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Control of Homodirectional and General Heterodirectional Linear Coupled Hyperbolic PDEs

Long Hu, Florent Di Meglio, Rafael Vazquez, Miroslav Krstic

***Abstract*—Research on stabilization of coupled hyperbolic PDEs has been dominated by the focus on pairs of counter-convecting ("heterodirectional") transport PDEs with distributed local coupling and with controls at one or both boundaries. A recent extension allows stabilization using only one control for a system containing an arbitrary number of coupled transport PDEs that convect at different speeds against the direction of the PDE whose boundary is actuated. In this paper we present a solution to the fully general case, in which the number of PDEs in either direction is arbitrary, and where actuation is applied on only one boundary (to all the PDEs that convect downstream from that boundary). To solve this general problem, we solve, as a special case, the problem of control of coupled "homodirectional" hyperbolic linear PDEs, where multiple transport PDEs convect in the same direction with arbitrary local coupling. Our approach is based on PDE backstepping and yields solutions to stabilization, by both full-state and observer-based output feedback, and trajectory tracking problems.**

I. Introduction

a) Background: Coupled first-order linear hyperbolic systems, typically formulated on a 1-D spatial domain normalized to the interval $(0,1)$, are common in modeling of traffic flow [2], heat exchangers [31], open channel flow [7], [10] or multiphase flow [11], [15], [16].

Research on stabilization of such PDEs has been dominated by the focus on pairs of counter-convecting transport PDEs with distributed local coupling. In [8] a first solution allowing actuation on only one boundary and permitting coupling coefficients of arbitrary size was presented. A recent extension [13] by three of the authors of the present paper allows stabilization using only one control for a system containing an arbitrary number of coupled transport PDEs that convect at different speeds against the direction of the PDE whose boundary is actuated.

L. Hu is with School of Mathematics, Shandong University, Jinan, Shandong 250100, China. Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France. School of Mathematical Sciences, Fudan University, Shanghai 200433, China. E-mail: hu@ann.jussieu.fr, hul10@fudan.edu.cn. This author was supported by the China Scholarship Council for Ph.D. study at UPMC (No. 201306100081) and was partially supported by ERC advanced grant 266907 (CPDENL) of the 7th Research Framework Programme (FP7).

F. Di Meglio is with MINES ParisTech, PSL Research University, CAS - Centre automatique et systèmes, 60 bd St Michel, 75006 Paris, France. florent.di_megliomines-paristech.fr

Rafael Vazquez is with the Department of Aerospace Engineering, Universidad de Sevilla, Camino de los Descubrimiento s.n., 41092 Sevilla, Spain. rvazquez1@us.es

M. Krstic is with the Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA 92093-0411, USA. krstic@ucsd.edu

In this paper we present a solution to the fully general case of coupled hyperbolic PDEs. We divide such PDE systems into two categories:

- *Homodirectional* systems of m transport PDEs, for which all the m transport velocities have the same signs, i.e., all of the PDEs convect in the same direction. Because of the finite length of the spatial domain, these are inherently stable but the coupling between states can cause undesirable transient behaviors and the trajectory planning problem is non-trivial.
- *Heterodirectional* systems of $n+m$ transport PDEs, for which there exist at least two transport velocities with opposite signs, i.e., where m PDEs convect in one direction and n PDEs convect in the opposite direction. The coupling between states traveling in opposite directions may cause instability.

In this paper we show control designs for the fully general case of coupled heterodirectional hyperbolic PDEs, allowing the numbers m and n of PDEs in either direction to be arbitrary, with actuation applied on only one boundary (to all the m PDEs that convect downstream from that boundary). To solve this general problem, we solve, as a special case, the heretofore unsolved problem of control of coupled homodirectional hyperbolic linear PDEs, where multiple transport PDEs convect in the same direction, have possibly distinct speeds, and arbitrary local coupling (see Section V).

Our approach is based on PDE backstepping and yields solutions to stabilization, by both full-state and observer-based output feedback, trajectory planning, and trajectory tracking problems.

b) Literature: Controllability of hyperbolic systems has first been investigated by using explicit computation of the solution along the characteristic curves in the framework of C^1 norm [17], [24], [26]. Later, the so-called Control Lyapunov Functions methods emerged, enabling the design of dissipative boundary conditions for nonlinear hyperbolic systems in the context of both C^1 norm and H^2 norm [5], [6], [9]. Further, using Lyapunov functions method, sufficient boundary conditions for the exponential stability of linear [14] or nonlinear [18], [19] hyperbolic systems of balance laws have been derived. All of these results impose restrictions on the magnitude of the coupling coefficients, which are responsible for potential instabilities.

In [8], a full-state feedback control law, with actuation only on one end of the domain, which achieves H^2 exponential stability of closed-loop 2–state heterodirectional linear and quasilinear hyperbolic systems is derived using a backstepping method. With a similar backstepping transformation, an output-feedback controller is designed in [13] for heterodirectional systems with $m = 1$ (controlled) negative velocity and n (arbitrary) positive ones. These results hold regardless of the (bounded) magnitude of the coupling coefficients. Unfortunately, the method presented in [8], [13] can not be extended to the case $m > 1$.

c) Contribution: The first step towards this paper's general solution for $m > 1$ was presented (but not published as a paper) in [27] for $m = 2$ and $n = 0$. In conference paper [21], an extension to $m = 2$ and $n = 1$ is achieved.

The contribution of this article is two-fold. First, we derive a stabilizing boundary feedback law that ensures finite-time convergence of all the states to zero. Then, we solve the tracking problem for an arbitrary output reference trajectory at the uncontrolled boundary.

Both designs rely on the backstepping approach. A particular choice of the target system, featuring a cascade structure similar to [8, Section 3.5], enables the use of a classical Volterra integral transformation. Well-posedness of the system of kernel equations, which is the main technical challenge of this paper, is proved by a method of successive approximations using a novel recursive bound.

When solving the null stabilization problem, the approach yields a full-state feedback law that would necessitate full distributed measurements to be implemented, which is not realistic in practice. For this reason, we derive an observer relying on measurements of the states at a single boundary (the anti-controlled one). Along with the full-state feedback law, this yields an output feedback controller amenable to implementation.

Solving the trajectory tracking problem also yields a full-state feedback law plus feedforward terms enabling the tracking of an output reference trajectory. Interestingly, in the case where the system only consists of two homodirectional states, the entire control law, including the backstepping kernels is explicitly expressed in terms of the system parameters.

d) Organization: In Section II we introduce the model equations. In Section III we present the stabilization result: the target system is presented in Section III-A while the backstepping transformation is derived in Section III-B. The design is summarized in Section III-C. In Section IV we present the boundary observer design. In Section V we present the trajectory tracking result. Section VI contains the main technical difficulty of the paper, i.e. the proof of well-posedness of the backstepping transformation. We finish in Section VII with some concluding remarks.

II. System description

We consider the following general linear hyperbolic system, which appear in Saint-Venant(-Exner) equations, plug flow chemical reactors equations, heat exchangers equations and other linear hyperbolic balance laws (see [3]),

$$u_t(t,x) + \Lambda^+ u_x(t,x) = \Sigma^{++}u(t,x) + \Sigma^{+-}v(t,x) \quad (1)$$

$$v_t(t,x) - \Lambda^- v_x(t,x) = \Sigma^{-+}u(t,x) + \Sigma^{--}v(t,x) \quad (2)$$

with the following linear boundary conditions

$$u(t,0) = Q_0 v(t,0), \qquad v(t,1) = R_1 u(t,1) + U(t) \quad (3)$$

where

$$u = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}^T, \qquad v = \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix}^T \quad (4)$$

$$\Lambda^+ = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \qquad \Lambda^- = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_m \end{pmatrix} \quad (5)$$

with constant speeds

$$-\mu_1 < \cdots < -\mu_m < 0 < \lambda_1 \leq \cdots \leq \lambda_n \quad (6)$$

and constant coupling matrices as well as the feedback control input

$$\Sigma^{++} = \left\{\sigma_{ij}^{++}\right\}_{1\leq i\leq n, 1\leq j\leq n}, \Sigma^{+-} = \left\{\sigma_{ij}^{+-}\right\}_{1\leq i\leq n, 1\leq j\leq m}, \quad (7)$$

$$\Sigma^{-+} = \left\{\sigma_{ij}^{-+}\right\}_{1\leq i\leq m, 1\leq j\leq n}, \Sigma^{--} = \left\{\sigma_{ij}^{--}\right\}_{1\leq i\leq m, 1\leq j\leq m} \quad (8)$$

$$Q_0 = \left\{q_{ij}\right\}_{1\leq i\leq n, 1\leq j\leq m}, \ R_1 = \left\{\rho_{ij}\right\}_{1\leq i\leq m, 1\leq j\leq n}, \quad (9)$$

$$U(t) = \begin{pmatrix} U_1(t) \cdots & U_m(t) \end{pmatrix}^T \quad (10)$$

Remark 1: The method presented here extends to spatially varying coefficients, with more involved technical developments.

Besides, we also make the following assumption without loss of generality

$$\forall j = 1, ..., m \qquad \sigma_{jj}^{--} = 0, \quad (11)$$

i.e. there are no (internal) diagonal coupling terms for v-system. Such coupling terms can be removed using a change of coordinates as presented in, e.g., [8] and [21]. This yields spatially-varying coupling terms, which is not an issue in the light of Remark 1.

Remark 2: In this paper, we only focus on the linear problem (both of the systems and boundary conditions are linear). The case of general quasililinear systems with non-linear boundary conditions will be the topic of future work (see [22]).

Remark 3: Throughout this paper, we only deal with one-sided actuation. In the case of two-sided actuation, with control variables at both boundaries $x = 0$ and $x = 1$, stronger controllability results hold [25]. In particular, it should be possible to perform state trajectory tracking since the system is exactly controllable. This study is however out of the scope of this paper.

III. Stabilization to zero

In this section, we derive a stabilizing feedback law for the general $(n + m)$–state system. Notice that this is interesting only in the case $n \neq 0$, since instability arises from coupling between states traveling in opposite directions. Following the backstepping approach, we seek to map system (1)–(3) to a target system with desirable stability properties using an invertible Volterra transformation.

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A. Target system

1) Target system design: We map system (1)–(3) to the following target system

$$\alpha_t(t,x) + \Lambda^+\alpha_x(t,x) = \Sigma^{++}\alpha(t,x) + \Sigma^{+-}\beta(t,x) + \int_0^x C^+(x,\xi)\alpha(t,\xi)d\xi + \int_0^x C^-(x,\xi)\beta(t,\xi)d\xi \quad (12)$$

$$\beta_t(t,x) - \Lambda^-\beta_x(t,x) = G(x)\beta(t,0) \quad (13)$$

with the following boundary conditions

$$\alpha(t,0) = Q_0\beta(t,0), \qquad \beta(t,1) = 0 \quad (14)$$

where C^+ and C^- are L^∞ matrix functions on the domain

$$\mathcal{T} = \{0 \le \xi \le x \le 1\}, \quad (15)$$

while $G \in L^\infty(0,1)$ is a lower triangular matrix with the following structures

$$G(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1}(x) & \cdots & g_{m,m-1}(x) & 0 \end{pmatrix}. \quad (16)$$

The coefficients of C^+, C^- and G will be determined in section III-B.

2) Convergence of the target system to zero: The following lemma asseses the finite-time convergence of the target system to zero.

Lemma 3.1: Consider system (12),(13) with boundary conditions (14). Its zero equilibrium is reached in finite time $t = t_F$, where

$$t_F := \frac{1}{\lambda_1} + \sum_{j=1}^{m}\frac{1}{\mu_j}. \quad (17)$$

Proof 3.2: Noting (13)-(14) and the particular structure of (16), we find that the β–system is in fact a cascade system, which allows us to explicitely solve it by recursion as follows. The explicit solution of β_1 is given by

$$\beta_1(t,x) = \begin{cases} \beta_1(0, x+\mu_1 t) & \text{if } t < \frac{1-x}{\mu_1}, \\ 0 & \text{if } t \ge \frac{1-x}{\mu_1}. \end{cases} \quad (18)$$

Notice in particular that β_1 is identically zero for $t \ge \mu_1^{-1}$. From the time $t \ge \mu_1^{-1}$ on, we have that $\beta_2(t,x)$ satisfies the following equation

$$\beta_{2t}(t,x) - \mu_2\beta_{2x}(t,x) = 0. \quad (19)$$

Similarly, by expressing the solution along the characteristic lines, one obtains that

$$\beta_2(t,x) \equiv 0 \quad \forall t \ge \mu_1^{-1} + \mu_2^{-1}. \quad (20)$$

Thus, by mathematical induction, one can easily get that $\beta_j (j = 1,\cdots,m)$ vanishes after

$$t = \sum_{k=1}^{j}\frac{1}{\mu_k}. \quad (21)$$

This yields that

$$\beta(t,x) \equiv 0, \quad t > \sum_{j=1}^{m}\frac{1}{\mu_j}. \quad (22)$$

When $t > \sum_{j=1}^{m}\frac{1}{\mu_j}$, the α–system becomes

$$\alpha_t(t,x) + \Lambda^+\alpha_x(t,x) = \Sigma^{++}\alpha(t,x) + \int_0^x C^+(x,\xi)\alpha(\xi)d\xi \quad (23)$$

with the boundary conditions (see Equation (14))

$$\alpha(t,0) = 0. \quad (24)$$

Since there are no zero transport velocities for the α–system (see (6)), we may change the status of t and x, and Equations (23) can be rewritten as

$$\alpha_x(t,x) + (\Lambda^+)^{-1}\alpha_t(t,x) = (\Lambda^+)^{-1}\Sigma^{++}\alpha(t,x) + \int_0^x (\Lambda^+)^{-1}C^+(x,\xi)\alpha(\xi)d\xi \quad (25)$$

with the null initial condition (24). Then by the uniqueness of the system (24),(25), and noting the order of the transport speeds of the α–system (see (6)), this yields that, no matter how large entries Σ appears in α-system which may result in increasing, α eventually identically vanishes for

$$t \ge \frac{1}{\lambda_1} + \sum_{j=1}^{m}\frac{1}{\mu_j} \quad (26)$$

This concludes the proof.

B. Backstepping transformation

To map system (1)–(3) to the target system (12)–(14), we consider the following backstepping (Volterra) transformation

$$\alpha(t,x) = u(t,x) \quad (27)$$

$$\beta(t,x) = v(t,x) - \int_0^x \left[K(x,\xi)u(\xi) + L(x,\xi)v(\xi)\right]d\xi \quad (28)$$

where the kernels to be determined K and L are defined on the triangular domain $\mathcal{T}$. Deriving (28) with respect to space and time, plugging into the target system equations and noticing that $\beta(t,0) \equiv v(t,0)$ yields the following system of kernel equations

$$0 = K(x,x)\Lambda^+ + \Lambda^-K(x,x) + \Sigma^{-+} \quad (29)$$

$$0 = \Lambda^-L(x,x) - L(x,x)\Lambda^- + \Sigma^{--} \quad (30)$$

$$0 = K(x,0)\Lambda^+Q_0 + G(x) - L(x,0)\Lambda^- \quad (31)$$

$$0 = \Lambda^-K_x(x,\xi) - K_\xi(x,\xi)\Lambda^+ - K(x,\xi)\Sigma^{++} - L(x,\xi)\Sigma^{-+} \quad (32)$$

$$0 = \Lambda^-L_x(x,\xi) + L_\xi(x,\xi)\Lambda^- - L(x,\xi)\Sigma^{--} - K(x,\xi)\Sigma^{+-} \quad (33)$$

and yields the following equations for $C^-(x,\xi)$ and $C^+(x,\xi)$

$$C^-(x,\xi) = \Sigma^{+-}L(x,\xi) + \int_\xi^x C^-(x,s)L(s,\xi)d\xi \quad (34)$$

$$C^+(x,\xi) = \Sigma^{+-}K(x,\xi) + \int_\xi^x C^-(x,s)K(s,\xi)d\xi \quad (35)$$

Remark 4: For each $x \in [0,1]$, Equation (34) is a Volterra equation of the second kind on $[0,x]$ with $C^-(x,\cdot)$ as the unknown. Besides, Equation (35) explicitly gives $C^+(x,\xi)$ as a function of $C^-(x,\xi)$ and $K(x,\xi)$. Therefore, provided the kernels K and L are well-defined and bounded, so are C^+ and C^-.

Developing equations (29)–(33) leads to the following set of kernel PDEs

for $1 \le i \le m$, $1 \le j \le n$

$$\mu_i\partial_x K_{ij}(x,\xi) - \lambda_j\partial_\xi K_{ij}(x,\xi) = \sum_{k=1}^{n}\sigma_{kj}^{++}K_{ik}(x,\xi) + \sum_{p=1}^{m}\sigma_{pj}^{-+}L_{ip}(x,\xi) \quad (36)$$

for $1 \le i \le m$, $1 \le j \le m$

$$\mu_i\partial_x L_{ij}(x,\xi) + \mu_j\partial_\xi L_{ij}(x,\xi) = \sum_{p=1}^{m}\sigma_{pj}^{--}L_{ip}(x,\xi) + \sum_{k=1}^{n}\sigma_{kj}^{+-}K_{ik}(x,\xi) \quad (37)$$

along with the following set of boundary conditions

$$\forall 1 \le i \le m, 1 \le j \le n, K_{ij}(x,x) = -\frac{\sigma_{ij}^{-+}}{\mu_i + \lambda_j} \triangleq k_{ij} \quad (38)$$

$$\forall 1 \le i,j \le m, i \ne j, \quad L_{ij}(x,x) = -\frac{\sigma_{ij}^{--}}{\mu_i - \mu_j} \triangleq l_{ij} \quad (39)$$

$$\forall 1 \le i \le j \le m,\ \mu_j L_{ij}(x,0) = \sum_{k=1}^{n}\lambda_k K_{ik}(x,0)q_{k,j} \quad (40)$$

Equations (36)–(40) do not uniquely define kernels K and L. To ensure well-posedness of the equations, we add the following artificial boundary conditions for $L_{ij}(i > j)$

$$L_{ij}(1,\xi) = l_{ij}, \text{ for } \ 1 \le j < i \le m \quad (41)$$

While the g_{ij}, for $1 \le j < i \le n$, are given by

$$g_{ij}(x) = \mu_j L_{ij}(x,0) - \sum_{p=1}^{n}\lambda_p q_{pj}K_{ip}(x,0) \quad (42)$$

provided the K and L kernels are properly defined by (36)–(41), which we prove in the next section.

Remark 5: The choice of imposing (41) as the boundary condition for L_{ij}, $(1 \le j < i \le m)$, on the boundary $x = 1$ is, a priori arbitrary. Importantly, the equations are well-posed regardless of this choice. An "appropriate" choice of boundary conditions is difficult to define. In [22], it is highlighted that continuity of the L_{ij} kernels at $(x,y) = (1,1)$ has to be imposed to deal with quasilinear systems. Notice that, in the present form (41), this continuity is imposed. For linear cases however, one could e.g. impose $L_{ij}(1,\xi) = 0$ $(1 \le j < i \le m)$, which may make $U(t)$ more concise. This degree of freedom in the control design has never appeared in previous backstepping designs for hyperbolic systems [8], [13]. The impact of the boundary values of L_{ij}, $1 \le j < i \le m$ on the transient behavior of the closed-loop system remains an open question, which we leave for future work.

Remark 6: As indicated in Equation (6), we consider in this paper distinct speeds for v-system. This ensures the denominator in (39) is non-zero. If two or more states of v have the same transport speeds (i.e. $\mu_i = \mu_j$ for some $i \ne j$) we refer to those states as *isotachic*. To deal with isotachic states, we consider the change of coordinates $\bar{v}(t,x) = A(x)v(t,x)$. The matrix $A(x)$ is a block-diagonal matrix, with $A_{ii} = 1$ if $\mu_i \ne \mu_j$ for $j \ne i$. If there is a set of n_i isotachic states (i.e. there is i such that $\mu_j = \mu_i$ for $j = i+1,\ldots,i+n_i-1$, then there is in $A(x)$ a corresponding block $B(x)$ of dimension $n_i \times n_i$ in $A(x)$. Each of these $B(x)$ is computed independently for each isotachic set of states. If we call Σ_{iso} the matrix of coupling coefficients among these isotachic states (i.e. with coefficients σ_{jk}^{--} for $j,k = i,i+1,\ldots,i+n_i-1$), then $B(x)$ is computed from the initial value problem $\frac{d}{dx}B(x) = 1/\mu_i B(x)\Sigma_{iso}$, $B(0) = I_{n_i\times n_i}$. It is easy to see that this transformation is invertible, since one can define a matrix $C(x)$ from $\frac{d}{dx}C(x) = -1/\mu_i\Sigma_{iso}C(x)$, $C(0) = I_{n_i\times n_i}$. One has that $C(x)$ is the inverse of $B(x)$ as $B(0)C(0) = I_{n_i\times n_i}$ and $\frac{d}{dx}B(x)C(x) = 0$. Applying this invertible transformation eliminates the coupling coefficients between isotachic states, but results in some spatially-varying coupling terms, which is not an issue as explained in Remark 1. If there are isotachic states, and the transformation explained in Remark 6 is applied, then the L_{ij} kernels for i,j corresponding to isotachic states ($\mu_i = \mu_j$) have all boundary conditions of the type (40) instead of (39)—which would become singular—or (41). The results that follow do not change, but we have omitted the case for the sake of brevity.

The well-posedness of the target system equations is assessed in the following Theorem.

Theorem 3.3: Consider system (36)–(41). There exists a unique solution K and L in $L^\infty(\mathcal{T})$. Moreover, all the boundary traces for the K-kernel and L-kernel are functions of $L^\infty(0,1)$. The proof of this Theorem is the main technical difficulty of the paper and is presented in Section VI.

C. Control law and main stabilization result

We are now ready to state the main stabilization result as follows.

Theorem 3.4: Consider system (1)-(2) with boundary conditions (3) and the following feedback control law

$$U(t) = -R_1u(t,1) + \int_0^1 \left[K(1,\xi)u(\xi) + L(1,\xi)v(\xi)\right]d\xi \quad (43)$$

For any initial condition $(u_0,v_0) \in (L^\infty(0,1))^{(n+m)\times(n+m)}$, the zero equilibrium is reached in finite time $t = t_F$, where t_F is given by (17).

Proof 3.5: First, notice that evaluating transformation (28) at $x = 1$ yields (43). Besides, rewriting transformation (28) as

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follows

$$\begin{pmatrix} \alpha(t,x) \\ \beta(t,x) \end{pmatrix} = \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} - \int_0^x \begin{pmatrix} 0 & 0 \\ K(x,\xi) & L(x,\xi) \end{pmatrix} \begin{pmatrix} u(t,\xi) \\ v(t,\xi) \end{pmatrix} d\xi. \quad (44)$$

one notices that it is a classical Volterra equation of the second kind. One can check from, e.g., [20] that there exists a unique matrix function $\mathcal{R} \in (L^\infty(\mathcal{T}))^{(n+m)\times(n+m)}$ such that

$$\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = \begin{pmatrix} \alpha(t,x) \\ \beta(t,x) \end{pmatrix} - \int_0^x \mathcal{R}(x,\xi) \begin{pmatrix} \alpha(t,\xi) \\ \beta(t,\xi) \end{pmatrix} d\xi. \quad (45)$$

Applying Lemma 3.1 implies that (α,β) go to zero in finite time $t = t_F$, therefore, by (45), (u, v) also converge to zero in finite time.

IV. Uncollocated observer design and output feedback controller

In this section, we derive an observer that relies on the measurement of the v states at the left boundary, i.e.

$$y(t) = v(t,0) \quad (46)$$

Then, using the estimates from the observer along with the control law (43), we derive an output feedback controller.

A. Observer design

The observer equations read as follows

$$\begin{aligned} \hat{u}_t(t,x) + \Lambda^+\hat{u}_x(t,x) =& \Sigma^{++}\hat{u}(t,x) + \Sigma^{+-}\hat{v}(t,x) \\ &- P^+(x)(\hat{v}(t,0) - v(t,0)) \quad (47) \\ \hat{v}_t(t,x) - \Lambda^-\hat{v}_x(t,x) =& \Sigma^{-+}\hat{u}(t,x) + \Sigma^{--}\hat{v}(t,x) \\ &- P^-(x)(\hat{v}(t,0) - v(t,0)) \quad (48) \end{aligned}$$

with the following boundary conditions

$$\hat{u}(t,0) = Q_0 v(t,0), \qquad \hat{v}(t,1) = R_1\hat{u}(t,1) + \hat{u}(t) \quad (49)$$

where $P^+(\cdot)$ and $P^-(\cdot)$ have yet to be designed. This yields the following error system

$$\begin{aligned} \tilde{u}_t(t,x) + \Lambda^+\tilde{u}_x(t,x) =& \Sigma^{++}\tilde{u}(t,x) + \Sigma^{+-}\tilde{v}(t,x) \\ &- P^+(x)\tilde{v}(t,0) \quad (50) \\ \tilde{v}_t(t,x) - \Lambda^-\tilde{v}_x(t,x) =& \Sigma^{-+}\tilde{u}(t,x) + \Sigma^{--}\tilde{v}(t,x) \\ &- P^-(x)\tilde{v}(t,0) \quad (51) \end{aligned}$$

with the following boundary conditions

$$\tilde{u}(t,0) = 0, \qquad \tilde{v}(t,1) = R_1\tilde{u}(t,1) \quad (52)$$

Remark 7: One should notice that the output is directly injected at the left boundary, which means potential sensor noise is only filtered throughout the spatial domain. Combining the approach of [13] and the cascade structure of (12)–(14), we now derive a target system and backstepping transformation to design observer gains $P^+(\cdot)$ and $P^-(\cdot)$ that yield finite-time stability of the error system (50)–(52).

B. Target system and backstepping tranformation

We map system (50)–(52) to the following target system

$$\begin{aligned} \tilde{\alpha}_t(t,x) + \Lambda^+\tilde{\alpha}_x(t,x) =& \Sigma^{++}\tilde{\alpha}(t,x) \\ &+ \int_0^x D^+(x,\xi)\tilde{\alpha}(\xi)d\xi \quad (53) \\ \tilde{\beta}_t(t,x) - \Lambda^-\tilde{\beta}_x(t,x) =& \Sigma^{-+}\tilde{\alpha}(t,x) \\ &+ \int_0^x D^-(x,\xi)\tilde{\alpha}(\xi)d\xi \quad (54) \end{aligned}$$

with the following boundary conditions

$$\tilde{\alpha}(t,0) = 0, \qquad \tilde{\beta}(t,1) = R_1\tilde{\alpha}(t,1) - \int_0^1 H(\xi)\tilde{\beta}(\xi)d\xi \quad (55)$$

where D^+ and D^- are L^∞ matrix functions on the domain $\mathcal{T}$ and $H \in L^\infty(0,1)$ is an upper triangular matrix with the following structure

$$H(x) = \begin{pmatrix} 0 & h_{1,2}(x) & \cdots & h_{1,m}(x) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & h_{m-1,m}(x) \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \quad (56)$$

all of which have yet to be determined.

Proposition 4.1: The solutions of system (53)–(14) converge to zero in finite time. More precisely, one has

$$\forall t \geq t_F, \quad \tilde{\alpha} \equiv \tilde{\beta} \equiv 0 \quad (57)$$

where t_F is defined by (17).

Proof The system consists in a cascade of the $\tilde{\alpha}$–system (that has zero input at the left boundary) into the $\tilde{\beta}$–system. Further, the $\tilde{\beta}$ is a cascade of its slow states into its fast states. The rigorous proof follows the same steps that the proof of Lemma 3.1 and is therefore omitted here.

To map system (50)–(52) to the target system (53)–(55), we consider the following backstepping (Volterra) transformation

$$\tilde{u}(t,x) = \tilde{\alpha}(t,x) + \int_0^x M(x,\xi)\tilde{\beta}(\xi)d\xi \quad (58)$$

$$\tilde{v}(t,x) = \tilde{\beta}(t,x) + \int_0^x N(x,\xi)\tilde{\beta}(\xi)d\xi \quad (59)$$

where the kernels to be determined M and N are defined on the triangular domain $\mathcal{T}$. Deriving (58),(59) with respect to space and time yields the following kernel equations

for $1 \leq i \leq n$, $1 \leq j \leq m$

$$\lambda_i\partial_x M_{ij}(x,\xi) - \mu_j\partial_\xi M_{ij}(x,\xi) = \sum_{k=1}^n \sigma_{ik}^{++}M_{kj}(x,\xi) + \sum_{p=1}^m \sigma_{ip}^{+-}N_{pj}(x,\xi) \quad (60)$$

for $1 \leq i \leq m$, $1 \leq j \leq m$

$$\mu_i\partial_x N_{ij}(x,\xi) + \mu_j\partial_\xi N_{ij}(x,\xi) = \sum_{k=1}^n \sigma_{ik}^{-+}M_{kj}(x,\xi) + \sum_{p=1}^m \sigma_{ip}^{--}N_{pj}(x,\xi) \quad (61)$$

along with the following set of boundary conditions

$$\forall 1 \leq i \leq m,\ 1 \leq j \leq n,\ M_{ij}(x,x) = \frac{\sigma_{ij}^{+-}}{\mu_i + \lambda_j} \triangleq m_{ij} \quad (62)$$

$$\forall 1 \leq i,j \leq m,\ i \neq j, \quad N_{ij}(x,x) = \frac{\sigma_{ij}^{--}}{\mu_i - \mu_j} \quad (63)$$

besides, evaluating (58),(59) at $x = 1$ yields

$$\forall 1 \leq j \leq i \leq m \quad N_{ij}(1,x) = \sum_{k=1}^{n} \rho_{ik} M_{kj}(1,x) \quad (64)$$

To ensure well-posedness of the kernel equations, we add the following artificial boundary conditions for $N_{ij} (i < j)$

$$\forall 1 \leq i < j \leq m, \quad N_{ij}(x,0) = 0 \quad (65)$$

while the d_{ij}^+, d_{ij}^- and h_{ij} are given by

$$h_{ij}(x) = N_{ij}(1,x) - \sum_{k=1}^{n} \rho_{ik} M_{kj}(1,x) \quad (66)$$

$$d_{ij}^+(x,\xi) = -\sum_{k=1}^{m} M_{ik}(x,\xi)\sigma_{kj}^{-+} + \int_{\xi}^{x} \sum_{k=1}^{m} M_{ik}(x,s) d_{kj}^-(s,\xi) ds \quad (67)$$

$$d_{ij}^-(x,\xi) = -\sum_{k=1}^{m} N_{ik}(x,\xi)\sigma_{kj}^{-+} + \int_{\xi}^{x} \sum_{k=1}^{m} N_{ik}(x,s) d_{kj}^-(s,\xi) ds \quad (68)$$

provided the M and N kernels are properly defined. Finally, the observer gains are given by

$$p_{ij}^+(x) = \mu_j m_{ij}(x,0) \quad (69)$$
$$p_{ij}^-(x) = \mu_j n_{ij}(x,0) \quad (70)$$

Interestingly, the well-posedness of the system of kernel equations of the observer (60)–(65) is equivalent to that of the controller kernels (36)–(41). Indeed, considering the following alternate variables

$$\bar{M}_{ij}(\chi,y) = M_{ij}(1-y,1-\chi) = M_{ij}(x,\xi), \quad (71)$$
$$\bar{N}_{ij}(\chi,y) = N_{ij}(1-y,1-\chi) = N_{ij}(x,\xi) \quad (72)$$

yields

for $1 \leq i \leq n,\ 1 \leq j \leq m$

$$\mu_j \partial_\chi \bar{M}_{ij}(\chi,y) - \lambda_i \partial_y \bar{M}_{ij}(\chi,y) = -\sum_{k=1}^{n} \sigma_{ik}^{++} \bar{M}_{kj}(\chi,y) - \sum_{p=1}^{m} \sigma_{ip}^{+-} \bar{N}_{pj}(\chi,y) \quad (73)$$

for $1 \leq i \leq m,\ 1 \leq j \leq m$

$$\mu_j \partial_\chi \bar{N}_{ij}(\chi,y) + \mu_i \partial_y \bar{N}_{ij}(\chi,y) = -\sum_{k=1}^{n} \sigma_{ik}^{-+} \bar{M}_{kj}(\chi,y) - \sum_{p=1}^{m} \sigma_{ip}^{--} \bar{N}_{pj}(\chi,y) \quad (74)$$

along with the following set of boundary conditions

$$1 \leq i \leq m,\ 1 \leq j \leq n\ \bar{M}_{ij}(\chi,\chi) = \frac{\sigma_{ij}^{+-}}{\mu_i + \lambda_j} \triangleq m_{ij}$$

$$\forall 1 \leq i,j \leq m,\ i \neq j, \quad \bar{N}_{ij}(\chi,\chi) = 0$$
$$\forall 1 \leq j \leq i \leq m, \quad \bar{N}_{ij}(\chi,0) = \sum_{k=1}^{n} \rho_{ik} \bar{M}_{kj}(\chi,0)$$
$$\forall 1 \leq i < j \leq m, \quad \bar{N}_{ij}(1,y) = 0$$

which has the exact same structure as the controller kernel system, the well-posedness of which is assessed in Theorem 3.3.

C. Output feedback controller

The estimates can be used in an observer-controller scheme to derive an output feedback law yielding finite-time stability of the zero equilibrium. More precisely, we have the following Lemma.

Lemma 4.2: Consider the system composed of the original (1)–(3) and target systems (47)–(49) with the following control law

$$U(t) = \int_0^1 \left[K(1,\xi)\hat{u}(\xi) + L(1,\xi)\hat{v}(\xi)\right] d\xi - R_1 \hat{u}(t,1) \quad (75)$$

where K and L are defined by (36)–(41). Its solutions $(u, v, \hat{u}, \hat{v})$ converge in finite time to zero.

Proof Proposition 4.1 along with the existence of the observer backstepping transformation (58),(59) yields convergence of the observer error states $\tilde{u}$, $\tilde{v}$ defined by (50)–(52) to zero for $t \geq t_F$[1]. Therefore, for $t \geq t_F$, one has $v(t,0) = \hat{v}(t,0)$ and Theorem 3.4 applies to the observer system (47)–(49). Therefore, for $t \geq 2t_F$, one has $(\tilde{u}, \tilde{v}, \hat{u}, \hat{v}) \equiv 0$ which also yields $(u,v) \equiv 0$.

V. Trajectory tracking

The motion planning problem for hyperbolic system has been investigated in [30], where an existence result is given. Here, we explictly solve the problem of trajectory tracking, defined in the next section.

A. Definition of the trajectory tracking problem

Consider the following (finite-time) *trajectory tracking* problem. Given $\Phi(t)$, a known function defined as

$$\Phi(t) = \begin{pmatrix} \Phi_1(t) & \cdots & \Phi_n(t) \end{pmatrix}^T, \quad (76)$$

find the value of $U(t)$ in (3) so that $v(t,0) = \Phi(t)$ for $t \geq t_M$, for some $t_M > 0$.

[1]the proof of this claim follows the exact same steps as in the controller case, see Section III-C

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B. Tracking control design

The following result solves the trajectory tracking problem.

Theorem 5.1: Consider system (1)-(2) with boundary conditions (3) and the following feedback control law

$$U(t) = -R_1 u(t,1) + \Phi_i\left(t + \frac{1}{\mu_i}\right) - \sum_{j=1}^{i-1} \int_0^1 \frac{\mu_j}{\mu_i} g_{ij}(\xi)\Phi_j\left(t + \frac{1-\xi}{\mu_i}\right) d\xi + \int_0^1 \left[K(1,\xi)u(\xi) + L(1,\xi)v(\xi)\right] d\xi \quad (77)$$

Then, $v(t,0) \equiv \Phi(t)$ if $t \geq t_M$, for $t_M = \sum_{j=1}^{m} \frac{1}{\mu_j}$.

Proof 5.2: We start by using the backstepping transformation (28) to map (1)-(2) into the target system (12)–(13) with the following boundary conditions

$$\alpha(t,0) = Q_0\beta(t,0), \qquad \beta(t,1) = B(t), \quad (78)$$

where $B(t)$ in (78) is a function defined as

$$B(t) = \begin{pmatrix} B_1(t) & \cdots & B_n(t) \end{pmatrix}^T, \quad (79)$$

with components to be determined. B represents an extra degree of freedom that did not appear in the target system for the stabilization problem (Equation (14)). It will be used to solve the motion planning problem. The presence of $B(t)$ in the boundary conditions does not change the backstepping transformation; however it modifies the feedback control law to

$$U(t) = B(t) + \int_0^1 \left[K(1,\xi)u(\xi) + L(1,\xi)v(\xi)\right] d\xi. \quad (80)$$

Now, noticing that if one sets $x = 0$ in the transformation (28) one obtains $v_i(t,0) = \beta_i(t,0)$, it is clear that we only need to solve the motion planning problem for the target β system by using $B(t)$. The next steps of the proof are devoted to finding the value of $B(t)$.

Using the method of characteristics, the explicit solution for each state $\beta_i(t,x)$ of (78) with boundary condition (78) at time $t \geq \frac{1-x}{\mu_i}$ is

$$\beta_i(t,x) = B_i\left(t + \frac{x-1}{\mu_i}\right) + \frac{1}{\mu_i}\int_x^1 G(\xi)\beta\left(t + \frac{x-\xi}{\mu_i}, 0\right) d\xi, \quad (81)$$

Using (16) , we obtain

$$\beta_i(t,x) = B_i\left(t + \frac{x-1}{\mu_i}\right) + \sum_{j=1}^{i-1}\int_x^1 \frac{\mu_j}{\mu_i} g_{ij}(\xi)\beta_j\left(t + \frac{x-\xi}{\mu_i}, 0\right) d\xi. \quad (82)$$

To solve now the motion planning problem, consider first (82) for $i = 1$ and $x = 0$, for $t \geq \frac{1}{\mu_1}$. Imposing $\beta_1(t,0) = \Phi_1(t)$, we obtain:

$$\Phi_1(t) = B_1\left(t - \frac{1}{\mu_1}\right), \quad (83)$$

thus, setting $B_1(t) = \Phi_1\left(t + \frac{1}{\mu_1}\right)$ for $t \geq 0$, we obtain the desired behavior for $\beta_1(t,0)$ for $t \geq \frac{1}{\mu_1}$. Now consider (82) for $i = 2$ and $x = 0$, for $t \geq \frac{1}{\mu_2}$. Imposing $\beta_2(t,0) = \Phi_2(t)$, we obtain:

$$\Phi_2(t) = B_2\left(t - \frac{1}{\mu_2}\right) + \int_0^1 \frac{\mu_2}{\mu_1} L_{21}(\xi,0)\beta_1\left(t - \frac{\xi}{\mu_2}, 0\right) d\xi. \quad (84)$$

Solving for B_2 as before

$$B_2(t) = \Phi_2\left(t + \frac{1}{\mu_2}\right) - \int_0^1 \frac{\mu_1}{\mu_2} L_{21}(\xi,0)\beta_1\left(t + \frac{1-\xi}{\mu_2}, 0\right) d\xi. \quad (85)$$

To be able to substitute $\beta_1(t,0)$ for $\Phi_1(t)$ in the whole domain of the integral in (85) we need to wait until $t = \frac{1}{\mu_1}$. Thus choosing

$$B_2(t) = \Phi_2\left(t + \frac{1}{\mu_2}\right) - \int_0^1 \frac{\mu_1}{\mu_2} L_{21}(\xi,0)\Phi_1\left(t + \frac{1-\xi}{\mu_2}, 0\right) d\xi, \quad (86)$$

we get that $\beta_2(t,0) = \Phi_2(t)$ for $t \geq \frac{1}{\mu_1} + \frac{1}{\mu_2}$ (as we have to wait an extra $\frac{1}{\mu_2}$ time for (86) to propagate). It is clear that this procedure can be continued for $i = 3, \ldots, m$. Thus we obtain that

$$B_i(t) = \Phi_i\left(t + \frac{1}{\mu_i}\right) - \sum_{j=1}^{i-1}\int_0^1 \frac{\mu_j}{\mu_i} L_{ij}(\xi,0)\Phi_j\left(t + \frac{1-\xi}{\mu_i}\right) d\xi \quad (87)$$

solves the motion problem for β_i for $t \geq \sum_{j=1}^{i} \frac{1}{\mu_j}$. Applying (87) for $i = 1, \ldots, m$ and substituting in (80) produces the feedback law (77), thus solving the motion planning problem in time $t_M = \sum_{j=1}^{m} \frac{1}{\mu_j}$.

Remark 8: Using Theorem 5.1 one can obtain a pure motion planning result. For that, one should take (82)—the explicit solutions of the target system obtained in the proof of the theorem—and substitute the values of B_i found in (87), so that the β_i's are explicit functions of the Φ_i's. The α system should be solved as well. Then, using the inverse backstepping transformation (45), find the u_i's and v_i's as explicit functions of the Φ_i's and substitute them in the control law (77), which would then be an exclusive function of the outputs. We omit this result for the sake of brevity.

C. An explicit motion planning example

To illustrate our results, we present an specific example of a motion planning problem for $n = 0, m = 2$. Although this example is presented here as an academic illustration, it presents the advantage to be explicitly solvable. In particular,

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as will appear, the backstepping kernels can be expressed in closed form (Equations (101)–(104)). Consider the plant

$$v_{1t}(t,x) - \mu_1 v_{1x}(t,x) = \sigma_{12} v_2(t,x), \tag{88}$$
$$v_{2t}(t,x) - \mu_2 v_{2x}(t,x) = \sigma_{21} v_1(t,x), \tag{89}$$

with boundary conditions

$$v_1(t,1) = U_1(t), \quad v_2(t,1) = U_2. \tag{90}$$

The objective is to design $U_1(t)$ and $U_2(t)$ so that $v_1(t,0) = \Phi_1(t)$ and $v_2(t,0) = \Phi_2(t)$ for some functions Φ_1, Φ_2 for $t \geq t_M$. Notice that since (88)–(90) is explicitly solvable, one might think that the inputs can be directly designed. Using the method of characteristics to explicitly write a solution of the system, one gets, after time $t = \frac{1}{\mu_2}$,

$$v_1(t,0) = U_1\left(t - \frac{1}{\mu_1}\right) + \frac{1}{\mu_1}\int_0^1 \sigma_{12} v_2\left(t - \frac{\xi}{\mu_1}, \xi\right) d\xi, \tag{91}$$
$$v_2(t,0) = U_2\left(t - \frac{1}{\mu_2}\right) + \frac{1}{\mu_2}\int_0^1 \sigma_{21} v_1\left(t - \frac{\xi}{\mu_2}, \xi\right) d\xi. \tag{92}$$

However, if one tries to proceed as in the proof of Theorem 5.1, by plugging in $\Phi_1(t)$ in (91) and $\Phi_2(t)$ in (92), and then solve for $U_1(t)$ and $U_2(t)$, one ends up with a feedback law that requires knowing *future* values of v_1 and v_2, i.e., a *non-causal* (and therefore not implementable) feedback law. Thus, a direct approach does not work even for the $m = 2$ case. To solve the motion planning problem, we resort to Theorem 5.1; in this particular case, the motion planning problem is solved by the inputs

$$U_1(t) = \Phi_1\left(t + \frac{1}{\mu_1}\right) + \int_0^1 L_{11}(1,\xi) v_1(\xi) d\xi + \int_0^1 L_{12}(1,\xi) v_2(\xi) d\xi, \tag{93}$$

$$U_2(t) = \Phi_2\left(t + \frac{1}{\mu_2}\right) - \int_0^1 \frac{\mu_1}{\mu_2} L_{21}(\xi,0)\Phi_1\left(t + \frac{1-\xi}{\mu_2}\right) d\xi + \int_0^1 L_{21}(1,\xi) v_1(\xi) d\xi + \int_0^1 L_{22}(1,\xi) v_2(\xi) d\xi, \tag{94}$$

where the kernels L_{11}, L_{12}, L_{21} and L_{22} satisfy

$$\mu_1 \partial_x L_{11}(x,\xi) + \mu_1 \partial_\xi L_{11}(x,\xi) = \sigma_{21} L_{12}(x,\xi) \tag{95}$$
$$\mu_1 \partial_x L_{12}(x,\xi) + \mu_2 \partial_\xi L_{12}(x,\xi) = \sigma_{12} L_{11}(x,\xi), \tag{96}$$
$$\mu_2 \partial_x L_{21}(x,\xi) + \mu_1 \partial_\xi L_{21}(x,\xi) = \sigma_{21} L_{22}(x,\xi) \tag{97}$$
$$\mu_2 \partial_x L_{22}(x,\xi) + \mu_2 \partial_\xi L_{22}(x,\xi) = \sigma_{12} L_{21}(x,\xi), \tag{98}$$

with boundary conditions

$$L_{11}(x,0) = L_{12}(x,0) = L_{22}(x,0) = 0, \tag{99}$$
$$L_{12}(x,x) = \frac{\sigma_{12}}{\mu_2 - \mu_1}, \quad L_{21}(x,x) = \frac{\sigma_{21}}{\mu_1 - \mu_2}, \tag{100}$$

plus the artificial boundary condition $L_{21}(1,\xi) = l_{21}(\xi)$, where the function l_{21} is arbitrary. These kernel PDEs can be explicitly solved using techniques akin to those used in [28]. The resulting kernels (whose validity can be verified by substitution in the kernel equations) are given by (101)–(104). where I_0 and I_1 are the modified Bessel functions of order 0 and 1, and J_0 and J_1 are the (regular) Bessel functions of order 0 and 1, respectively.

The kernels appearing in (91)–(92) are depicted in Fig 1 for the case $\mu_1 = 1$, $\mu_2 = 0.2$ and $\sigma_{12} = 2$, $\sigma_{21} = 5$. It can be seen that $L_{11}(1,\xi)$ and $L_{12}(1,\xi)$ have a monotone behaviour (they are always negative or zero), whereas $L_{21}(1,\xi)$, $L_{21}(\xi,0)$, and $L_{22}(1,\xi)$ are oscillatory. Fig. 2 shows L_{11} and L_{12} in the whole domain $\mathcal{T}$; notice that $L_{12}(x,\xi)$ is discontinuous along the line $\xi = \frac{\mu_2}{\mu_1}$ (which is the lower domain on Figure 6), whereas $L_{11}(x,\xi)$ is not discontinuous. On the other hand, it is evident that $l_{21}(\xi) = L_{21}(1,\xi)$ is rather non-trivial. In fact, the procedure that was followed to find these explicit solutions was not setting a value of l_{21} a priori, but rather *extending* the domain shown in Figure 4 up to $x = \frac{\mu_1}{\mu_1 - \mu_2}$, so that boundary condition (100) can be used to actually find the value of l_{21}.

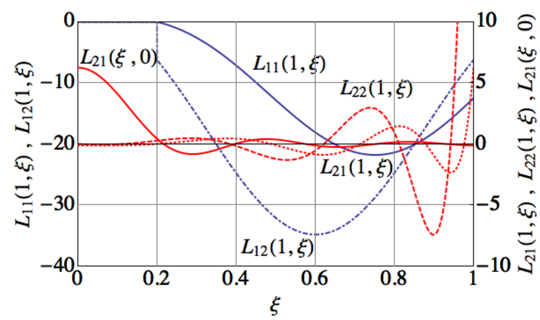

Fig. 1: Motion planning kernels ($n = 0$, $m = 2$). Solid: $L_{11}(1,\xi)$ and $L_{21}(\xi,0)$. Dash-dotted: $L_{12}(1,\xi)$. Dotted: $L_{21}(1,\xi)$. Dashed: $L_{22}(1,\xi)$.

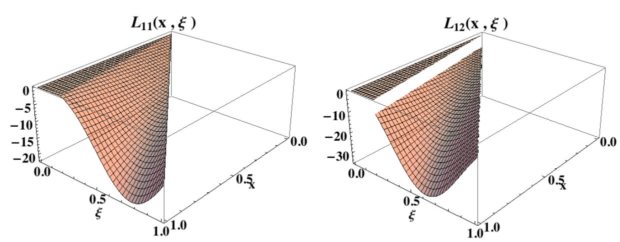

Fig. 2: Motion planning kernels $L_{11}(x,\xi)$ and $L_{12}(x,\xi)$ ($n = 0$, $m = 2$).

VI. Proof of Theorem 3.3: well-posedness of the kernel equations

To prove well-posedness of the kernel equations, we classically transform them into integral equations and use the method of successive approximations. Although this approach to solving hyperbolic equations is classical (see [23] and [29]), the particular geometry of the problem and the coupling between kernels, both in-domain and at the boundaries, could lead the equations to explode in finite space on the triangle. For this reason, we provide a detailed proof of well-posedness.

Remark 9: Similar proofs have been derived for less general systems, e.g. in [8] or [13]. The proof is more involved

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$$L_{11}(x,\xi) = \begin{cases} \frac{\sqrt{\sigma_{12}\sigma_{21}}}{\mu_2-\mu_1} \sqrt{\frac{\mu_1\xi-\mu_2 x}{\mu_1(x-\xi)}} \mathrm{I}_1\left(\frac{2}{\mu_1-\mu_2}\sqrt{\frac{\sigma_{12}\sigma_{21}(x-\xi)(\mu_1\xi-\mu_2 x)}{\mu_1}}\right), & \xi \geq \frac{\mu_2}{\mu_1}x \\ 0, & \xi < \frac{\mu_2}{\mu_1}x \end{cases} \tag{101}$$

$$L_{12}(x,\xi) = \begin{cases} \frac{\sigma_{21}}{\mu_2-\mu_1} \mathrm{I}_0\left(\frac{2}{\mu_1-\mu_2}\sqrt{\frac{\sigma_{12}\sigma_{21}(x-\xi)(\mu_1\xi-\mu_2 x)}{\mu_1}}\right), & \xi \geq \frac{\mu_2}{\mu_1}x \\ 0, & \xi < \frac{\mu_2}{\mu_1}x \end{cases} \tag{102}$$

$$L_{21}(x,\xi) = \frac{\sigma_{21}\xi}{\mu_1 x-\mu_2\xi}\mathrm{J}_0\left(\frac{2}{\mu_1-\mu_2}\sqrt{\frac{\sigma_{12}\sigma_{21}(x-\xi)(\mu_1 x-\mu_2\xi)}{\mu_2}}\right) + \mu_1\sqrt{\frac{\sigma_{21}\mu_2(x-\xi)}{\sigma_{12}(\mu_1 x-\mu_2\xi)^3}}\mathrm{J}_1\left(\frac{2}{\mu_1-\mu_2}\sqrt{\frac{\sigma_{12}\sigma_{21}(x-\xi)(\mu_1 x-\mu_2\xi)}{\mu_2}}\right), \tag{103}$$

$$L_{22}(x,\xi) = \xi\sqrt{\frac{\sigma_{12}\sigma_{21}}{\mu_2(x-\xi)(\mu_1 x-\mu_2\xi)}}\mathrm{J}_1\left(\frac{2}{\mu_1-\mu_2}\sqrt{\frac{\sigma_{12}\sigma_{21}(x-\xi)(\mu_1 x-\mu_2\xi)}{\mu_2}}\right) \tag{104}$$

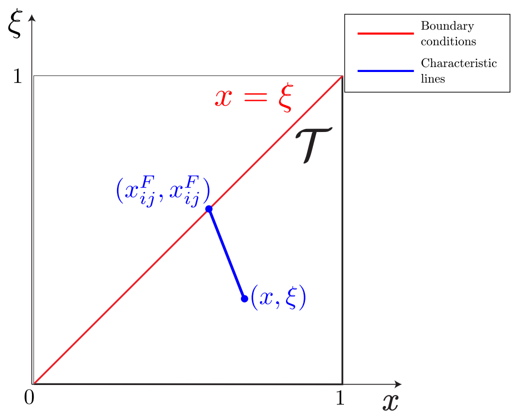

Fig. 3: Characteristic lines of the K kernels

here due to the existence of *homodirectional controlled* states, which lead to the homodirectional kernel PDEs (37).

A. Method of characteristics

1) Characteristics of the K kernels: For each $1 \leq i \leq m$, $1 \leq j \leq n$, and $(x,\xi) \in \mathcal{T}$, we define the following characteristic lines $(x_{ij}(x,\xi;\cdot), \xi_{ij}(x,\xi;\cdot))$ corresponding to Equations (36)

$$\begin{cases} \frac{dx_{ij}}{ds}(x,\xi;s) = -\mu_i, \quad s \in \left[0, s^F_{ij}(x,\xi)\right] \\ x_{ij}(x,\xi;0) = x, \;\; x_{ij}(x,\xi;s^F_{ij}(x,\xi)) = x^F_{ij}(x,\xi) \end{cases}, \tag{105}$$

$$\begin{cases} \frac{d\xi_{ij}}{ds}(x,\xi;s) = \lambda_j, \quad s \in \left[0, s^F_{ij}(x,\xi)\right] \\ \xi_{ij}(x,\xi;0) = \xi, \;\; \xi_{ij}(x,\xi;s^F_{ij}(x,\xi)) = x^F_{ij}(x,\xi) \end{cases} \tag{106}$$

These lines, depicted on Figure 3, originate at the point (x,ξ) and terminate on the hypothenuse at the point $\left(x^F_{ij}(x,\xi), x^F_{ij}(x,\xi)\right)$. The expressions of $x_{ij}(x,\xi;s)$, $\xi_{ij}(x,\xi;s)$ $s^F_{ij}(x,\xi)$ and $x^F_{ij}(x,\xi)$ are omitted here for simplicity, but are straightforward to compute. Integrating (36) along these characteristic lines and plugging in the boundary condition (38) yields

$$K_{ij}(x,\xi) = k_{ij} + \int_0^{s^F_{ij}(x,\xi)} \left[\sum_{k=1}^n \sigma^{++}_{kj} K_{ik}(x_{ij}(x,\xi;s), \xi_{ij}(x,\xi;s)) + \sum_{p=1}^m \sigma^{-+}_{pj} L_{ip}(x_{ij}(x,\xi;s), \xi_{ij}(x,\xi;s))\right] ds \tag{107}$$

2) Characteristics of the L kernels: For each $1 \leq i \leq m$, $1 \leq j \leq m$, and $(x,\xi) \in \mathcal{T}$, we define the following characteristic lines $(\chi_{ij}(x,\xi;\cdot), \zeta_{ij}(x,\xi;\cdot))$ corresponding to Equations (37)

$$\begin{cases} \frac{d\chi_{ij}}{d\nu}(x,\xi;\nu) = \epsilon_{ij}\mu_i, \quad \nu \in \left[0, \nu^F_{ij}(x,\xi)\right] \\ \chi_{ij}(x,\xi;0) = x, \;\; \chi_{ij}(x,\xi;\nu^F_{ij}(x,\xi)) = \chi^F_{ij}(x,\xi) \end{cases}, \tag{108}$$

$$\begin{cases} \frac{d\zeta_{ij}}{d\nu}(x,\xi;\nu) = \epsilon_{ij}\mu_j, \quad \nu \in \left[0, \nu^F_{ij}(x,\xi)\right] \\ \zeta_{ij}(x,\xi;0) = \xi, \;\; \zeta_{ij}(x,\xi;\nu^F_{ij}(x,\xi)) = \zeta^F_{ij}(x,\xi) \end{cases} \tag{109}$$

where ϵ_{ij} is defined by

$$\epsilon_{ij}(x,\xi) = \begin{cases} 1 & \text{if } i > j \\ -1 & \text{otherwise} \end{cases} \tag{110}$$

These lines all originate at (x,ξ) and terminate on $\partial\mathcal{T}$ at the point $\left(\chi^F_{ij}(x,\xi), \zeta^F_{ij}(x,\xi)\right)$. They are depicted on Figures 4–6 in the three distinct cases $i < j$, $i = j$ and $i > j$. The detailed expressions of $\chi_{ij}(x,\xi;s)$, $\zeta_{ij}(x,\xi;s)$ $\nu^F_{ij}(x,\xi)$, $\chi^F_{ij}(x,\xi)$ and $\zeta^F_{ij}(x,\xi)$ are, again, omitted here because of space constraints. Integrating (37) along these characteristics and plugging in the boundary conditions (39),(40) and (41) yields

$$L_{ij}(x,\xi) = \delta_{ij}(x,\xi) l_{ij} + \left(1-\delta_{ij}(x,\xi)\right)\frac{1}{\mu_j}\sum_{r=1}^n \lambda_r q_{rj} K_{ir}(\chi^F_{ij}(x,\xi),0) - \epsilon_{ij}\int_0^{\nu^F_{ij}(x,\xi)} \left[\sum_{p=1}^m \sigma^{--}_{pj} L_{ip}\left(\chi_{ij}(x,\xi;\nu), \zeta_{ij}(x,\xi;\nu)\right) + \sum_{k=1}^n \sigma^{+-}_{kj} K_{ik}\left(\chi_{ij}(x,\xi;\nu), \zeta_{ij}(x,\xi;\nu)\right)\right] d\nu \tag{111}$$

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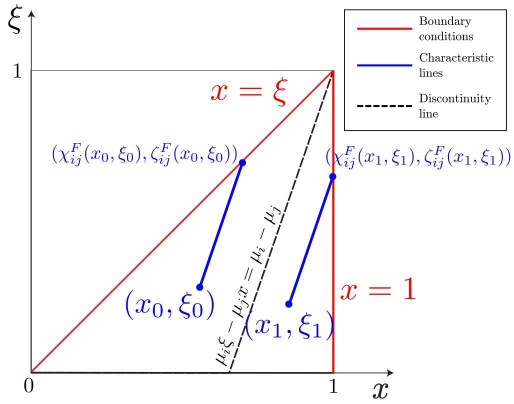

Fig. 4: Characteristic lines of the kernels L_{ij} for $i > j$

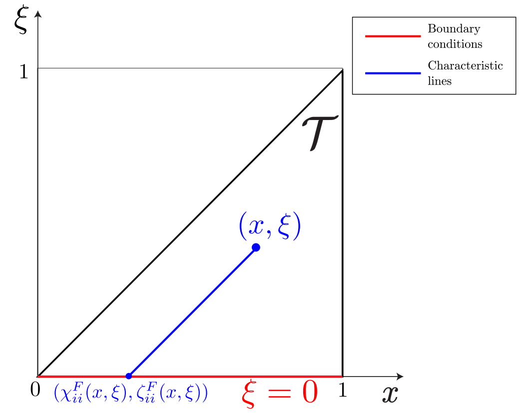

Fig. 5: Characteristic lines of the kernels L_{ii}

where the coefficient $\delta_{ij}(x,\xi)$, defined by

$$\delta_{ij}(x,\xi) = \begin{cases} 0 & \text{if } i = j \\ 0 & \text{if } i < j \text{ and } \mu_i\xi - \mu_j x \leq 0 \\ 1 & \text{otherwise} \end{cases}, \tag{112}$$

reflects the fact that some characteristics terminate on the $\xi = 0$ boundary of $\mathcal{T}$, while others terminate on the hypotenuse or on the $x = 1$ boundary of $\mathcal{T}$. Plugging in (107) evaluated at $(\chi^F_{ij}(x,\xi), 0)$ yields

$$\begin{aligned} L_{ij}(x,\xi) &= \delta_{ij}(x,\xi) l_{ij} + \left(1 - \delta_{ij}(x,\xi)\right) \frac{1}{\mu_j} \sum_{r=1}^{n} \lambda_r q_{rj} k_{ir} \\ &+ \left(1 - \delta_{ij}(x,\xi)\right) \frac{1}{\mu_j} \sum_{r=1}^{n} \lambda_r q_{rj} \int_0^{s^F_{ir}(\chi^F_{ij}(x,\xi),0)} \\ &\Bigg[\sum_{k=1}^{n} \sigma^{++}_{kr} K_{ik}(x_{ir}(\chi^F_{ij}(x,\xi),0;s), \xi_{ir}(\chi^F_{ij}(x,\xi),0;s)) \\ &+ \sum_{p=1}^{m} \sigma^{-+}_{pr} L_{ip}(x_{ir}(\chi^F_{ij}(x,\xi),0;s), \xi_{ir}(\chi^F_{ij}(x,\xi),0;s)) \Bigg] ds \\ &- \epsilon_{ij} \int_0^{\nu^F_{ij}(x,\xi)} \Bigg[\sum_{p=1}^{m} \sigma^{--}_{pj} L_{ip}\left(\chi_{ij}(x,\xi;\nu), \zeta_{ij}(x,\xi;\nu)\right) + \\ &\sum_{k=1}^{n} \sigma^{+-}_{kj} K_{ik}\left(\chi_{ij}(x,\xi;\nu), \zeta_{ij}(x,\xi;\nu)\right) \Bigg] d\nu \end{aligned} \tag{113}$$

B. Method of successive approximations

We now use the method of successive approximations to solve equations (107),(113). Define first

$$\forall 1 \leq i \leq m, 1 \leq j \leq n, \quad \varphi_{ij}(x,\xi) = k_{ij}, \tag{114}$$

$$\forall 1 \leq i \leq m, 1 \leq j \leq m,$$

$$\psi_{ij}(x,\xi) = \delta_{ij}(x,\xi) l_{ij} + \left(1 - \delta_{ij}(x,\xi)\right) \frac{1}{\mu_j} \sum_{r=1}^{n} \lambda_r q_{rj} k_{ir} \tag{115}$$

Besides, we define $\mathbf{H}$ as the vector containing all the kernels, reordered line by line and stacked up, and similarly $\boldsymbol{\phi}$, as

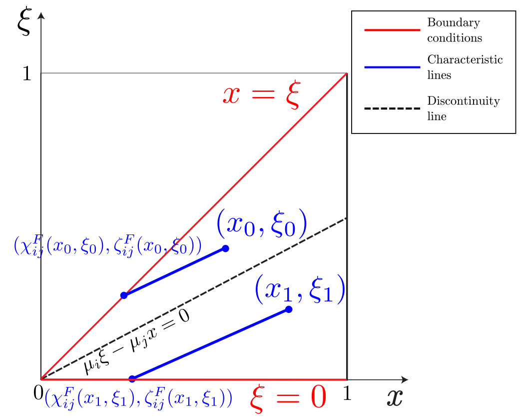

Fig. 6: Characteristic lines of the kernels L_{ij} for $i < j$

follows

$$\mathbf{H} = \begin{pmatrix} H_1 \\ \vdots \\ H_{nm} \\ H_{nm+1} \\ \vdots \\ H_{nm+m^2} \end{pmatrix} = \begin{pmatrix} K_{11} \\ \vdots \\ K_{mn} \\ L_{11} \\ \vdots \\ L_{mm} \end{pmatrix}, \quad \boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{nm} \\ \phi_{nm+1} \\ \vdots \\ \phi_{nm+m^2} \end{pmatrix} = \begin{pmatrix} \varphi_{11} \\ \vdots \\ \varphi_{mn} \\ \psi_{11} \\ \vdots \\ \psi_{mm} \end{pmatrix} \tag{116}$$

We consider the following linear operators acting on $\mathbf{H}$, for $1 \leq i \leq m, 1 \leq j \leq n$

$$\begin{aligned} \Phi_{ij}[\mathbf{H}](x,\xi) = & \\ \int_0^{s^F_{ij}(x,\xi)} & \Bigg[\sum_{k=1}^{n} \sigma^{++}_{kj} K_{ik}(x_{ij}(x,\xi;s), \xi_{ij}(x,\xi;s)) \\ & + \sum_{p=1}^{m} \sigma^{-+}_{pj} L_{ip}(x_{ij}(x,\xi;s), \xi_{ij}(x,\xi;s)) \Bigg] ds \end{aligned} \tag{117}$$

and for $1 \leq i \leq m, 1 \leq j \leq m$

$$\begin{aligned}\Psi_{ij}[\mathbf{H}](x,\xi) = &\\ &\Big(1-\delta_{ij}(x,\xi)\Big)\frac{1}{\mu_j}\sum_{r=1}^{n}\lambda_r q_{rj}\int_0^{s_{ir}^F(\chi_{ij}^F(x,\xi),0)}\\ &\Bigg[\sum_{k=1}^{n}\sigma_{kr}^{++}K_{ik}(x_{ir}(\chi_{ij}^F(x,\xi),0;s),\xi_{ir}(\chi_{ij}^F(x,\xi),0;s))\\ &+\sum_{p=1}^{m}\sigma_{pr}^{-+}L_{ip}(x_{ir}(\chi_{ij}^F(x,\xi),0;s),\xi_{ir}(\chi_{ij}^F(x,\xi),0;s))\Bigg]ds\\ &-\epsilon_{ij}\int_0^{\nu_{ij}^F(x,\xi)}\Bigg[\sum_{p=1}^{m}\sigma_{pj}^{--}L_{ip}\Big(\chi_{ij}(x,\xi;\nu),\zeta_{ij}(x,\xi;\nu)\Big)+\\ &\sum_{k=1}^{n}\sigma_{kj}^{+-}K_{ik}\Big(\chi_{ij}(x,\xi;\nu),\zeta_{ij}(x,\xi;\nu)\Big)\Bigg]d\nu. \quad (118)\end{aligned}$$

Define then the following sequence

$$\mathbf{H}^0(x,\xi) = 0, \quad (119)$$

$$\mathbf{H}^q(x,\xi) = \boldsymbol{\phi}(x,\xi) + \mathbf{\Phi}[\mathbf{H}^{q-1}](x,\xi) \quad (120)$$

$$= \begin{pmatrix} \varphi_{11}(x,\xi)+\Phi_{11}[\mathbf{H}^{q-1}](x,\xi)\\ \vdots\\ \varphi_{1n}(x,\xi)+\Phi_{1n}[\mathbf{H}^{q-1}](x,\xi)\\ \varphi_{21}(x,\xi)+\Phi_{21}[\mathbf{H}^{q-1}](x,\xi)\\ \vdots\\ \varphi_{mn}(x,\xi)+\Phi_{mn}[\mathbf{H}^{q-1}](x,\xi)\\ \psi_{11}(x,\xi)+\Psi_{11}[\mathbf{H}^{q-1}](x,\xi)\\ \vdots\\ \psi_{mm}(x,\xi)+\Psi_{mm}[\mathbf{H}^{q-1}](x,\xi)\end{pmatrix} \quad (121)$$

One should notice that if the limit exists, then $\mathbf{H} = \lim_{q\to+\infty}\mathbf{H}^q(x,\xi)$ is a solution of the integral equations, and thus solves the original hyperbolic system. Besides, define for $q \geq 1$ the increment $\Delta\mathbf{H}^q = \mathbf{H}^q - \mathbf{H}^{q-1}$, with $\Delta\mathbf{H}^0 = \boldsymbol{\phi}$ by definition. Since the functional $\mathbf{\Phi}$ is linear, the following equation $\Delta\mathbf{H}^q(x,\xi) = \mathbf{\Phi}[\mathbf{H}^{q-1}](x,\xi)$ holds. Using the definition of $\Delta\mathbf{H}^q$, it follows that if the sum $\sum_{q=0}^{+\infty}\Delta\mathbf{H}^q(x,\xi)$ is finite, then

$$\mathbf{H}(x,\xi) = \sum_{q=0}^{+\infty}\Delta\mathbf{H}^q(x,\xi) \quad (122)$$

In the next section, we prove convergence of the series in L^∞.

C. Convergence of the successive approximation series

To prove convergence of the series, we look for a recursive upper bound, similarly to, e.g. [13]. More precisely, let ϵ be such that

$$0 < \epsilon < 1 - \max_{1\leq j<i\leq m}\left\{\frac{\mu_i}{\mu_j}\right\}. \quad (123)$$

Then, the following result holds

Proposition 6.1: For $q \geq 1$, assume that

$$\forall (x,\xi)\in\mathcal{T},\ \forall i = 1,...,nm+m^2$$
$$\left|\Delta H_i^q(x,\xi)\right| \leq \bar{\phi}\frac{M^q(x-(1-\epsilon)\xi)^q}{q!} \quad (124)$$

where $\Delta H_i^q(x,\xi)$ denotes the i-th ($i = 1,\cdots,mn+m^2$) component of $\Delta H^q(x,\xi)$, then it follows that

$$\forall (x,\xi)\in\mathcal{T},\ \forall i = 1,...,m,\ \forall j = 1,...,n,$$
$$\left|\Phi_{ij}[\Delta\mathbf{H}^q](x,\xi)\right| \leq \bar{\phi}\frac{M^{q+1}(x-(1-\epsilon)\xi)^{q+1}}{(q+1)!} \quad (125)$$

and

$$\forall (x,\xi)\in\mathcal{T},\ \forall i = 1,...,m,\ \forall j = 1,...,m,$$
$$\left|\Psi_{ij}[\Delta\mathbf{H}^q](x,\xi)\right| \leq \bar{\phi}\frac{M^{q+1}(x-(1-\epsilon)\xi)^{q+1}}{(q+1)!}. \quad (126)$$

The proof of this proposition relies on the following Lemma, which is crucial and different from previous works (see [8, Lemma A.4]).

Lemma 6.2: For $q \in \mathbb{N}$, $(x,\xi)\in\mathcal{T}$, and $s_{ij}^F(x,\xi)$, $\nu_{ij}^F(x,\xi)$, $x_{ij}(x,\xi,\cdot)$, $\xi_{ij}(x,\xi,\cdot)$, $\chi_{ij}(x,\xi,\cdot)$, $\zeta_{ij}(x,\xi,\cdot)$ defined as in (105),(106),(108),(109), respectively, the following inequalities holds

$\forall 1\leq i\leq m$, $\forall 1\leq j\leq n$

$$\int_0^{s_{ij}^F(x,\xi)}\Big(x_{ij}(x,\xi;s)-(1-\epsilon)\xi_{ij}(x,\xi;s)\Big)^q ds \leq M_\lambda\frac{(x-(1-\epsilon)\xi)^{q+1}}{q+1} \quad (127)$$

$\forall 1\leq i,j\leq m$

$$\int_0^{\nu_{ij}^F(x,\xi)}\Big(\chi_{ij}(x,\xi;\nu)-(1-\epsilon)\zeta_{ij}(x,\xi;\nu)\Big)^q d\nu \leq M_\lambda\frac{(x-(1-\epsilon)\xi)^{q+1}}{q+1} \quad (128)$$

where

$$M_\lambda = \max_{i,p=1,...,m,j=1,...,n}\left\{\frac{1}{\mu_i+(1-\epsilon)\lambda_j},\frac{1}{-\epsilon_{ij}\Big(\mu_i-(1-\epsilon)\mu_p\Big)}\right\} \quad (129)$$

Proof 6.3: Consider the following change of variables, noting (105),(106),

$$\begin{aligned}\tau &= x_{ij}(x,\xi;s)-(1-\epsilon)\xi_{ij}(x,\xi;s),\\ d\tau &= \left[\frac{dx_{ij}}{ds}(x,\xi;s)-(1-\epsilon)\frac{d\xi_{ij}}{ds}(x,\xi;s)\right]ds\\ &= \Big(-\mu_i-(1-\epsilon)\lambda_j\Big)ds\end{aligned}$$

The left-hand-side of (127) becomes

$$\begin{aligned}&\int_0^{s_{ij}^F(x,\xi)}\Big(x_{ij}(x,\xi;s)-(1-\epsilon)\xi_{ij}(x,\xi;s)\Big)^q ds\\ &= \int_{x-(1-\epsilon)\xi}^{x_{ij}^F(x,\xi)-(1-\epsilon)\xi_{ij}^F(x,\xi)}\frac{-\tau^q}{\mu_i+(1-\epsilon)\lambda_j}d\tau\\ &= \frac{(x-(1-\epsilon)\xi)^{q+1}-\Big(x_{ij}^F(x,\xi)-(1-\epsilon)\xi_{ij}^F(x,\xi)\Big)^{q+1}}{(\mu_i+(1-\epsilon)\lambda_j)(q+1)}\\ &\leq M_\lambda\frac{(x-(1-\epsilon)\xi)^{q+1}}{q+1}\end{aligned}$$

where we have used the fact that for all $1 \leq i \leq m$, $1 \leq j \leq n$, one has

$$x_{ij}^F(x,\xi) - (1-\epsilon)\xi_{ij}^F(x,\xi) \geq 0 \tag{130}$$

which is trivially satisfied since $(x_{ij}^F(x,\xi), \xi_{ij}^F(x,\xi)) \in \partial\mathcal{T}$ and $\epsilon > 0$. Consider now the following change of variables

$$\begin{aligned}\tau &= \chi_{ij}(x,\xi;s) - (1-\epsilon)\zeta_{ij}(x,\xi;s),\\ d\tau &= \left[\frac{d\chi_{ij}}{ds}(x,\xi;s) - (1-\epsilon)\frac{d\zeta_{ij}}{ds}(x,\xi;s)\right]ds\\ &= \epsilon_{ij}\left(\mu_i - (1-\epsilon)\mu_j\right)ds\end{aligned}$$

Thus, the left-hand-side of (128) becomes

$$\begin{aligned}&\int_0^{\nu_{ij}^F(x,\xi)} \left(\chi_{ij}(x,\xi;\nu) - (1-\epsilon)\zeta_{ij}(x,\xi;\nu)\right)^q d\nu\\ &= \int_{x-(1-\epsilon)\xi}^{\chi_{ij}^F(x,\xi)-(1-\epsilon)\zeta_{ij}^F(x,\xi)} \frac{\tau^q}{\epsilon_{ij}\left(\mu_i - (1-\epsilon)\mu_j\right)} d\tau \qquad (131)\\ &= \frac{(x-(1-\epsilon)\xi)^{q+1} - \left(\chi_{ij}^F(x,\xi) - (1-\epsilon)\zeta_{ij}^F(x,\xi)\right)^{q+1}}{-\epsilon_{ij}\left(\mu_i - (1-\epsilon)\mu_j\right)(q+1)}\end{aligned}$$

Given the definition of ϵ_{ij} given by (110), one has

$$-\epsilon_{ij}\left(\mu_i - (1-\epsilon)\mu_j\right) = \begin{cases}\mu_i - (1-\epsilon)\mu_j & \text{if } i \leq j\\ (1-\epsilon)\mu_j - \mu_i & \text{if } i > j\end{cases}$$

Therefore, given the definition of ϵ (Equation (123)) in the case $i > j$ and the ordering of the μ_i in the case $i \leq j$, one has

$$-\epsilon_{ij}\left(\mu_i - (1-\epsilon)\mu_j\right) > 0 \tag{132}$$

Besides, since $(\chi_{ij}^F(x,\xi), \zeta_{ij}^F(x,\xi)) \in \mathcal{T}$, one has $\left(\chi_{ij}^F(x,\xi) - (1-\epsilon)\zeta_{ij}^F(x,\xi)\right) > 0$ and (131) becomes

$$\int_0^{\nu_{ij}^F(x,\xi)} \left(\chi_{ij}(x,\xi;\nu) - (1-\epsilon)\zeta_{ij}(x,\xi;\nu)\right)^q d\nu \leq M_\lambda \frac{(x-(1-\epsilon)\xi)^{q+1}}{q+1} \tag{133}$$

which concludes the proof.

Remark 10: Notice that (132) also implies that, for any $(x,\xi) \in \mathcal{T}$ and $1 \leq i \leq m$, $1 \leq j \leq n$ the function

$$\nu \in [0, \nu_{ij}^F(x,\xi)] \mapsto \chi_{ij}(x,\xi;\nu) - (1-\epsilon)\zeta_{ij}(x,\xi;\nu) \tag{134}$$

is strictly decreasing, in particular the following inequality holds

$$0 \leq \chi_{ij}^F(x,\xi) - (1-\epsilon)\zeta_{ij}^F(x,\xi) \leq x - (1-\epsilon)\xi \tag{135}$$

which will be useful in the proof of Proposition 6.1.

Proof 6.4 (Proof of Proposition 6.1): Define

$$\begin{aligned}\bar{\lambda} &= \max\{\lambda_n, \mu_1\}, \quad \underline{\lambda} = \max\left\{\frac{1}{\lambda_1}, \frac{1}{\mu_n}\right\},\\ \bar{\sigma} &= \max_{i,j}\{\sigma^{++}, \sigma^{-+}, \sigma^{+-}, \sigma^{--}\}, \quad \bar{q} = \max_{i,j}\{q_{ij}\}\\ M &= \left(n\bar{\lambda}\underline{\lambda}\bar{q} + 1\right)(n+m)\bar{\sigma}M_\lambda,\\ \bar{\phi} &= \max_{i,j}\max_{(x,\xi)\in\mathcal{T}}\left\{|\varphi_{i,j}(x,\xi)|,\ |\psi_{i,j}(x,\xi)|\right\}\end{aligned}$$

Let now $q \in \mathbb{N}$ and assume that

$$\forall (x,\xi) \in \mathcal{T},\ \forall i = 1, ..., nm + m^2 \quad \left|\Delta H_i^q(x,\xi)\right| \leq \bar{\phi}\frac{M^q(x-(1-\xi))^q}{q!} \tag{136}$$

Then, for $1 \leq i \leq m$, $1 \leq j \leq n$, $(x,\xi) \in \mathcal{T}$ one has

$$\begin{aligned}&\left|\Phi_{ij}[\Delta\mathbf{H}^q](x,\xi)\right|\\ &\leq \int_0^{s_{ij}^F(x,\xi)} \Bigg|\sum_{k=1}^n \sigma_{kj}^{++}\Delta K_{ik}^q(x_{ij}(x,\xi;s), \xi_{ij}(x,\xi;s))\\ &\qquad + \sum_{p=1}^m \sigma_{pj}^{-+}\Delta L_{ip}^q(x_{ij}(x,\xi;s), \xi_{ij}(x,\xi;s))\Bigg| ds \qquad (137)\end{aligned}$$

using (127) and (136), this yields

$$\begin{aligned}\left|\Phi_{ij}[\Delta\mathbf{H}^q](x,\xi)\right| \leq&\ (n+m)\bar{\sigma}\\ &\cdot \int_0^{s_{ij}^F(x,\xi)} \bar{\phi}\frac{M^q\left(x_{ij}(x,\xi;s) - (1-\epsilon)\xi_{ij}(x,\xi;s)\right)^q}{q!}ds\\ \leq&\ (n+m)\bar{\sigma}\frac{\bar{\phi}M^q}{q!}M_\lambda\frac{(x-(1-\xi))^{q+1}}{q+1}\\ \leq&\ \bar{\phi}\frac{M^{q+1}(x-(1-\epsilon)\xi)^{q+1}}{(q+1)!} \qquad (138)\end{aligned}$$

Similarly, for $1 \leq i, j \leq m$, one gets, using (136)

$$\begin{aligned}&\left|\Psi_{ij}[\Delta\mathbf{H}^q](x,\xi)\right| \leq \bar{\lambda}\underline{\lambda}\bar{q}(n+m)\bar{\sigma}\sum_{r=1}^n \int_0^{s_{ir}^F(\chi_{ij}^F(x,\xi),0)}\\ &\bar{\phi}\frac{M^q\left(x_{ir}(\chi_{ij}^F(x,\xi),0;s) - (1-\epsilon)\xi_{ir}(\chi_{ij}^F(x,\xi),0;s)\right)^q}{q!}ds\\ &\qquad + (n+m)\bar{\sigma}\int_0^{\nu_{ij}^F(x,\xi)}\\ &\qquad \bar{\phi}\frac{M^q\left(\chi_{ij}(x,\xi;\nu) - (1-\epsilon)\zeta_{ij}(x,\xi;\nu)\right)^q}{q!}d\nu \qquad (139)\end{aligned}$$

Then, using (127) at $(x,\xi) = (\chi_{ij}^F(x,\xi), 0)$ and (128) yields

$$\begin{aligned}&\left|\Psi_{ij}[\Delta\mathbf{H}^q](x,\xi)\right|\\ &\leq \bar{\lambda}\underline{\lambda}\bar{q}(n+m)\bar{\sigma}n\bar{\phi}M_\lambda M^q\frac{\left(\chi_{ij}^F(x,\xi) - (1-\epsilon)\zeta_{ij}^F(x,\xi)\right)^{q+1}}{(q+1)!}\\ &\qquad + (n+m)\bar{\sigma}\bar{\phi}\frac{M^q M_\lambda(x-(1-\epsilon)\xi)^{q+1}}{(q+1)!} \qquad (140)\end{aligned}$$

Inequality (135) yields

$$\begin{aligned}&\left|\Psi_{ij}[\Delta\mathbf{H}^q](x,\xi)\right|\\ &\leq \left(n\bar{\lambda}\underline{\lambda}\bar{q} + 1\right)(n+m)\bar{\sigma}\bar{\phi}M_\lambda\frac{M^q(x-(1-\epsilon)\xi)^{q+1}}{(q+1)!}\\ &\leq \bar{\phi}\frac{M^{q+1}(x-(1-\epsilon)\xi)^{q+1}}{(q+1)!} \qquad (141)\end{aligned}$$

which concludes the proof.

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Proposition 6.1 directly leads to Theorem 3.3, since by the same procedures presented in [8] and [13], one has that (122) converges and

$$|\mathbf{H}(x,\xi)| = \left|\sum_{q=0}^{+\infty} \Delta\mathbf{H}^q(x,\xi)\right| \leq \bar{\phi} e^{M(x-(1-\epsilon)\xi)}. \qquad (142)$$

VII. Concluding remarks

We have presented boundary control designs for a general class of linear first-order hyperbolic systems: an output-feedback law for stabilization to zero and a control law ensuring output trajectory tracking.

These results bridge the gap with the results of, e.g. [25], where the null (or weak) controllability of $(n+m)$–state hyperbolic systems is proved but no explicit design is given.

Our results open the door for a large number of related problems to be solved, e.g. collocated observer design, disturbance rejection, similarly to [1], parameter identification as in [12], output-feedback adaptive control as in [4], and stabilization of quasilinear systems as in [8] and [22].

Another important question concerns the degree of freedom given by Equation (41) in the control design. The effect of the boundary value of the kernels on the transient performances of the closed-loop system is non-trivial, yet crucial for applications.

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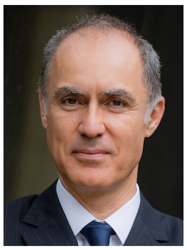
Long Hu received his Ph.D. degrees in Applied Mathematics both from Fundan University, Shanghai, China in June 2015 and from Université Pierre et Marie Curie, Paris, France in September 2015. He is currently an Assistant Professor at Shandong University, Jinan, China. His research interests include nonlinear control theory and nonlinear partial differential equations.

Florent Di Meglio received M.S. (2008) and Ph.D. in Mathematics and Automatic Control (2011) from MINES ParisTech, in France. He currently holds an Associate Professor (Matre-Assistant) position at the Centre Automatique et Systmes, MINES ParisTech, PSL Research University. His research interests lie in Control of Distributed Parameter systems, dynamical systems analysis, estimation and identification with applications in Oil & Gas production and drilling.



Miroslav Krstic holds the Alspach chair and serves as director of the Cymer Center for Control Systems and Dynamics and as Associate Vice Chancellor for Research at UC San Diego. Krstic is a Fellow of IEEE, IFAC, ASME, SIAM, and IET (UK), and Associate Fellow of AIAA. He has coauthored eleven books on adaptive, nonlinear, and stochastic control, extremum seeking, control of PDE systems including turbulent flows, and control of delay systems.

Rafael Vazquez (S'05-M'08-SM'15) received the M.S. and Ph.D. degrees in aerospace engineering from the University of California, San Diego (USA), and degrees in electrical engineering and mathematics from the University of Seville (Spain). He is currently an Associate Professor in the Aerospace Engineering Department, University of Seville. His research interests include control theory, distributed parameter systems, and optimization, with applications to flow control, ATM, UAVs, and orbital mechanics. He is coauthor of the book Control of Turbulent and Magnetohydrodynamic Channel Flows (Birkhauser, 2007). He currently serves as Associate Editor for Automatica.