,\vec{\theta}\,)$.

The solution is a combination of the controller and observer design. Use the control law that we found but using the observer estimates

$$
\begin{aligned}
U &= -\frac{1}{\mathrm{Area}(S^{n-1})}\sqrt{\frac{\lambda}{\varepsilon}}\int_{B^n(R)}\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\|\vec{\xi}\|^2)}\right]\frac{\sqrt{R^2-\|\vec{\xi}\|^2}}{\|\vec{x}-\vec{\xi}\|^n}\hat{u}(t,\vec{\xi})d\vec{\xi},\\
\hat{u}_t &= \varepsilon\triangle_n\hat{u}+\lambda\hat{u}-\frac{\sqrt{\lambda\varepsilon}}{\mathrm{Area}(S^{n-1})}\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\|\vec{x}\|^2)}\right]\sqrt{R^2-\|\vec{x}\|^2}\int_{S^{n-1}(R)}\frac{u_r(t,\vec{\xi})-\hat{u}_r(t,\vec{\xi})}{\|\vec{x}-\vec{\xi}\|^n}d\vec{\xi}
\end{aligned}
$$

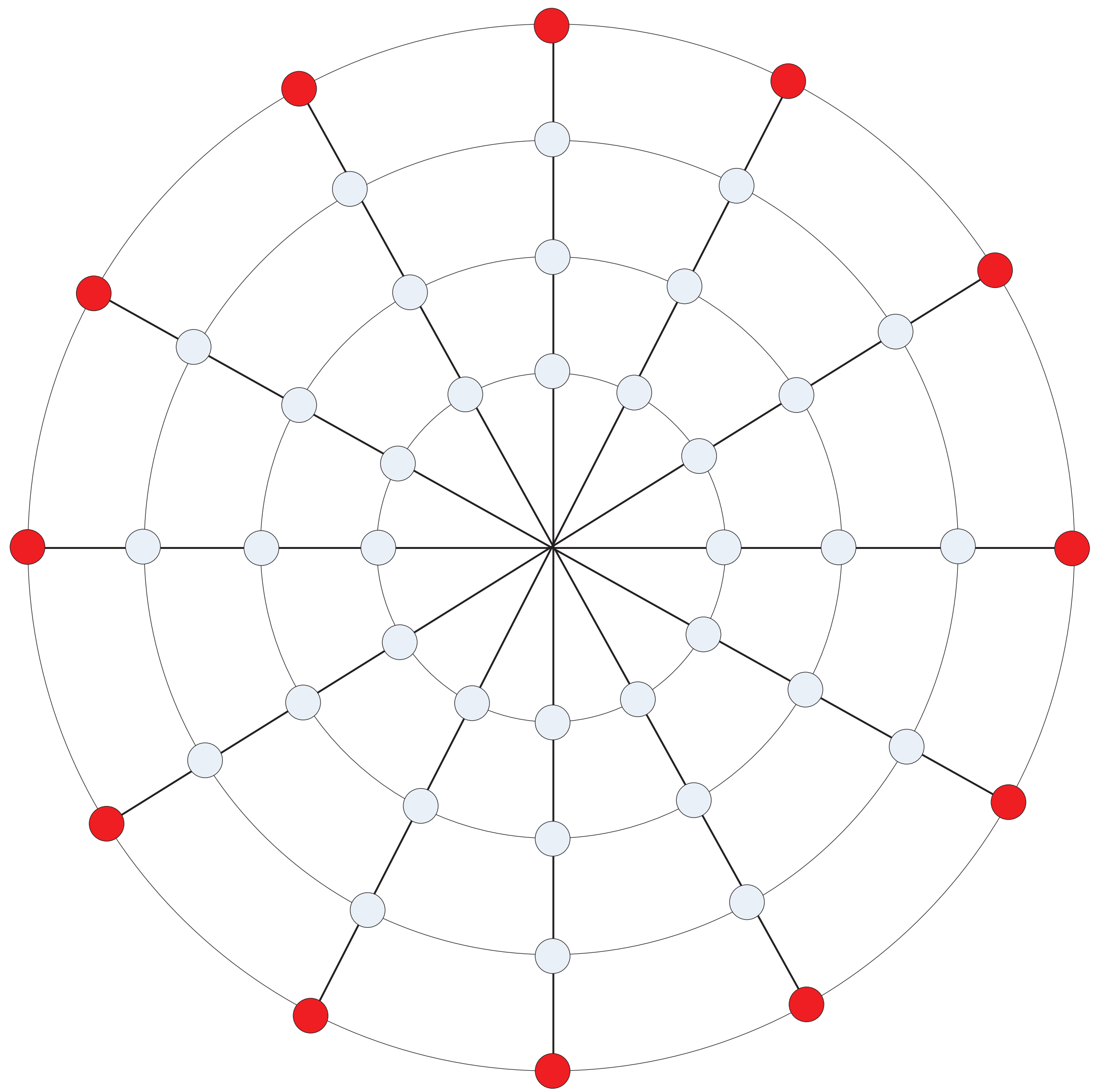
It can be proved that (provided some compatibility conditions are fulfilled) the states $(u,\hat{u})$ exponentially converge to zero in the H^1 norm.

Application

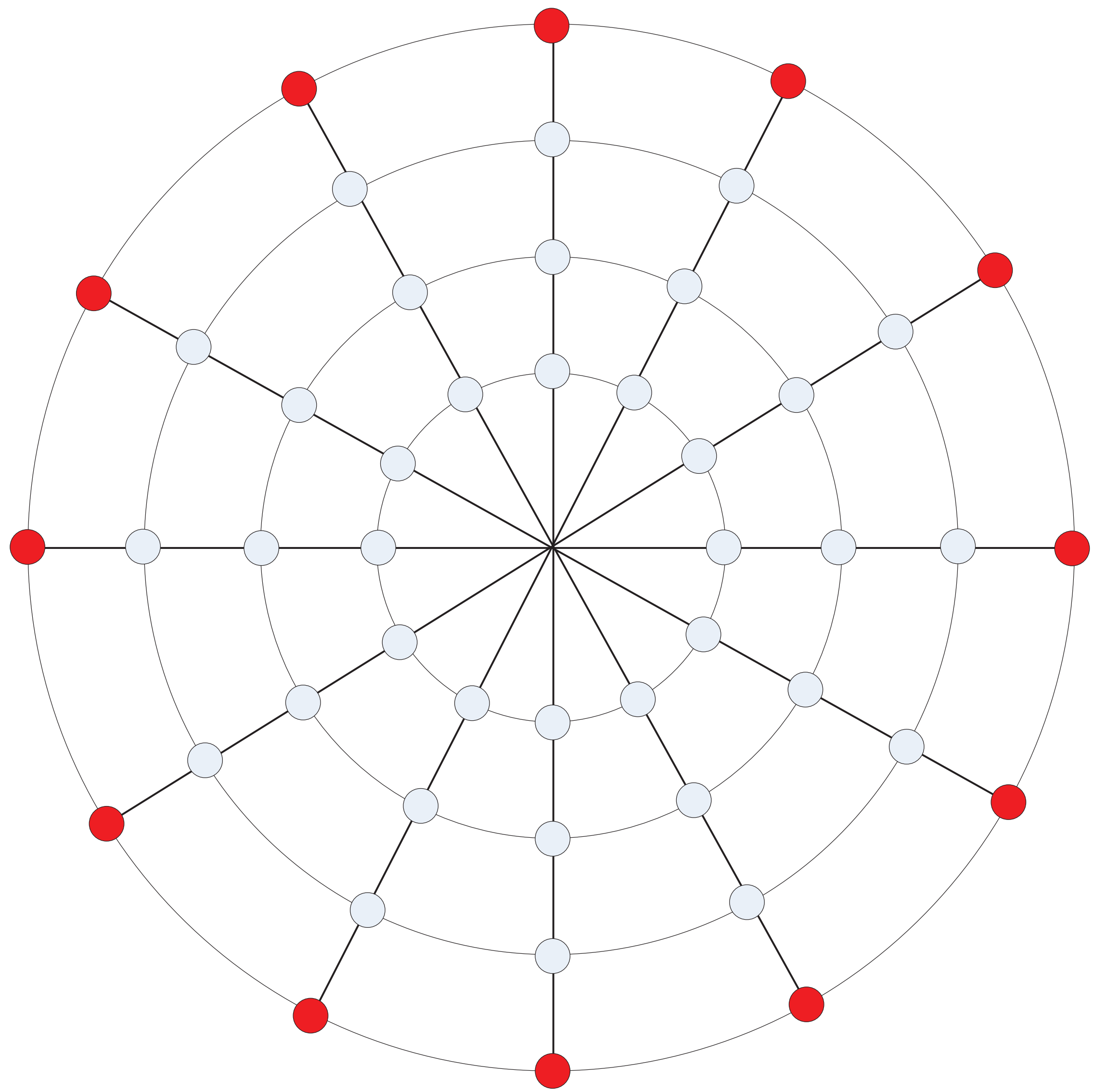
Application to motion planning: Multi-agent deployment using unstable PDEs

Joint work with Jie Qi (Donghua Univ., Shanghai, China)

Communication topology (polar/disk)



Communication topology (polar/disk) (actuated agents in red)



Typical model of inter-agent interaction: heat PDE

Limited—can achieve only equidistant deployment (in Cartesian topology)

(u = agent's complex-valued position)

$$u_t(t,r,\theta) \;=\; \frac{\varepsilon}{r}(r u_r(t,r,\theta))_r + \frac{\varepsilon}{r^2} u_{\theta\theta}(t,r,\theta)$$

Typical model of inter-agent interaction: heat PDE

Limited—can achieve only equidistant deployment (in Cartesian topology)

Reaction-diffusion model of inter-agent interaction:

(u = agent's complex-valued position)

$$u_t(t,r,\theta) = \frac{\varepsilon}{r}(ru_r(t,r,\theta))_r + \frac{\varepsilon}{r^2}u_{\theta\theta}(t,r,\theta) + \lambda u(t,r,\theta)$$

$\varepsilon, \lambda \in \mathbb{C}$

Rich deployment shapes but unstable

Typical model of inter-agent interaction: heat PDE

Limited—can achieve only equidistant deployment (in Cartesian topology)

Reaction-diffusion model of inter-agent interaction:

(u = agent's complex-valued position)

$$u_t(t,r,\theta) = \frac{\varepsilon}{r}(ru_r(t,r,\theta))_r + \frac{\varepsilon}{r^2}u_{\theta\theta}(t,r,\theta) + \lambda u(t,r,\theta)$$

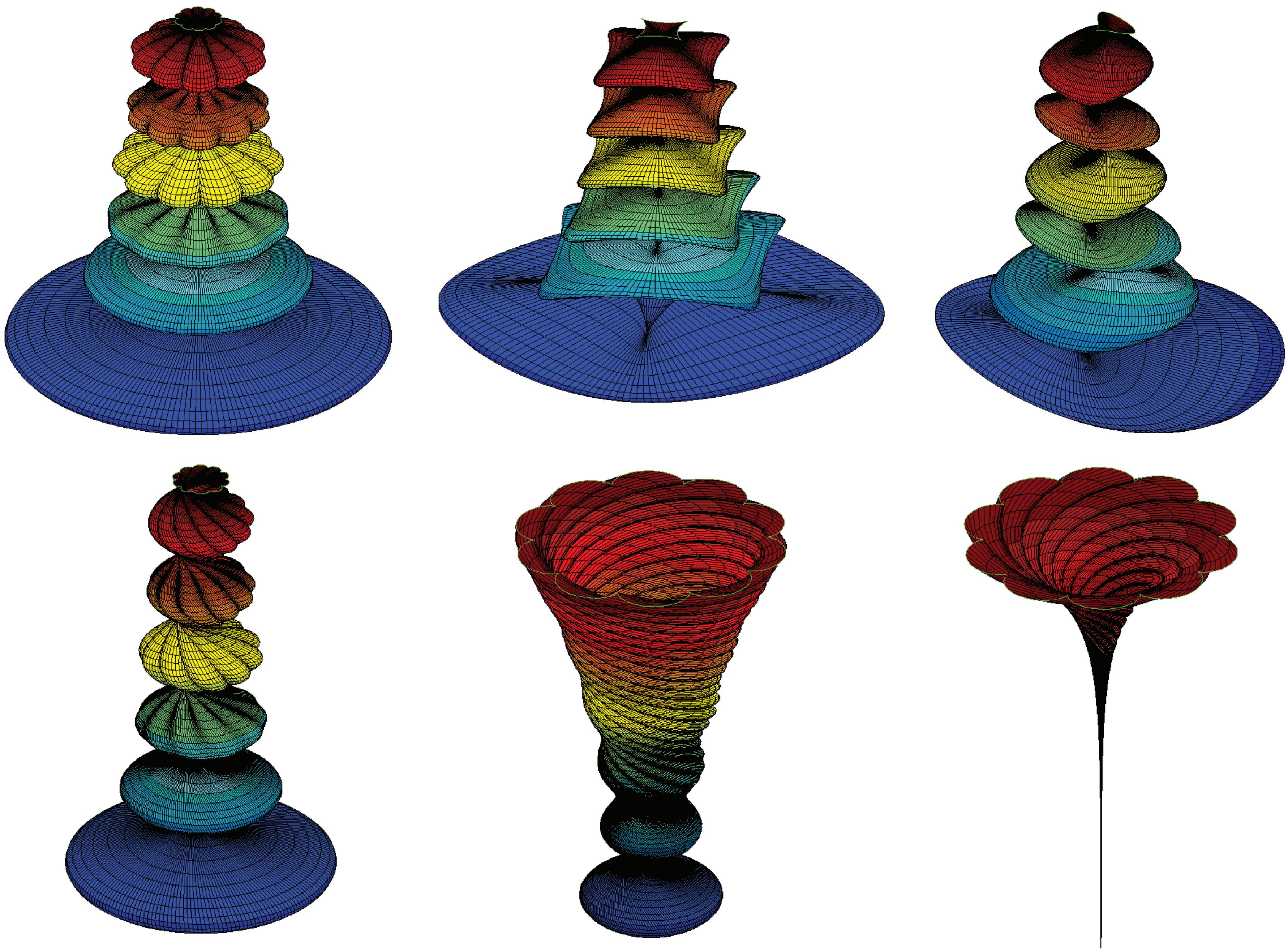
$\varepsilon, \lambda \in \mathbb{C}$

Rich deployment shapes but unstable

Follower agents' deployment positions as a function of leader agents' positions:

$$\bar{u}(r,\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} \frac{\mathrm{J}_n\left(\sqrt{\frac{\lambda}{\varepsilon}}r\right)}{\mathrm{J}_n\left(\sqrt{\frac{\lambda}{\varepsilon}}R\right)} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{j}n(\theta-\vartheta)}\bar{U}(\vartheta)d\vartheta$$

Deployment examples



Backstepping controller:

$$U(t,\theta) = \bar{U}(\theta) - \mathcal{K}\{\bar{u}\}(\theta) + \mathcal{K}\{u\}(t,\theta)$$

$$\boxed{\mathcal{K}\{u\}(t,\theta) = -\frac{\lambda}{4\pi^2\varepsilon}\int_0^R \underbrace{\rho\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}}}_{\text{Smyshlyaev kernel}}\int_{-\pi}^{\pi}\underbrace{\frac{1-\frac{\rho^2}{R^2}}{1+\frac{\rho^2}{R^2}-2\frac{\rho}{R}\cos(\theta-\psi)}}_{\text{Poisson kernel}}u(t,\rho,\psi)d\psi d\rho}$$

Qi, Vazquez, K (TAC 2015)

Extensions and open problems

Consider now the same problem but with spatially-varying coefficient λ:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \varepsilon \triangle_n u + \lambda(\vec{x}) u, \\ u(t,\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} &= U(t,\vec{x}) \end{aligned}$$

the question is: what can be done?

Extensions and open problems

Consider now the same problem but with spatially-varying coefficient λ:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \varepsilon \triangle_n u + \lambda(\vec{x})u, \\ u(t,\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} &= U(t,\vec{x}) \end{aligned}$$

the question is: what can be done?

Consider two cases:

- General $\lambda(\vec{x})$
- Radially-varying $\lambda(\|\vec{x}\|)$.

We will concentrate in the $2-D$ and/or $3-D$ cases, to simplify:

2-D case—general $\lambda(r,\theta)$

$$u_t = \frac{\varepsilon}{r}(ru_r)_r + \frac{\varepsilon}{r^2}u_{\theta\theta} + \lambda(r,\theta)u,$$

It is not possible to use spherical harmonics (they are no longer eigenfunctions that decouple the problem).

Pose a physical-space transformation:

$$w = u - \int_0^r \int_{-\pi}^{\pi} K(r,\rho,\theta,\psi)u(\rho,\psi)d\psi d\rho,$$

to transform the u equation into the target system

$$w_t = \frac{\varepsilon}{r}(rw_r)_r + \frac{\varepsilon}{r^2}w_{\theta\theta},$$

2-D case—general $\lambda(r,\theta)$

The kernel verifies the ultrahyperbolic equation

$$K_{rr} + \frac{K_r}{r} - K_{\rho\rho} + \frac{K_\rho}{\rho} - \frac{K}{\rho^2} - \frac{K_{\theta\theta}}{r^2} + \frac{K_{\psi\psi}}{\rho^2} = \frac{\lambda(\rho,\psi)}{\varepsilon} K$$

with BC

$$\begin{aligned} K(r,\rho,0,\psi) &= K(r,\rho,\pi,\psi) \\ K(r,\rho,\theta,0) &= K(r,\rho,\theta,\pi) \\ K(r,0,\theta,\psi) &= 0, \\ \int_{-\pi}^{\pi} K(r,r,\theta,\psi) u(r,\psi) d\psi &= -\int_0^r \frac{\lambda(\rho,\theta)}{2\varepsilon} d\rho\, u(r,\theta), \end{aligned}$$

and the second boundary condition can be verified if

$$\lim_{\rho \to r} K(r,\rho,\theta,\psi) = -\delta(\theta-\psi) \int_0^r \frac{\lambda(\rho,\theta)}{2\varepsilon} d\rho.$$

2-D case—general $\lambda(r,\theta)$

The kernel verifies the ultrahyperbolic equation

$$K_{rr}+\frac{K_r}{r}-K_{\rho\rho}+\frac{K_\rho}{\rho}-\frac{K}{\rho^2}-\frac{K_{\theta\theta}}{r^2}+\frac{K_{\psi\psi}}{\rho^2}=\frac{\lambda(\rho,\psi)}{\varepsilon}K$$

with BC

$$\begin{aligned}
K(r,\rho,0,\psi) &= K(r,\rho,\pi,\psi) \\
K(r,\rho,\theta,0) &= K(r,\rho,\theta,\pi) \\
K(r,0,\theta,\psi) &= 0, \\
\int_{-\pi}^{\pi} K(r,r,\theta,\psi)u(r,\psi)d\psi &= -\int_0^r \frac{\lambda(\rho,\theta)}{2\varepsilon}d\rho u(r,\theta),
\end{aligned}$$

and the second boundary condition can be verified if

$$\lim_{\rho\to r} K(r,\rho,\theta,\psi) = -\delta(\theta-\psi)\int_0^r \frac{\lambda(\rho,\theta)}{2\varepsilon}d\rho.$$

We don't know how to solve, only know there is a solution for constant λ!

$$K(r,\rho,\theta,\psi)=-\rho\frac{\lambda}{2\pi\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}}\frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cos(\theta-\psi)}$$

2-D case—radially-varying $\lambda(r)$

Now

$$u_t = \frac{\varepsilon}{r}(ru_r)_r + \frac{\varepsilon}{r^2}u_{\theta\theta} + \lambda(r)u,$$

and we can apply Spherical Harmonics (Fourier series in 2-D) to try to solve the problem.

Kernel equations are

$$K_{nrr} + \frac{K_{nr}}{r} - K_{n\rho\rho} + \frac{K_{n\rho}}{\rho} - \frac{K_n}{\rho^2} - n^2\left(\frac{1}{r^2} - \frac{1}{\rho^2}\right)K_n = \frac{\lambda(\rho)}{\varepsilon}K_n, \quad n \in \mathbb{Z}.$$

with BC

$$\begin{aligned} K_n(r,0) &= 0, \\ K_n(r,r) &= -\int_0^r \frac{\lambda(\rho)}{2\varepsilon}d\rho, \quad n \in \mathbb{Z}. \end{aligned}$$

Due to the singular terms, we don't know how to prove this equation is solvable (or how to solve it), except for a very special case: $n = 0$.

2-D and 3-D cases, $n = 0$—totally symmetric problem

The $n = 0$ case is of some physical interest: if the initial conditions are symmetric (do not depend on the angle or angles in 3-D), this is the only mode that plays a role. It is a typical engineering simplification.

Then the equation is, in 2-D:

$$u_t = \frac{\varepsilon}{r}(ru_r)_r + \lambda(r)u$$

and in 3-D:

$$u_t = \frac{\varepsilon}{r^2}\left(r^2 u_r\right)_r + \lambda(r)u$$

We apply the method as before but only one kernel (corresponding to the constant Fourier mode or Spherical Harmonic) is needed.

3-D case—totally symmetric problem

Kernel equation is:

$$
\begin{aligned}
K_{rr} + 2\frac{K_r}{r} - K_{\rho\rho} + 2\frac{K_\rho}{\rho} - 2\frac{K}{\rho^2} &= \frac{\lambda(r)}{\varepsilon}K \\
K(r,0) &= K_\rho(r,0) = 0, \\
K(r,r) &= -\frac{\lambda r}{2\varepsilon},
\end{aligned}
$$

3-D case—totally symmetric problem

Kernel equation is:

$$\begin{aligned} K_{rr} + 2\frac{K_r}{r} - K_{\rho\rho} + 2\frac{K_\rho}{\rho} - 2\frac{K}{\rho^2} &= \frac{\lambda(r)}{\varepsilon}K \\ K(r,0) &= K_\rho(r,0) = 0, \\ K(r,r) &= -\frac{\lambda r}{2\varepsilon}, \end{aligned}$$

Define $K(r,\rho) = \frac{\rho}{r}\bar{K}(r,\rho)$. Then:

$$\begin{aligned} \bar{K}_{rr} - \bar{K}_{\rho\rho} &= \frac{\lambda(r)}{\varepsilon}\bar{K} \\ \bar{K}(r,0) &= 0, \\ \bar{K}(r,r) &= -\frac{\lambda r}{2\varepsilon}, \end{aligned}$$

which is the 1-D backstepping equation! Can be proved solvable by successive approximations (classical backstepping papers).

3-D case—totally symmetric problem

For instance if λ is constant we directly get:

$$K(r,\rho) = \frac{\rho}{r}\bar{K}(r,\rho) = \frac{\rho^2}{r}\frac{c}{\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{c}{\varepsilon}\left(r^2-\rho^2\right)}\right]}{\sqrt{\frac{c}{\varepsilon}\left(r^2-\rho^2\right)}}$$

2-D case—totally symmetric problem

Interestingly, the 2-D case is harder than the 3-D case. Kernel equations are

$$
\begin{aligned}
K_{rr}+\frac{K_r}{r}-K_{\rho\rho}+\frac{K_\rho}{\rho}-\frac{K}{\rho^2} &= \frac{\lambda(\rho)}{\varepsilon}K, \\
K(r,0) &= 0, \\
K(r,r) &= -\int_0^r \frac{\lambda(\rho)}{2\varepsilon}d\rho
\end{aligned}
$$

2-D case—totally symmetric problem

Interestingly, the 2-D case is harder than the 3-D case. Kernel equations are

$$\begin{aligned} K_{rr} + \frac{K_r}{r} - K_{\rho\rho} + \frac{K_\rho}{\rho} - \frac{K}{\rho^2} &= \frac{\lambda(\rho)}{\varepsilon} K, \\ K(r,0) &= 0, \\ K(r,r) &= -\int_0^r \frac{\lambda(\rho)}{2\varepsilon} d\rho \end{aligned}$$

Define $G = \sqrt{\frac{r}{\rho}} K$. Then, for G we have:

$$\begin{aligned} G_{rr} - G_{\rho\rho} + \frac{G}{4r^2} - \frac{G}{4\rho^2} &= \frac{\lambda(\rho)}{\varepsilon} G \\ G(r,0) &= 0, \\ G(r,r) &= -\int_0^r \frac{\lambda(\rho)}{2\varepsilon} d\rho. \end{aligned}$$

and we can try to prove existence & uniqueness of a solution by using the classical successive approximation method.

2-D case—totally symmetric problem

Define new variables $\alpha = r + \rho$, $\beta = r - \rho$. The G equations become

$$\begin{aligned}
4G_{\alpha\beta} + \frac{G}{(\alpha+\beta)^2} - \frac{G}{(\alpha-\beta)^2} &= \frac{\lambda\left(\frac{\alpha-\beta}{2}\right)}{\varepsilon} G \\
G(\beta,\beta) &= 0, \\
G(\alpha,0) &= -\int_0^{\alpha/2} \frac{\lambda(\rho)}{2\varepsilon} d\rho.
\end{aligned}$$

2-D case—totally symmetric problem

Define new variables $\alpha = r + \rho$, $\beta = r - \rho$. The G equations become

$$\begin{aligned}
4G_{\alpha\beta} + \frac{G}{(\alpha+\beta)^2} - \frac{G}{(\alpha-\beta)^2} &= \frac{\lambda\left(\frac{\alpha-\beta}{2}\right)}{\varepsilon} G \\
G(\beta,\beta) &= 0, \\
G(\alpha,0) &= -\int_0^{\alpha/2} \frac{\lambda(\rho)}{2\varepsilon} d\rho.
\end{aligned}$$

This can be transformed into the (singular) integral equation

$$\begin{aligned}
G(\alpha,\beta) &= -\int_{\beta/2}^{\alpha/2} \frac{\lambda(\rho)}{2\varepsilon} d\rho + \int_\beta^\alpha \int_0^\beta \frac{\lambda\left(\frac{\eta-\sigma}{2}\right)}{4\varepsilon} G(\eta,\sigma)\, d\sigma\, d\eta \\
&\quad + \int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} G(\eta,\sigma)\, d\sigma\, d\eta
\end{aligned}$$

2-D case—totally symmetric problem

Try the successive approximations scheme, by defining

$$G_0(\alpha,\beta) \quad = \quad -\int_{\beta/2}^{\alpha/2} \frac{\lambda(\rho)}{2\varepsilon} d\rho$$

and for $k > 0$,

$$G_k(\alpha,\beta) \quad = \quad \int_{\beta}^{\alpha}\int_{0}^{\beta} \frac{\lambda\left(\frac{\eta-\sigma}{2}\right)}{4\varepsilon} G_{k-1}(\eta,\sigma)\, d\sigma d\eta + \int_{\beta}^{\alpha}\int_{0}^{\beta} \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} G_{k-1}(\eta,\sigma)\, d\sigma d\eta$$

then, the solution to the integral equation would be

$$G = \sum_{k=0}^{\infty} G_k(\alpha,\beta)$$

if the series converges.

2-D case—totally symmetric problem

Call $\bar{\lambda} = \max_{(\alpha,\beta)\in\mathcal{T}'} \left| \frac{\lambda\left(\frac{\alpha-\beta}{2}\right)}{4\varepsilon} \right|$.

Then one clearly obtains $|G_0(\alpha,\beta)| \leq \bar{\lambda}(\alpha-\beta)$.

However when trying to substitute in G_1 even the first integral is not so easy to perform.

2-D case—totally symmetric problem

Call $\bar{\lambda} = \max_{(\alpha,\beta)\in\mathcal{T}'} \left| \frac{\lambda\left(\frac{\alpha-\beta}{2}\right)}{4\varepsilon} \right|$.

Then one clearly obtains $|G_0(\alpha,\beta)| \leq \bar{\lambda}(\alpha-\beta)$.

However when trying to substitute in G_1 even the first integral is not so easy to perform. We use an alternative approach based on the following Lemma:

Define, for $n \geq 0, k \geq 0$,

$$F_{nk}(\alpha,\beta) = \frac{\bar{\lambda}^{n+1}\alpha^n\beta^n}{n!(n+1)!}(\alpha-\beta)\frac{\log^k\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}{k!}.$$

and $F_{nk} = 0$ if $n < 0$ or $k < 0$. Then F_{nk} is well-defined and nonnegative in the integration domain for all n,k, $F_{nk}(\beta,\beta) = 0$ for all n and k, $F_{nk}(\alpha,0) = 0$ if $n \geq 1$ or $k \geq 1$ and $F_{00}(\alpha,0) = \alpha$, and we have the following identity valid for $n \geq 1$ or $k \geq 1$.

$$F_{nk} = \int_\beta^\alpha \int_0^\beta \bar{\lambda} F_{(n-1)k}(\eta,\sigma)d\sigma d\eta + 4\int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2}\left(F_{n(k-1)}(\eta,\sigma) - F_{n(k-2)}(\eta,\sigma)\right)d\sigma d\eta$$

2-D case—totally symmetric problem

We use the lemma to try to find estimates for the terms in the successive approximation series:

$$|G_0| \leq F_{00}$$

next

$$|G_1| \leq \int_\beta^\alpha \int_0^\beta \bar{\lambda} F_{00}(\eta,\sigma)\,d\sigma\,d\eta + \int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} F_{00}(\eta,\sigma)\,d\sigma\,d\eta = F_{10} + \frac{F_{01}}{4}$$

where we have used the formulas of the lemma. The next term is

$$\begin{aligned} |G_2| &\leq \int_\beta^\alpha \int_0^\beta \bar{\lambda}\left(F_{10}+\frac{F_{01}}{4}\right) d\sigma\,d\eta + \int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2}\left(F_{10}+\frac{F_{01}}{4}\right) d\sigma\,d\eta \\ &= F_{20} + \frac{F_{11}}{4} + \frac{F_{01}+F_{02}}{16} \end{aligned}$$

If we keep going we find

$$|G_3| \leq F_{30} + \frac{F_{21}}{4} + \frac{F_{11}+F_{12}}{16} + \frac{2F_{01}+2F_{02}+F_{03}}{64}$$

2-D case—totally symmetric problem

The key to find these numbers is the following. Call:

$$\begin{aligned} I_1[F] &= \int_\beta^\alpha \int_0^\beta \bar{\lambda} F(\eta,\sigma)\, d\sigma d\eta \\ I_2[F] &= \int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} F(\eta,\sigma)\, d\sigma d\eta \end{aligned}$$

For instance, to find a bound on G_4 we find the following:

$$\begin{aligned} I_1[F_{30}] &= F_{40} \\ I_2[F_{30}] + \frac{I_1[F_{21}]}{4} &= \frac{F_{31}}{4} \\ \frac{I_2[F_{21}]}{4} + \frac{I_1[F_{11}+F_{12}]}{16} &= \frac{F_{21}+F_{22}}{16} \\ \frac{I_2[F_{11}+F_{12}]}{16} + \frac{I_1[2F_{01}+2F_{02}+F_{03}]}{64} &= \frac{2F_{11}+2F_{12}+F_{13}}{64} \\ \frac{I_2[2F_{01}+2F_{02}+F_{03}]}{64} &= \frac{5F_{01}+5F_{02}+3F_{03}+F_{04}}{256} \end{aligned}$$

Thus,

$$|G_4| \le F_{40} + \frac{F_{31}}{4} + \frac{F_{21}+F_{22}}{16} + \frac{2F_{11}+2F_{12}+F_{13}}{64} + \frac{5F_{01}+5F_{02}+3F_{03}+F_{04}}{256}$$

2-D case—totally symmetric problem

Based on this structure, we propose the following recursive formula for $n > 0$:

$$|G_n| \leq F_{n0} + \sum_{i=0}^{n-1} \sum_{j=1}^{j=n-i} \frac{C_{(n-i)j}}{4^{n-i}} F_{ij}$$

where C_{ij} verifies $C_{ij} = C_{(i-1)(j-1)} + C_{i(j+1)}$, taking $C_{11} = 1$, $C_{i0} = 0$, and $C_{ij} = 0$ if $j > i$, for all i. This set of numbers, known as the "Catalan's Triangle", verifies many interesting properties.

In particular it can be shown

$$\begin{aligned} C_{ii} &= 1. \\ C_{ij} &= \sum_{k=j-1}^{i-1} C_{(i-1)k}. \end{aligned}$$

which allows us to write the recursive formula

2-D case—totally symmetric problem

Let us show in a table the first few numbers.

C_{ij}	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$
$i=1$	1									
$i=2$	1	1								
$i=3$	2	2	1							
$i=4$	5	5	3	1						
$i=5$	14	14	9	4	1					
$i=6$	42	42	28	14	5	1				
$i=7$	132	132	90	48	20	6	1			
$i=8$	429	429	297	165	75	27	7	1		
$i=9$	1430	1430	1001	572	275	110	35	8	1	
$i=10$	4862	4862	3432	2002	1001	429	154	44	9	1

Catalan's Triangle

2-D case—totally symmetric problem

Now, since the solution verifies

$$|G| \leq \sum_{n=0}^{\infty} |G_n(\alpha, \beta)|$$

and we found

$$|G_n| \leq F_{n0} + \sum_{i=0}^{n-1} \sum_{j=1}^{j=n-i} \frac{C_{(n-i)j}}{4^{n-i}} F_{ij}$$

We get

$$|G| \leq \sum_{n=0}^{\infty} F_{n0} + \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \sum_{j=1}^{j=n-i} \frac{C_{(n-i)j}}{4^{n-i}} F_{ij}$$

and we only need to prove convergence of this series.

2-D case—totally symmetric problem

First term of the series:

$$\sum_{n=0}^{\infty} F_{n0} = \frac{\bar{\lambda}^{n+1}\alpha^n\beta^n}{n!(n+1)!}(\alpha-\beta) = \frac{\sqrt{\bar{\lambda}}}{2}(\alpha-\beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}}$$

For the next term, we use the fact that

$$\sum_{n=1}^{\infty}\sum_{i=0}^{n-1} H(n,i) = \sum_{i=0}^{\infty}\sum_{l=1}^{\infty} H(l+i,i)$$

Therefore

$$\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\sum_{j=1}^{j=n-i}\frac{C_{(n-i)j}}{4^{n-i}}F_{ij} = \sum_{i=0}^{\infty}\sum_{l=1}^{\infty}\sum_{j=1}^{j=l}\frac{C_{lj}}{4^l}F_{ij} = \sum_{i=0}^{\infty}\sum_{j=1}^{j=\infty}\left(\sum_{l=j}^{\infty}\frac{C_{lj}}{4^l}\right)F_{ij}$$

It turns out that the parenthesis can be calculated and gives an exact sum for each j.

2-D case—totally symmetric problem

To find the sum, consider first the generating function of the Catalan numbers C_{l1}:

$$f_1(x) = \frac{2}{1+\sqrt{1-4x}}$$

Remember that a generating function of a sequence of number is a function such that the coefficients of its power series is exactly those of the sequence of numbers.

Thus,

$$f_1(x) = C_{11} + C_{21}x + C_{31}x^2 + \ldots = \sum_{l=1}^{\infty} C_{l1}x^{l-1}$$

Therefore if we evaluate the function at $x = 1/4$ we find that

$$f_1(\frac{1}{4}) = \sum_{l=1}^{\infty} C_{l1}\frac{1}{4^{l-1}}$$

thus we find

$$\sum_{l=1}^{\infty} \frac{C_{l1}}{4^l} = \frac{1}{4}\sum_{l=1}^{\infty} \frac{C_{lj}}{4^{l-1}} = \frac{f_1(\frac{1}{4})}{4} = \frac{1}{2}$$

2-D case—totally symmetric problem

Following the previous argument, it is clear that

$$\sum_{l=j}^{\infty} \frac{C_{lj}}{4^l} = \frac{1}{4} \sum_{l=j}^{\infty} \frac{C_{lj}}{4^{l-1}} = \frac{f_j(\frac{1}{4})}{4}$$

where we define the generating function f_j as

$$f_j(x) = \sum_{l=j}^{\infty} C_{lj} x^{l-1}$$

Now since $C_{l2} = C_{l1}$ but obviously $C_{12} = 0$, it is clear that $f_2 = f_1 - C_{11} = f_1 - 1$. Thus $f_2(1/4) = 1$ and we find

$$\sum_{l=2}^{\infty} \frac{C_{l2}}{4^l} = \frac{f_2(\frac{1}{4})}{4} = \frac{1}{4}$$

2-D case—totally symmetric problem

To find successive generating functions we use the properties of the Catalan's Triangle and make the following claim:

$$f_n(x) = f_{n-1}(x) - x f_{n-2}(x)$$

Based on this fact, we can now prove that

$$\sum_{l=j}^{\infty} \frac{C_{lj}}{4^l} = \frac{1}{2^j}$$

Thus we obtain

$$\begin{aligned} |G| &\leq \frac{\sqrt{\bar{\lambda}}}{2}(\alpha-\beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}} + \sum_{i=0}^{\infty}\sum_{j=1}^{j=\infty}\frac{F_{ij}}{2^j} \\ &= \frac{\sqrt{\bar{\lambda}}}{2}(\alpha-\beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}} + \sum_{i=0}^{\infty}\sum_{j=1}^{j=\infty}\frac{\bar{\lambda}^{i+1}\alpha^i\beta^i}{i!(i+1)!}(\alpha-\beta)\frac{\log^j\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}{2^j j!} \end{aligned}$$

2-D case—totally symmetric problem

Summing the series

$$|G| \;\le\; \frac{\sqrt{\bar\lambda}}{2}(\alpha-\beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar\lambda\alpha\beta}\right]}{2\sqrt{\alpha\beta}}\left(\sum_{j=0}^{j=\infty}\frac{\log^j\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}{2^j j!}\right),$$

therefore

$$|G| \;\le\; \frac{\sqrt{\bar\lambda}}{2}(\alpha-\beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar\lambda\alpha\beta}\right]}{2\sqrt{\alpha\beta}}\mathrm{e}^{\log\left(\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\right)} = \frac{\sqrt{\bar\lambda}}{2}\sqrt{\alpha^2-\beta^2}\frac{\mathrm{I}_1\left[2\sqrt{\bar\lambda\alpha\beta}\right]}{2\sqrt{\alpha\beta}}$$

In physical variables r and ρ:

$$|G| \;\le\; \sqrt{\bar\lambda}\sqrt{r\rho}\frac{\mathrm{I}_1\left[2\sqrt{\bar\lambda(r^2-\rho^2)}\right]}{2\sqrt{r^2-\rho^2}}$$

2-D case—totally symmetric problem

Finally, going back to the original K, we find

$$|K(r,\rho)| \leq \rho\sqrt{\bar{\lambda}}\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}(r^2-\rho^2)}\right]}{2\sqrt{r^2-\rho^2}}$$

Thus, we have shown that the successive approximation series converges, with the solution K verifying the above bound. Uniqueness can be proved easily from the successive approximation series.

Unfortunately, this approach does not seem to be extensible for other Fourier coefficients.

Final remarks

We have found explicit stabilizing control laws for a constant-coefficient reaction-diffusion equation on a ball in arbitrary dimension

Final remarks

We have found explicit stabilizing control laws for a constant-coefficient reaction-diffusion equation on a ball in arbitrary dimension

Use of Spherical Harmonics and the backstepping method

Final remarks

We have found explicit stabilizing control laws for a constant-coefficient reaction-diffusion equation on a ball in arbitrary dimension

Use of Spherical Harmonics and the backstepping method

Very similar to backstepping control law for 1-D reaction-diffusion equation

Final remarks

We have found explicit stabilizing control laws for a constant-coefficient reaction-diffusion equation on a ball in arbitrary dimension

Use of Spherical Harmonics and the backstepping method

Very similar to backstepping control law for 1-D reaction-diffusion equation

The well-known Poisson kernel appears in the control law

Final remarks

We have found explicit stabilizing control laws for a constant-coefficient reaction-diffusion equation on a ball in arbitrary dimension

Use of Spherical Harmonics and the backstepping method

Very similar to backstepping control law for 1-D reaction-diffusion equation

The well-known Poisson kernel appears in the control law

Easily extended to observer design, output-feedback control laws with measurements on the boundary

Final remarks

We have found explicit stabilizing control laws for a constant-coefficient reaction-diffusion equation on a ball in arbitrary dimension

Use of Spherical Harmonics and the backstepping method

Very similar to backstepping control law for 1-D reaction-diffusion equation

The well-known Poisson kernel appears in the control law

Easily extended to observer design, output-feedback control laws with measurements on the boundary

For $n = 2$ the proof can be extended to the Sobolev space $H^k(B^2(R))$ for arbitrary k

Final remarks

We have found explicit stabilizing control laws for a constant-coefficient reaction-diffusion equation on a ball in arbitrary dimension

Use of Spherical Harmonics and the backstepping method

Very similar to backstepping control law for 1-D reaction-diffusion equation

The well-known Poisson kernel appears in the control law

Easily extended to observer design, output-feedback control laws with measurements on the boundary

For $n = 2$ the proof can be extended to the Sobolev space $H^k(B^2(R))$ for arbitrary k

Open problems: higher regularity for $n > 2$, space-varying $\lambda(r, \vec{\theta})$ (partial solution for radially-varying λ), more complicated domains

Merci!

Questions?

Some references:

- J. Qi, R. Vazquez, M. Krstic, ”Multi-agent Deployment in 3-D via PDE Control,” IEEE Transactions on Automatic Control, vol. 60 (4), pp. 891-906, 2015.

- R. Vazquez and M. Krstic, ”Explicit boundary control of a reaction-diffusion equation on a disk,” IFAC World Congress, 2014.

- R. Vazquez and M. Krstic, ”Explicit boundary control of reaction-diffusion PDEs on arbitrary-dimensonal balls,” 2015 European Control Conference.

- R. Vazquez and M. Krstic, ”Explicit output-feedback boundary control of reaction-diffusion PDEs on arbitrary-dimensional balls,” to be submitted, 2015.

- R. Vazquez and M. Krstic, ”Boundary control of reaction-diffusion equations on the disk and the sphere under revolution symmetry conditions,” under preparation, 2015.