Control of the longitudinal flight dynamics of an UAV using adaptive backstepping

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Outline

Outline

Introduction

Introductory slides on Backstepping and Adaptive Control by Prof. Miroslav Krstic

Adaptive Nonlinear Control—A Tutorial

- Backstepping
- Tuning Functions Design

main source:

Nonlinear and Adaptive Control Design (Wiley, 1995)

M. Krstić, I. Kanellakopoulos and P. V. Kokotović

Backstepping is an explicit method to design controllers for linear and nonlinear systems with a very specific structure:

System Description (*Strict feedback form*):

$$
\begin{aligned}
\dot{x}_1 &= F_1(x_1) + G_1(x_1)x_2 \\
\dot{x}_2 &= F_2(x_1, x_2) + G_2(x_1, x_2)x_3 \\
\dot{x}_3 &= F_3(x_1, x_2, x_3) + G_3(x_1, x_2, x_3)x_4 \\
\dots &= \dots \\
\dot{x}_m &= F_m(x_1, x_2, \dots, x_m) + G_m(x_1, x_2, \dots, x_m)u
\end{aligned}
$$

Example: $\dot{x}_1 = x_1^2 - x_1^3 + x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = u$

$\rightarrow$ recursive design

x_1^2 "bad" nonlinearity

$-x_1^3$ "good" nonlinearity

Backstepping (nonadaptive)

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^{\mathrm{T}}\theta, \qquad \varphi(0) = 0 \\ \dot{x}_2 &= u \end{aligned}$$

where θ is known parameter vector and $\varphi(x_1)$ is smooth nonlinear function.

Goal: stabilize the equilibrium $x_1 = 0$, $x_2 = -\varphi(0)^{\mathrm{T}}\theta = 0$.

virtual control for the x_1-equation:

$$\alpha_1(x_1) = -c_1 x_1 - \varphi(x_1)^{\mathrm{T}}\theta, \qquad c_1 > 0$$

error variables:

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 - \alpha_1(x_1), \end{aligned}$$

System in error coordinates:

$$
\begin{aligned}
\dot{z}_1 &= \dot{x}_1 = x_2 + \varphi^{\mathrm{T}}\theta = z_2 + \alpha_1 + \varphi^{\mathrm{T}}\theta = -c_1 z_1 + z_2 \\
\dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 = u - \frac{\partial \alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\theta\right).
\end{aligned}
$$

Need to design $u = \alpha_2(x_1, x_2)$ to stabilize $z_1 = z_2 = 0$.

Choose Lyapunov function

$$
V(x_1, x_2) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2
$$

we have

$$
\begin{aligned}
\dot{V} &= z_1(-c_1 z_1 + z_2) + z_2\left[u - \frac{\partial \alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\theta\right)\right] \\
&= -c_1 z_1^2 + z_2 \underbrace{\left[u + z_1 - \frac{\partial \alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\theta\right)\right]}_{= -c_2 z_2}
\end{aligned}
$$

$$
\Rightarrow \dot{V} = -c_1 z_1^2 - c_2 z_2^2
$$

$z = 0$ is globally asymptotically stable

invertible change of coordinates

$$\Downarrow$$

$x = 0$ is globally asymptotically stable

The closed-loop system in z-coordinates is linear:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Tuning Functions Design

Introductory examples:

A

$$\dot{x}_1 = u + \varphi(x_1)^{\mathrm{T}}\theta$$

B

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^{\mathrm{T}}\theta \\ \dot{x}_2 &= u \end{aligned}$$

C

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi(x_1)^{\mathrm{T}}\theta \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned}$$

where θ is unknown parameter vector and $\varphi(0) = 0$.

Degin A. Let $\hat{\theta}$ be the estimate of θ and $\tilde{\theta} = \theta - \hat{\theta}$,

Using

$$u = -c_1 x_1 - \varphi(x_1)^{\mathrm{T}}\hat{\theta}$$

gives

$$\dot{x}_1 = -c_1 x_1 + \varphi(x_1)^{\mathrm{T}}\tilde{\theta}$$

To find update law for $\hat{\theta}(t)$, choose

$$V_1(x,\hat{\theta}) = \frac{1}{2}x_1^2 + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}$$

then

$$\begin{aligned}\dot{V}_1 &= -c_1x_1^2 + x_1\varphi(x_1)^{\mathrm{T}}\tilde{\theta} - \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\hat{\theta}} \\ &= -c_1x_1^2 + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\underbrace{\left(\Gamma\varphi(x_1)x_1 - \dot{\hat{\theta}}\right)}_{=0}\end{aligned}$$

Update law:

$$\dot{\hat{\theta}} = \Gamma\varphi(x_1)x_1, \qquad \varphi(x_1)\text{—regressor}$$

gives

$$\dot{V}_1 = -c_1x_1^2 \le 0.$$

By Lasalle's invariance theorem, $x_1 = 0, \hat{\theta} = \theta$ is stable and

$$\lim_{t\to\infty} x_1(t) = 0$$

Design B. replace θ by $\hat{\theta}$ in the nonadaptive design:

$$z_2 = x_2 - \alpha_1(x_1, \hat{\theta}), \qquad \alpha_1(x_1, \hat{\theta}) = -c_1 z_1 - \varphi^{\mathrm{T}} \hat{\theta}$$

and strengthen the control law by $\nu_2(x_1, x_2, \hat{\theta})$(to be designed)

$$u = \alpha_2(x_1, x_2, \hat{\theta}) = -c_2 z_2 - z_1 + \frac{\partial \alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}} \hat{\theta}\right) + \nu_2(x_1, x_2, \hat{\theta})$$

error system

$$\begin{aligned}
\dot{z}_1 &= z_2 + \alpha_1 + \varphi^{\mathrm{T}}\theta = -c_1 z_1 + z_2 + \varphi^{\mathrm{T}}\tilde{\theta} \\
\dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\theta\right) - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} \\
&= -z_1 - c_2 z_2 - \frac{\partial \alpha_1}{\partial x_1}\varphi^{\mathrm{T}}\tilde{\theta} - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} + \nu_2(x_1, x_2, \hat{\theta}),
\end{aligned}$$

or

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \varphi^{\mathrm{T}} \\ -\frac{\partial \alpha_1}{\partial x_1}\varphi^{\mathrm{T}} \end{bmatrix} \tilde{\theta} + \underbrace{\begin{bmatrix} 0 \\ -\frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} + \nu_2(x_1, x_2, \hat{\theta}) \end{bmatrix}}_{=0}$$

remaining: design adaptive law.

Choose

$$V_2(x_1, x_2, \hat{\theta}) = V_1 + \frac{1}{2} z_2^2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$$

we have

$$\begin{aligned}
\dot{V}_2 &= -c_1 z_1^2 - c_2 z_2^2 + [z_1,\ z_2] \left[\begin{array}{c} \varphi^T \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi^T \end{array} \right] \tilde{\theta} - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \\
&= -c_1 z_1^2 - c_2 z_2^2 + \tilde{\theta}^T \Gamma^{-1} \left(\Gamma \left[\varphi,\ -\frac{\partial \alpha_1}{\partial x_1} \varphi \right] \left[\begin{array}{c} z_1 \\ z_2 \end{array} \right] - \dot{\hat{\theta}} \right).
\end{aligned}$$

The choice

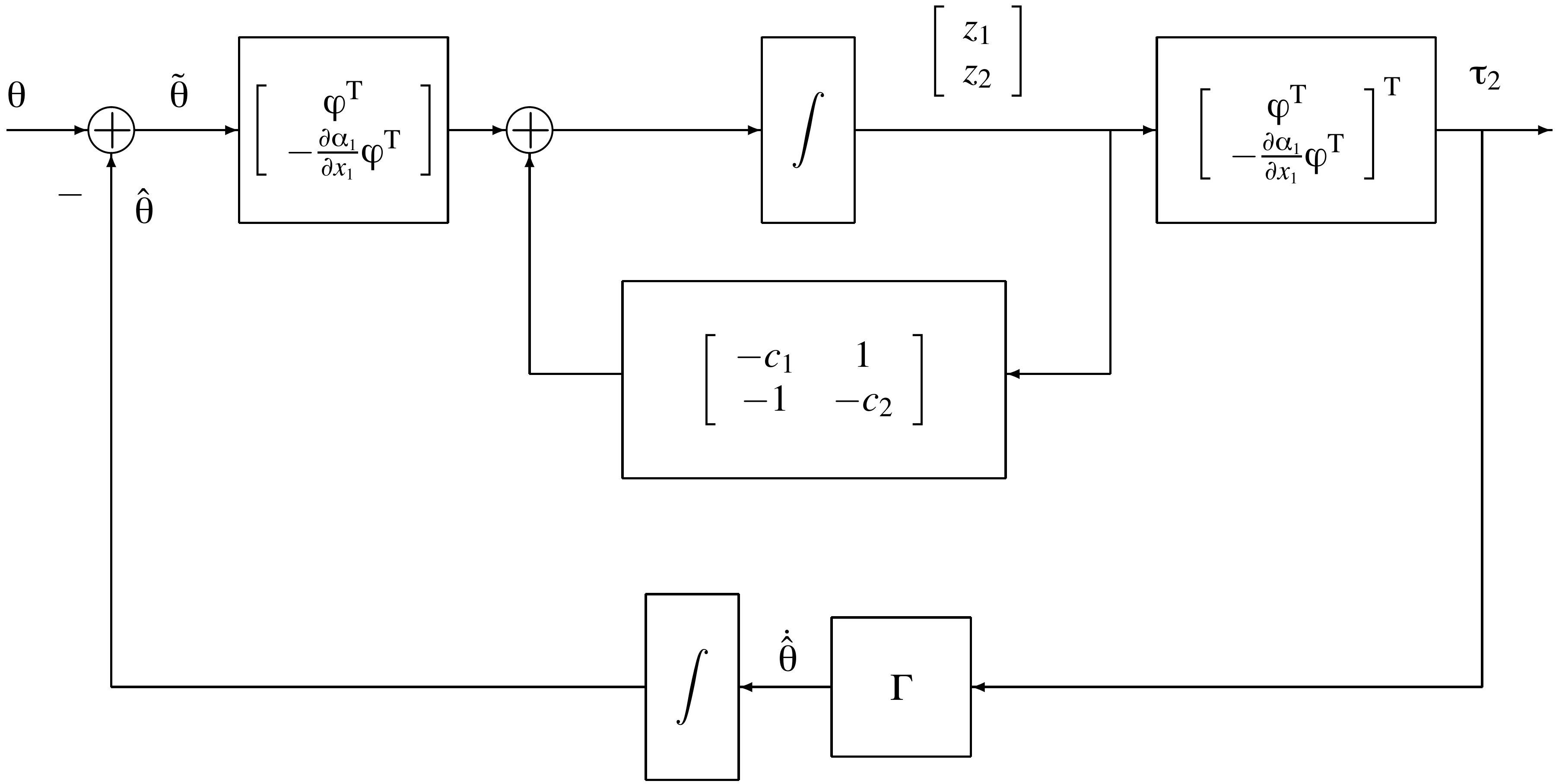
$$\dot{\hat{\theta}} = \Gamma \tau_2(x, \hat{\theta}) = \Gamma \left[\varphi,\ -\frac{\partial \alpha_1}{\partial x_1} \varphi \right] \left[\begin{array}{c} z_1 \\ z_2 \end{array} \right] = \Gamma \underbrace{\left(\overbrace{\varphi z_1}^{\tau_1} - \frac{\partial \alpha_1}{\partial x_1} \varphi z_2 \right)}_{\tau_2}$$

($\tau_1,\ \tau_2$ are called tuning functions)

makes

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2,$$

thus $z = 0,\ \tilde{\theta} = 0$ is GS and $x(t) \to 0$ as $t \to \infty$.



The closed-loop adaptive system

Design C.

We have one more integrator, so we define the third error coordinate and replace $\dot{\hat{\theta}}$ in design B by potential update law,

$$z_3 = x_3 - \alpha_2(x_1, x_2, \hat{\theta})$$

$$\nu_2(x_1, x_2, \hat{\theta}) = \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2(x_1, x_2, \hat{\theta}).$$

Now the z_1, z_2-system is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ z_3 + \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) \end{bmatrix}$$

and

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) + \tilde{\theta}^T (\tau_2 - \Gamma^{-1} \dot{\hat{\theta}}).$$

z_3-equation is given by

$$\begin{aligned}
\dot{z}_3 &= u - \frac{\partial \alpha_2}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\theta\right) - \frac{\partial \alpha_2}{\partial x_2}x_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}} \\
&= u - \frac{\partial \alpha_2}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\hat{\theta}\right) - \frac{\partial \alpha_2}{\partial x_2}x_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial \alpha_2}{\partial x_1}\varphi^{\mathrm{T}}\tilde{\theta}\,.
\end{aligned}$$

Choose

$$V_3(x,\hat{\theta}) = V_2 + \frac{1}{2}z_3^2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}z_3^2 + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\mathbf{\Gamma}^{-1}\tilde{\theta}$$

we have

$$\begin{aligned}
\dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 + z_2\frac{\partial \alpha_1}{\partial \hat{\theta}}(\mathbf{\Gamma}\tau_2 - \dot{\hat{\theta}}) \\
&\quad + z_3\left[z_2 + u - \frac{\partial \alpha_2}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\hat{\theta}\right) - \frac{\partial \alpha_2}{\partial x_2}x_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}}\right] \\
&\quad + \tilde{\theta}^{\mathrm{T}}\left(\tau_2 - \frac{\partial \alpha_2}{\partial x_1}\varphi z_3 - \mathbf{\Gamma}^{-1}\dot{\hat{\theta}}\right)\,.
\end{aligned}$$

Pick update law

$$\dot{\hat{\theta}} = \Gamma\tau_3(x_1,x_2,x_3,\hat{\theta}) \;=\; \Gamma\left(\tau_2 - \frac{\partial\alpha_2}{\partial x_1}\varphi z_3\right) = \Gamma\left[\varphi, \frac{\partial\alpha_1}{\partial x_1}\varphi,\; -\frac{\partial\alpha_2}{\partial x_1}\varphi\right]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

and control law

$$u = \alpha_3(x_1,x_2,x_3,\hat{\theta}) \;=\; -z_2 - c_3 z_3 + \frac{\partial\alpha_2}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\hat{\theta}\right) + \frac{\partial\alpha_2}{\partial x_2}x_3 + \nu_3,$$

results in

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}(\Gamma\tau_2 - \dot{\hat{\theta}}) + z_3\left(\nu_3 - \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}}\right).$$

Notice

$$\dot{\hat{\theta}} - \Gamma\tau_2 = \dot{\hat{\theta}} - \Gamma\tau_3 - \Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi z_3$$

we have

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3\underbrace{\left(\nu_3 - \frac{\partial\alpha_2}{\partial\hat{\theta}}\Gamma\tau_3 + \frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi z_2\right)}_{=0}.$$

Stability and regulation of x to zero follows.

Further insight:

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 \\ 0 & -1 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial\alpha_1}{\partial x_1}\varphi^T \\ -\frac{\partial\alpha_2}{\partial x_1}\varphi^T \end{bmatrix} \tilde\theta + \begin{bmatrix} 0 \\ \frac{\partial\alpha_1}{\partial\hat\theta}(\Gamma\tau_2 - \dot{\hat\theta}) \\ \nu_3 - \frac{\partial\alpha_2}{\partial\hat\theta}\Gamma\tau_3 \end{bmatrix}.$$

$$\Downarrow \dot{\hat\theta} - \Gamma\tau_2 = \dot{\hat\theta} - \Gamma\tau_3 - \Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi z_3$$

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 + \frac{\partial\alpha_1}{\partial\hat\theta}\Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi \\ 0 & -1 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial\alpha_1}{\partial x_1}\varphi^T \\ -\frac{\partial\alpha_2}{\partial x_1}\varphi^T \end{bmatrix} \tilde\theta + \begin{bmatrix} 0 \\ 0 \\ \nu_3 - \frac{\partial\alpha_2}{\partial\hat\theta}\Gamma\tau_3 \end{bmatrix}$$

$\Downarrow$ seletion of ν_3

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 + \frac{\partial\alpha_1}{\partial\hat\theta}\Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi \\ 0 & -1 - \frac{\partial\alpha_1}{\partial\hat\theta}\Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial\alpha_1}{\partial x_1}\varphi^T \\ -\frac{\partial\alpha_2}{\partial x_1}\varphi^T \end{bmatrix} \tilde\theta.$$

General Recursive Design Procedure

parametric strict-feedback system:

$$\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_1(x_1)^{\mathrm{T}}\theta \\
\dot{x}_2 &= x_3 + \varphi_2(x_1, x_2)^{\mathrm{T}}\theta \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \varphi_{n-1}(x_1, \ldots, x_{n-1})^{\mathrm{T}}\theta \\
\dot{x}_n &= \beta(x)u + \varphi_n(x)^{\mathrm{T}}\theta \\
y &= x_1
\end{aligned}$$

where β and φ_i are smooth.

Objective: asymptotically track reference output $y_r(t)$, with $y_r^{(i)}(t), i = 1, \cdots, n$ known, bounded and piecewise continuous.

Tuning functions design for tracking ($z_0 \stackrel{\triangle}{=} 0$, $\alpha_0 \stackrel{\triangle}{=} 0$, $\tau_0 \stackrel{\triangle}{=} 0$)

$$
\begin{aligned}
z_i &= x_i - y_{\mathrm{r}}^{(i-1)} - \alpha_{i-1} \\
\alpha_i(\bar{x}_i, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(i-1)}) &= -z_{i-1} - c_i z_i - w_i^{\mathrm{T}}\hat{\theta} + \sum_{k=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_k}x_{k+1} + \frac{\partial\alpha_{i-1}}{\partial y_{\mathrm{r}}^{(k-1)}}y_{\mathrm{r}}^{(k)}\right) + \nu_i \\
\nu_i(\bar{x}_i, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(i-1)}) &= +\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma\tau_i + \sum_{k=2}^{i-1}\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\Gamma w_i z_k \\
\tau_i(\bar{x}_i, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(i-1)}) &= \tau_{i-1} + w_i z_i \\
w_i(\bar{x}_i, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(i-2)}) &= \varphi_i - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}\varphi_k
\end{aligned}
$$

$$i = 1, \ldots, n$$

$$\bar{x}_i = (x_1, \ldots, x_i), \qquad \bar{y}_{\mathrm{r}}^{(i)} = (y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i)})$$

Adaptive control law:

$$u = \frac{1}{\beta(x)}\left[\alpha_n(x, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(n-1)}) + y_{\mathrm{r}}^{(n)}\right]$$

Parameter update law:

$$\dot{\hat{\theta}} = \Gamma\tau_n(x, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(n-1)}) = \Gamma W z$$

Closed-loop system

$$\begin{aligned}\dot{z} &= A_z(z,\hat{\theta},t)z + W(z,\hat{\theta},t)^{\mathrm{T}}\tilde{\theta} \\ \dot{\hat{\theta}} &= \Gamma W(z,\hat{\theta},t)z,\end{aligned}$$

where

$$A_z(z,\hat{\theta},t) = \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 & 1+\sigma_{23} & \cdots & \sigma_{2n} \\ 0 & -1-\sigma_{23} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1+\sigma_{n-1,n} \\ 0 & -\sigma_{2n} & \cdots & -1-\sigma_{n-1,n} & -c_n \end{bmatrix}$$

$$\sigma_{jk}(x,\hat{\theta}) = -\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_k$$

This structure ensures that the Lyapunov function

$$V_n = \frac{1}{2} z^{\mathrm{T}} z + \frac{1}{2} \tilde{\theta}^{\mathrm{T}} \Gamma^{-1} \tilde{\theta}$$

has derivative

$$\dot{V}_n = -\sum_{k=1}^{n} c_k z_k^2.$$

Major Applications of Adaptive Nonlinear Control

- **Electric Motors Actuating Robotic Loads**

  Nonlinear Control of Electric Machinery, Dawson, Hu, Burg, 1998.
- **Marine Vehicles** (ships, UUVs; dynamic positioning, way point tracking, maneuvering)

  Marine Control Systems, Fossen, 2002
- **Automotive Vehicles** (lateral and longitudinal control, traction, overall dynamics)

  The groups of Tomizuka and Kanellakopoulos.

Dozens of other occasional applications, including: aircraft wing rock, compressor stall and surge, satellite attitude control.

Other Books on Adaptive NL Control Theory Inspired by [KKK]

1. Marino and Tomei (1995),
   Nonlinear Control Design: Geometric, Adaptive, and Robust

2. Freeman and Kokotovic (1996),
   Robust Nonlinear Control Design: State Space and Lyapunov Techniques

3. Qu (1998),
   Robust Control of Nonlinear Uncertain Systems

4. Krstic and Deng (1998),
   Stabilization of Nonlinear Uncertain Systems

5. Ge, Hang, Lee, Zhang (2001),
   Stable Adaptive Neural Network Control

6. Spooner, Maggiore, Ordonez, and Passino (2002),
   Stable Adaptive Control and Estimation for Nonlinear Systems: Neural and Fuzzy Approximation Techniques

7. French, Szepesvari, Rogers (2003),
   Performance of Nonlinear Approximate Adaptive Controllers

Adaptive NL Control/Backstepping Coverage in Major Texts

1. Khalil (1995/2002),
   Nonlinear Systems

2. Isidori (1995),
   Nonlinear Control Systems

3. Sastry (1999),
   Nonlinear Systems: Analysis, Stability, and Control

4. Astrom and Wittenmark (1995),
   Adaptive Control

Outline

Application

Our goal:

Design airplane flight control systems for any operating point, with minimum aerodynamic modeling

The challenge:

- Nonlinear equations
- Aerodynamic models are difficult to obtain and often inaccurate

Our approach:

- Consider only airplane longitudinal motion
- Separate controllers to regulate the aerodynamic velocity and the flight path angle
- Adaptive control to estimate unknown model parameters
- Backstepping scheme to exploit system structure

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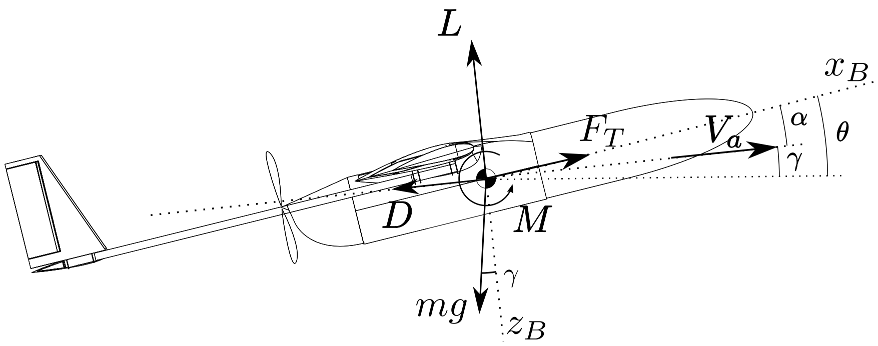
Application

Lecture based on two papers:

- F. Gavilan, J. A. Acosta, R. Vazquez, "Control of the longitudinal flight dynamics of an UAV using adaptive backstepping," IFAC World Congress, 2011. Initial Design and Ideas
- F. Gavilan, R. Vazquez and J. A. Acosta, "Adaptive Backstepping Control for UAV Longitudinal Flight Dynamics with Thrust Saturation," Journal of Guidance, Control and Dynamics, Vol. 38, No. 4, pp. 651-661, 2015. Including saturation in thrust and more realistic measurements

Outline

Longitudinal aircraft equations of motion in wind axes.



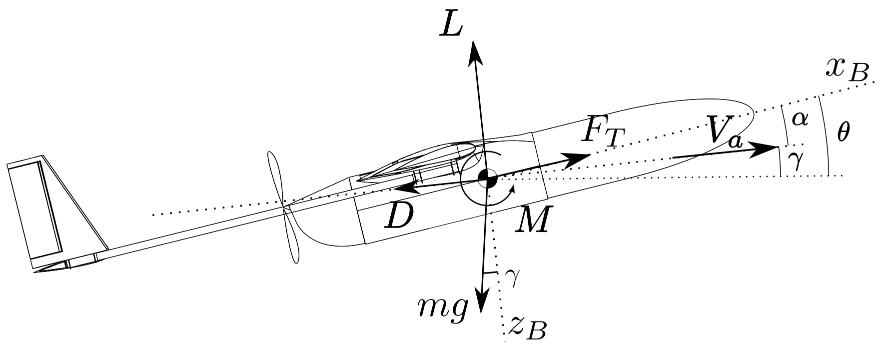
$$
\begin{aligned}
\dot{V}_a &= \frac{1}{m}(-D + F_T\cos\alpha - mg\sin\gamma) \\
\dot{\gamma} &= \frac{1}{mV_a}(L + F_T\sin\alpha - mg\cos\gamma) \\
\dot{\theta} &= q \\
\dot{q} &= \frac{M}{I_y} \\
\alpha &= \theta - \gamma
\end{aligned}
$$

Where:

- m and I_y are the aircraft mass and inertia.
- V_a is the aerodynamic speed.
- α, γ and θ are the angles of attack, flight path and pitch.
- q is the pitch velocity.
- D, L and M are the drag, lift and aerodynamic pitching moment.
- F_T is the thrust.

Aircraft model

Longitudinal aircraft equations of motion in wind axes.



$$\begin{aligned}
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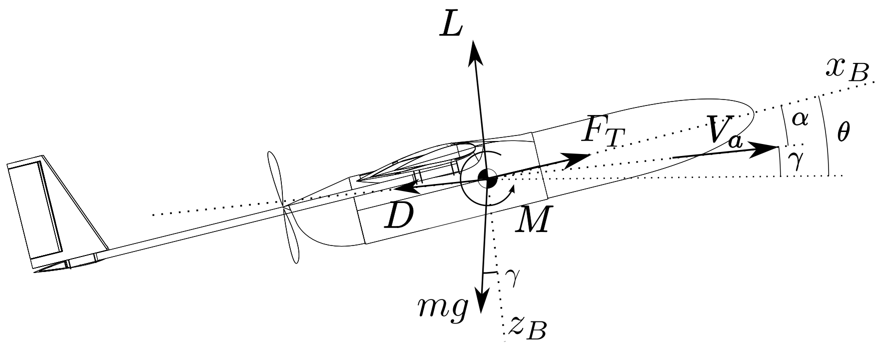
\dot{\theta} &= q \\
\dot{q} &= \frac{M}{I_y} \\
\alpha &= \theta - \gamma
\end{aligned}$$

Regulated variables:

- Aerodynamic velocity (V_a)
- Flight path angle (γ)

Longitudinal aircraft equations of motion in wind axes.

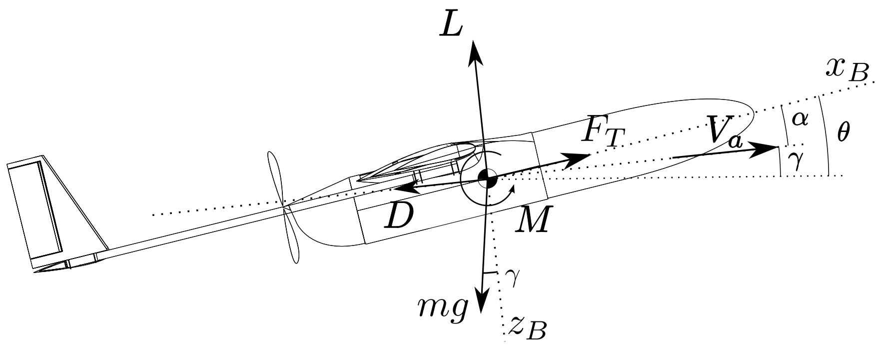


$$
\begin{aligned}
\dot{V}_a &= \frac{1}{m}\left(-D + F_T\cos\alpha - mg\sin\gamma\right)\\
\dot{\gamma} &= \frac{1}{mV_a}\left(L + F_T\sin\alpha - mg\cos\gamma\right)\\
\dot{\theta} &= q\\
\dot{q} &= \frac{M}{I_y}\\
\alpha &= \theta - \gamma
\end{aligned}
$$

Regulated variables:

- Aerodynamic velocity (V_a)
- Flight path angle (γ)

Longitudinal aircraft equations of motion in wind axes.



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Control signals:

- Thrust (F_T).
- Elevator deflection (δ_e)

$$M = f(\delta_e)$$

Aircraft model

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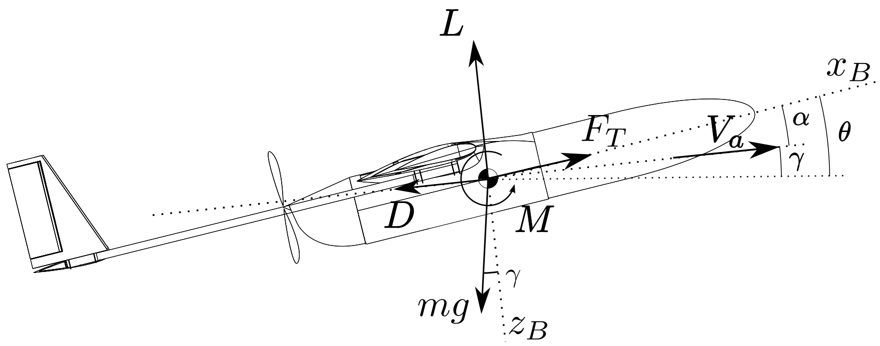

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Aerodynamic coefficients definition:

$$L = \frac{1}{2}\rho V_a^2 S C_L; \quad D = \frac{1}{2}\rho V_a^2 S C_D; \quad M = \frac{1}{2}\rho V_a^2 S \bar{c} C_m$$

✓ C_L, C_D and C_m are very difficult to model accurately.
✓ They are nonlinear functions of the aircraft state and the control signals.

A classic approach: linear aerodynamics

$$\begin{aligned} C_L &= C_{L_0} + C_{L_\alpha}\alpha + C_{L_q} q + C_{L_{\delta_e}}\delta_e \\ C_D &= C_{D_0} + C_{D_\alpha}\alpha + C_{D_{\delta_e}}\delta_e \\ C_m &= C_{m_0} + C_{m_\alpha}\alpha + C_{m_q} q + C_{m_{\delta_e}}\delta_e \end{aligned}$$

- Linear aerodynamic models are widely used in aircraft control.
- The coefficients depend on the flight condition. Controllers are designed for **one** operating point.
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- Stability derivatives are airplane-dependent.

Aerodynamic model

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✓ **Our challenge:** exploit the knowledge of aerodynamic properties to design controllers with minimum aerodynamic modeling.

Aerodynamic model used:

$$\begin{aligned} C_L &= f(\alpha) \\ C_D &= C_{D_0} + k_1\alpha + k_2\alpha^2 \\ C_m &= C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}q + C_{m_{\delta_e}}\delta_e \end{aligned}$$

- The lift coefficient satisfies $\alpha \cdot C_L(\alpha) \geq 0$
- Parabolic drag model, with unknown coefficients. It is considered that $C_D > 0$
- Coefficients in the aerodynamic moment model are unknown, except $C_{m_{\delta_e}}$

Aerodynamic model applicable to most conventional airplanes, in normal flight conditions

Aerodynamic model: Our approach

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Problem statement

Goal:

Design aerodynamic speed and flight path angle controllers such that

- Previous identification of C_L, C_D and C_m is not required.
- Global stability in any flight condition.

We split the system to design the controllers separately

Airspeed controller:

- Thrust (F_T) as control signal.
- Drag model coefficients unknown.
- Adaptive control.

Flight path angle controller:

- Elevator (δ_E) as control signal.
- Lift model unknown
- Aerodynamic moment coefficients unknown (except $C_{m_{\delta_e}}$)
- Adaptive backstepping approach

Measured variables:

$$V_a, \quad \gamma, \quad \alpha, \quad \theta, \quad q$$

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Extensions

In this lecture:

- The initial result (IFAC WC paper) will be presented in full detail.
- The design is Lyapunov-based adaptive control and backstepping.
- The JGCD paper extensions will then be introduced.

Extension:

- Saturations in thrust: hybrid control (but easy).
- Drop the assumption that $C_{m_{\delta_e}}$ is known.
- Drop the assumption that α_0 (angle of attack trim angle) is known.

Outline

Airspeed controller

System:
$$\dot{V}_a = \frac{1}{m}\left(-\frac{1}{2}\rho V_a^2 S C_D + F_T \cos\alpha - mg\sin\gamma\right)$$

Define $z_V = V_a - V_{ref}$.
Then:

$$\dot{z}_V = \frac{1}{m}\left(-\frac{1}{2}\rho (z_V + V_{ref})^2 S C_D + F_T \cos\alpha - mg\sin\gamma\right) - \dot{V}_{ref}$$

Remember hypothesis $C_D = C_{D_0} + k_1\alpha + k_2\alpha^2 \geq 0$. Denoting

$$\boldsymbol{\varphi}(\alpha) = \begin{bmatrix} 1 & \alpha & \alpha^2 \end{bmatrix}^T \qquad \boldsymbol{\theta}_V = \begin{bmatrix} C_{D_0} & k_1 & k_2 \end{bmatrix}^T$$

we get $C_D = \boldsymbol{\varphi}(\alpha)^T \boldsymbol{\theta}_V$.
Finally, define $\beta_1 = \frac{\rho S}{2m}$. Then we reach:

$$\dot{z}_V = -\beta_1\left(z_V^2 + V_{ref}^2 + 2 z_V V_{ref}\right)\boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + F_T \frac{\cos\alpha}{m} - g\sin\gamma - \dot{V}_{ref}$$

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Airspeed controller — non-adaptive

$$\dot{z}_V = -\beta_1 \left(z_V^2 + V_{ref}^2 + 2 z_V V_{ref}\right) \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + F_T \frac{\cos\alpha}{m} - g \sin\gamma - \dot{V}_{ref}$$

Consider first the non-adaptive problem
Lyapunov function (nonadaptive):

$$W_V = \frac{1}{2} z_V^2$$

Computing the derivative:

$$\dot{W}_V = z_V \left[-\beta_1 \left(z_V^2 + V_{ref}^2 + 2 z_V V_{ref}\right) \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + F_T \frac{\cos\alpha}{m} - g \sin\gamma - \dot{V}_{ref}\right]$$

Note that $V_a = z_V + V_{ref} \geq 0$. Thus $z_V \geq -V_{ref}$. So $-z_V \leq V_{ref}$. Using this idea, the term $-z_V(\beta_1 z_V^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V)$ is smaller that $V_{ref}(\beta_1 z_V^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V)$ because the parenthesis is positive. Reaching:

$$\dot{W}_V \leq z_V \left[-\beta_1 V_{ref} \left(V_{ref} + z_V\right) \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + F_T \frac{\cos\alpha}{m} - g \sin\gamma - \dot{V}_{ref}\right]$$

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$$F_T = \frac{m}{\cos\alpha} \left(g \sin\gamma + \dot{V}_{ref} + \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - \kappa_{V_1} z_V\right) \rightarrow \dot{W}_V \leq -\kappa_{V_1} z_V^2 - \beta_1 z_V^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V$$

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Airspeed controller — non-adaptive

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Airspeed controller — adaptive case

$$\dot{z}_V = -\beta_1\left(z_V^2 + V_{ref}^2 + 2z_V V_{ref}\right)\boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + F_T\frac{\cos\alpha}{m} - g\sin\gamma - \dot{V}_{ref}$$

Use the same control law but using an estimate of $\boldsymbol{\theta}_V$, denoted as $\hat{\boldsymbol{\theta}}_V$:
Control law (adaptive):

$$F_T = \frac{m}{\cos\alpha}\left(g\sin\gamma + \dot{V}_{ref} + \beta_1 V_{ref}^2\boldsymbol{\varphi}(\alpha)^T \cdot \hat{\boldsymbol{\theta}}_V - \kappa_{V_1} z_V\right)$$

If we now use the same Lyapunov function we get:

$$\dot{W}_V \leq -\kappa_{V_1} z_V^2 - \beta_1 z_V^2 V_{ref}\boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - z_V\beta_1 V_{ref}^2\boldsymbol{\varphi}(\alpha)^T \cdot \tilde{\boldsymbol{\theta}}_V$$

where $\tilde{\boldsymbol{\theta}}_V = \boldsymbol{\theta}_V - \hat{\boldsymbol{\theta}}_V$ is the estimate error.
To account for this term, consider a new Lyapunov function
Lyapunov function (adaptive):

$$W_V = \frac{1}{2}z_V^2 + \frac{1}{2}\tilde{\boldsymbol{\theta}}_V^T\boldsymbol{\Gamma}_V{}^{-1}\tilde{\boldsymbol{\theta}}_V$$

where $\boldsymbol{\Gamma}_V$ is a symmetric, positive definite matrix.

$$\dot{z}_V = -\beta_1 \left(z_V^2 + V_{ref}^2 + 2 z_V V_{ref}\right) \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + F_T \frac{\cos\alpha}{m} - g \sin\gamma - \dot{V}_{ref}$$

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where the assumption $\dot{\boldsymbol{\theta}}_V = 0$ has been made. Now one has to design an adaptation law (how $\dot{\hat{\boldsymbol{\theta}}}_V$).

Choosing:

Adaptation law:

$$\dot{\hat{\boldsymbol{\theta}}}_V = -\beta_1 z_V V_{ref}^2 \boldsymbol{\Gamma_V} \boldsymbol{\varphi}_V(\alpha)$$

we obtain:

$$\dot{W}_V \leq -\kappa_{V_1} z_V^2 - \beta_1 z_V^2 V_{ref} \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V < 0$$

Airspeed Controller Properties:

- Global Asymptotic stability for z_V, implies that $V_a \to V_{ref}$.
- Stability for $\tilde{\theta}_V$. Does NOT imply that the parameter estimates converge.

Now:

$$\dot{W}_V \leq -\kappa_{V_1} z_V^2 - \beta_1 z_V^2 V_{ref} \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - z_V \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \tilde{\boldsymbol{\theta}}_V - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma_V}^{-1} \dot{\hat{\boldsymbol{\theta}}}_V$$

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Airspeed controller — adaptive case, with saturation

It is realistic to consider that $F_T \in [0, \bar{F}_T]$. The thrust cannot be negative or exceed its maximum value, no matter what the control law commands.

Assumption: the reference velocity V_{ref} is reachable for a feasible value of thrust.

This implies, in the velocity equation, that if we substitute V_{ref}:

$$\dot{V}_{ref} = \left(-\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + F_T \frac{\cos\alpha}{m} - g \sin\gamma\right)$$

Then, $F_T \in [0, \bar{F}_T]$.

Two inequalities are implied, since $F_T = \frac{m}{\cos\alpha}\left[\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + g\sin\gamma + \dot{V}_{ref}\right]$.

Ineq 1: $\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + g\sin\gamma + \dot{V}_{ref} \geq 0$

Ineq 2: $\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + g\sin\gamma + \dot{V}_{ref} \leq \bar{F}_T \frac{\cos\alpha}{m}$

These inequalities will be used to deduce how to modify the control-adaptation law when saturations happen.

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$$\dot{V}_{ref} = \left(-\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + F_T \frac{\cos\alpha}{m} - g\sin\gamma\right)$$

Then, $F_T \in [0, \bar{F}_T]$.

Two inequalities are implied, since $F_T = \frac{m}{\cos\alpha}\left[\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + g\sin\gamma + \dot{V}_{ref}\right]$.

Ineq 1: $\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + g\sin\gamma + \dot{V}_{ref} \geq 0$

Ineq 2: $\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + g\sin\gamma + \dot{V}_{ref} \leq \bar{F}_T \frac{\cos\alpha}{m}$

These inequalities will be used to deduce how to modify the control-adaptation law when saturations happen.

Airspeed controller — adaptive case, with saturation

It is realistic to consider that $F_T \in [0, \bar{F}_T]$. The thrust cannot be negative or exceed its maximum value, no matter what the control law commands.

Assumption: the reference velocity V_{ref} is reachable for a feasible value of thrust.

This implies, in the velocity equation, that if we substitute V_{ref}:

$$\dot{V}_{ref} = \left(-\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + F_T \frac{\cos\alpha}{m} - g \sin\gamma\right)$$

Then, $F_T \in [0, \bar{F}_T]$.

Two inequalities are implied, since $F_T = \frac{m}{\cos\alpha}\left[\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + g\sin\gamma + \dot{V}_{ref}\right]$.

Ineq 1: $\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + g\sin\gamma + \dot{V}_{ref} \geq 0$

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These inequalities will be used to deduce how to modify the control-adaptation law when saturations happen.

Three possible situations:

1. The control law for F_T does not saturate. Then, use the control-adaptation law that has been computed.
2. The control law for F_T saturate above (the commanded F_T is greater than $\bar{F}_T$).
3. The control law for F_T saturate below (the commanded F_T is lower than 0).

Case 2: If the commanded F_T is greater than $\bar{F}_T$, this means that $\frac{m}{\cos\alpha}\left(g\sin\gamma + \dot{V}_{ref} + \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \hat{\boldsymbol{\theta}}_V - \kappa_{V_1} z_V\right) > \bar{F}_T$

Also, $F_T = \bar{F}_T$ since it is its maximum possible value.

In the Lyapunov function that we computed, we find:

$$\dot{W}_V \leq z_V\left[-\beta_1 V_{ref}\left(V_{ref}+z_V\right)\boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + \bar{F}_T\frac{\cos\alpha}{m} - g\sin\gamma - \dot{V}_{ref}\right] - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma}_{\boldsymbol{V}}^{-1}\dot{\hat{\boldsymbol{\theta}}}_V$$

Three possible situations:

1. The control law for F_T does not saturate. Then, use the control-adaptation law that has been computed.
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$$\dot{W}_V \leq z_V \left[-\beta_1 V_{ref}\left(V_{ref} + z_V\right)\boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V + \bar{F}_T \frac{\cos\alpha}{m} - g\sin\gamma - \dot{V}_{ref}\right] - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma}_{\boldsymbol{V}}{}^{-1}\dot{\hat{\boldsymbol{\theta}}}_V$$

Airspeed controller — adaptive case, with saturation $(F_T > \bar{F}_T)$

Analyzing the first term, there are two sub-cases.

1. If z_V is zero or positive, then since $\bar{F}_T < \frac{m}{\cos\alpha}\left(g\sin\gamma + \dot{V}_{ref} + \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \hat{\boldsymbol{\theta}}_V - \kappa_{V_1} z_V\right)$ we get

$$\dot{W}_V \leq -\kappa_{V_1} z_V^2 - \beta_1 V_{ref} z_V^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - z_V \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \tilde{\boldsymbol{\theta}}_V - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma_V}^{-1} \dot{\hat{\boldsymbol{\theta}}}_V$$

   and using the original adaptation law we get the result.

2. If z_V is negative, then $z_V\left[-\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - g\sin\gamma - \dot{V}_{ref}\right]$ is positive and less than $-z_V \bar{F}_T \frac{\cos\alpha}{m}$. Thus, $\dot{W}_V < -\beta_1 V_{ref} z_V^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma_V}^{-1} \dot{\hat{\boldsymbol{\theta}}}_V$ and setting to zero the adaptation law one gets the result.

Airspeed controller — adaptive case, with saturation $(F_T > \bar{F}_T)$

Analyzing the first term, there are two sub-cases.

1. If z_V is zero or positive, then since $\bar{F}_T < \frac{m}{\cos\alpha}\left(g\sin\gamma + \dot{V}_{ref} + \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \hat{\boldsymbol{\theta}}_V - \kappa_{V_1} z_V\right)$ we get

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   and using the original adaptation law we get the result.

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Case 3: If the commanded F_T is less than 0, this means that $g\sin\gamma + \dot V_{ref} + \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \hat{\boldsymbol{\theta}}_V - \kappa_{V_1} z_V < 0$.

Also, $F_T = 0$ since it is its minimum possible value.

In the Lyapunov function that we computed, we find:

$$\dot W_V \le z_V \left[-\beta_1 V_{ref}\left(V_{ref} + z_V\right)\boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - g\sin\gamma - \dot V_{ref}\right] - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma}_{\boldsymbol{V}}{}^{-1}\dot{\hat{\boldsymbol{\theta}}}_V$$

Again, there are two sub-cases.

1. If z_V is zero or negative, then since $g\sin\gamma + \dot V_{ref} + \beta_1(z_V^2 + V_{ref}^2)\boldsymbol{\varphi}(\alpha)^T \cdot \hat{\boldsymbol{\theta}}_V < \kappa_{V_1} z_V$ we get

$$\dot W_V \le -\kappa_{V_1} z_V^2 - \beta_1 V_{ref} z_V^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - z_V \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \tilde{\boldsymbol{\theta}}_V - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma}_{\boldsymbol{V}}{}^{-1}\dot{\hat{\boldsymbol{\theta}}}_V$$

   and using the original adaptation law we get the result.

2. If z_V is positive, then $z_V\left[-\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - g\sin\gamma - \dot V_{ref}\right]$ is negative and less than 0. Thus, $\dot W_V < -\beta_1 V_{ref} z_V^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma}_{\boldsymbol{V}}{}^{-1}\dot{\hat{\boldsymbol{\theta}}}_V$ and setting to zero the adaptation law one gets the result.

Case 3: If the commanded F_T is less than 0, this means that $g\sin\gamma + \dot{V}_{ref} + \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \hat{\boldsymbol{\theta}}_V - \kappa_{V_1} z_V < 0$.

Also, $F_T = 0$ since it is its minimum possible value.

In the Lyapunov function that we computed, we find:

$$\dot{W}_V \leq z_V \left[-\beta_1 V_{ref} \left(V_{ref} + z_V \right) \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - g\sin\gamma - \dot{V}_{ref} \right] - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma_V}^{-1} \dot{\hat{\boldsymbol{\theta}}}_V$$

Again, there are two sub-cases.

1. If z_V is zero or negative, then since $g\sin\gamma + \dot{V}_{ref} + \beta_1 (z_V^2 + V_{ref}^2) \boldsymbol{\varphi}(\alpha)^T \cdot \hat{\boldsymbol{\theta}}_V < \kappa_{V_1} z_V$ we get

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and using the original adaptation law we get the result.

2. If z_V is positive, then $z_V \left[-\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - g\sin\gamma - \dot{V}_{ref} \right]$ is negative and less than 0. Thus, $\dot{W}_V < -\beta_1 V_{ref} z_V^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma_V}^{-1} \dot{\hat{\boldsymbol{\theta}}}_V$ and setting to zero the adaptation law one gets the result.

Airspeed controller — adaptive case, with saturation $(F_T < 0)$

Case 3: If the commanded F_T is less than 0, this means that $g\sin\gamma + \dot{V}_{ref} + \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \hat{\boldsymbol{\theta}}_V - \kappa_{V_1} z_V < 0$.

Also, $F_T = 0$ since it is its minimum possible value.

In the Lyapunov function that we computed, we find:

$$\dot{W}_V \leq z_V \left[-\beta_1 V_{ref}\left(V_{ref} + z_V\right)\boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - g\sin\gamma - \dot{V}_{ref}\right] - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma_V}^{-1}\dot{\hat{\boldsymbol{\theta}}}_V$$

Again, there are two sub-cases.

1. If z_V is zero or negative, then since $g\sin\gamma + \dot{V}_{ref} + \beta_1(z_V^2 + V_{ref}^2)\boldsymbol{\varphi}(\alpha)^T \cdot \hat{\boldsymbol{\theta}}_V < \kappa_{V_1} z_V$ we get

$$\dot{W}_V \leq -\kappa_{V_1} z_V^2 - \beta_1 V_{ref} z_V^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - z_V \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \tilde{\boldsymbol{\theta}}_V - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma_V}^{-1}\dot{\hat{\boldsymbol{\theta}}}_V$$

   and using the original adaptation law we get the result.
2. If z_V is positive, then $z_V\left[-\beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - g\sin\gamma - \dot{V}_{ref}\right]$ is negative and less than 0. Thus, $\dot{W}_V < -\beta_1 V_{ref} z_V^2 \boldsymbol{\varphi}(\alpha)^T \cdot \boldsymbol{\theta}_V - \tilde{\boldsymbol{\theta}}_V^T \boldsymbol{\Gamma_V}^{-1}\dot{\hat{\boldsymbol{\theta}}}_V$ and setting to zero the adaptation law one gets the result.

Airspeed controller — adaptive case, with saturation

In summary, the following controller-adaptation law guarantees global asymptotic stability of the $z_V = 0$ equilibrium (this is, convergence of V_a to the reference).

Control law:

$$F_T = \frac{m}{\cos\alpha}\left(g\sin\gamma + \dot{V}_{ref} + \beta_1 V_{ref}^2 \boldsymbol{\varphi}(\alpha)^T \cdot \hat{\boldsymbol{\theta}}_V - \kappa_{V_1} z_V\right)$$

Adaptation law:

$$\dot{\hat{\boldsymbol{\theta}}}_V = \begin{cases} 0 & F_T < 0, z_V > 0 \\ 0 & F_T > \bar{F}_T, z_V < 0 \\ -\beta_1 z_V V_{ref}^2 \boldsymbol{\Gamma_V} \boldsymbol{\varphi}_V(\alpha) & \text{otherwise} \end{cases}$$

Flight path angle controller I

System:

$$\left.\begin{aligned}\dot\gamma &= \tfrac{1}{mV_a}\left(L + F_T\sin\alpha - mg\cos\gamma\right)\\ \dot\theta &= q\\ \dot q &= M/I_y\end{aligned}\right\}$$

Assumptions:

- $\cos\gamma \approx \cos\gamma_{ref}$.
- Let $f(\alpha) = \frac{1}{mV_a}\left(L + F_T\sin\alpha - mg\cos\gamma\right)$.
- Aerodynamic property:

$$(\alpha - \alpha_0) f(\alpha) \geq 0$$

where α_0 is the trim angle of attack

$$f(\alpha_0) = 0$$

Error coordinates:

$$z_1 = \gamma - \gamma_{ref} \quad z_2 = \theta - \gamma_{ref} - \alpha_0 \quad z_3 = q$$

$$\begin{aligned}\dot\gamma &= f(\alpha) = f(\theta - \gamma)\\ \dot\theta &= q\\ \dot q &= \frac{\rho V_a^2 S}{2I_y}\left(C_{m_0} + C_{m_\alpha}\alpha\right.\\ &\quad \left. + C_{m_q} q + C_{m_{\delta_e}}\delta_e\right)\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}\dot z_1 &= \eta(z_2 - z_1)\\ \dot z_2 &= z_3\\ \dot z_3 &= \beta_2\left(C_{m_0} + C_{m_\alpha}(z_2 - z_1 + \alpha_0)\right.\\ &\quad \left. + C_{m_q} z_3 + C_{m_{\delta_e}}\delta_e\right)\end{aligned}$$

Where

$$\eta(x) := f(x + \alpha_0), \quad x \cdot \eta(x) \geq 0$$

System:

$$\left.\begin{array}{l}\dot{\gamma} = \frac{1}{mV_a}\left(L + F_T \sin\alpha - mg\cos\gamma\right) \\ \dot{\theta} = q \\ \dot{q} = M/I_y\end{array}\right\} \rightarrow$$

$$\begin{aligned}\dot{\gamma} &= f(\alpha) = f(\theta - \gamma) \\ \dot{\theta} &= q \\ \dot{q} &= \frac{\rho V_a^2 S}{2I_y}\left(C_{m_0} + C_{m_\alpha}\alpha \right. \\ &\quad \left. + C_{m_q} q + C_{m_{\delta_e}}\delta_e\right)\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}\dot{z}_1 &= \eta(z_2 - z_1) \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \beta_2\left(C_{m_0} + C_{m_\alpha}(z_2 - z_1 + \alpha_0)\right. \\ &\quad \left. + C_{m_q} z_3 + C_{m_{\delta_e}}\delta_e\right)\end{aligned}$$

Assumptions:

- $\cos\gamma \approx \cos\gamma_{ref}$.
- Let $f(\alpha) = \frac{1}{mV_a}\left(L + F_T \sin\alpha - mg\cos\gamma\right)$.
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where α_0 is the trim angle of attack

$$f(\alpha_0) = 0$$

Where

$$\eta(x) := f(x + \alpha_0), \quad x \cdot \eta(x) \geq 0$$

Error coordinates:

$$z_1 = \gamma - \gamma_{ref} \quad z_2 = \theta - \gamma_{ref} - \alpha_0 \quad z_3 = q$$

System:

$$\left.\begin{array}{l}\dot\gamma = \frac{1}{mV_a}\left(L + F_T \sin\alpha - mg\cos\gamma\right)\\ \dot\theta = q\\ \dot q = M/I_y\end{array}\right\} \to$$

$$\begin{aligned}\dot\gamma &= f(\alpha) = f(\theta-\gamma)\\ \dot\theta &= q\\ \dot q &= \frac{\rho V_a^2 S}{2I_y}\left(C_{m_0} + C_{m_\alpha}\alpha\right.\\ &\quad \left.+ C_{m_q} q + C_{m_{\delta_e}}\delta_e\right)\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}\dot z_1 &= \eta(z_2 - z_1)\\ \dot z_2 &= z_3\\ \dot z_3 &= \beta_2\left(C_{m_0} + C_{m_\alpha}(z_2 - z_1 + \alpha_0)\right.\\ &\quad \left.+ C_{m_q} z_3 + C_{m_{\delta_e}}\delta_e\right)\end{aligned}$$

Assumptions:

- $\cos\gamma \approx \cos\gamma_{ref}$.
- Let $f(\alpha) = \frac{1}{mV_a}\left(L + F_T \sin\alpha - mg\cos\gamma\right)$.
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Error coordinates:

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Where

$$\eta(x) := f(x + \alpha_0), \quad x \cdot \eta(x) \geq 0$$

Goal:

Equilibrium $(z_1, z_2, z_3) = (0, 0, 0)$ Global Asymptotically Stable

Adaptive backstepping scheme

Step 1:

Reduced system:

$$\dot{z}_1 = \eta(z_2 - z_1)$$

Virtual control law:

$$z_2 = u_1(z_1) = -\kappa_{\gamma_1} z_1$$

Lyapunov function:

$$W_1 = \frac{1}{2} z_1^2, \qquad \dot{W}_1|_{z_2 = u_1(z_1)} = z_1 \eta(-(1 + \kappa_{\gamma_1}) z_1) \leq 0$$

No more information about $\eta(x)$, except the property $x \cdot \eta(x) \geq 0$

Goal:

Equilibrium $(z_1, z_2, z_3) = (0, 0, 0)$ Global Asymptotically Stable

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No more information about $\eta(x)$, except the property $x \cdot \eta(x) \geq 0$

Goal:

Equilibrium $(z_1, z_2, z_3) = (0, 0, 0)$ Global Asymptotically Stable

Adaptive backstepping scheme

Step 1:

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No more information about $\eta(x)$, except the property $x \cdot \eta(x) \geq 0$

Flight path angle controller II

Goal:

Equilibrium $(z_1, z_2, z_3) = (0, 0, 0)$ Global Asymptotically Stable

Adaptive backstepping scheme

Step 1:

Reduced system:

$$\dot{z}_1 = \eta(z_2 - z_1)$$

Virtual control law:

$$z_2 = u_1(z_1) = -\kappa_{\gamma_1} z_1$$

Lyapunov function:

$$W_1 = \frac{1}{2} z_1^2, \qquad \dot{W}_1|_{z_2 = u_1(z_1)} = z_1 \eta(-(1 + \kappa_{\gamma_1}) z_1) \leq 0$$

No more information about $\eta(x)$, except the property $x \cdot \eta(x) \geq 0$

Flight path angle controller III

Step 2: Define $\tilde{z}_2 = z_2 - u_1(z_1) = z_2 + \kappa_{\gamma_1} z_1$

$$\begin{aligned}
\dot{z}_1 &= \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right) \\
\dot{\tilde{z}}_2 &= z_3 + \kappa_{\gamma_1}\eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right)
\end{aligned}$$

Virtual control law:

$$z_3 = u_2(\tilde{z}_2) = -\kappa_{\gamma_2}\tilde{z}_2, \quad \kappa_{\gamma_2} > 0$$

Lyapunov function:

$$\begin{aligned}
W_2 &= c_1 W_1 + \frac{1}{2}\tilde{z}_2^2 + c_2\int_0^{-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2} \eta(s)ds, c_1, c_2 \geq 0 \\
\dot{W}_2 &= \left(c_1 z_1 + \kappa_{\gamma_1}\tilde{z}_2 - c_2\eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right)\right)\eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right) \\
&\quad + \left(\tilde{z}_2 + c_2\eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right)\right) z_3
\end{aligned}$$

Calling $\xi = -\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2$ and by selecting $c_1 = -(1+\kappa_{\gamma_1})(\kappa_{\gamma_1} - \kappa_{\gamma_2} c_2)$, with $\kappa_{\gamma_2} c_2 > \kappa_{\gamma_1}$:

$$\boxed{\dot{W}_2 = -(\kappa_{\gamma_2} c_2 - \kappa_{\gamma_1})\xi\eta\left(\xi\right) - c_2\eta^2\left(\xi\right) - \kappa_{\gamma_2}\tilde{z}_2^2 \leq 0}$$

Step 2: Define $\tilde{z}_2 = z_2 - u_1(z_1) = z_2 + \kappa_{\gamma_1} z_1$

$$\begin{aligned} \dot{z}_1 &= \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right) \\ \dot{\tilde{z}}_2 &= z_3 + \kappa_{\gamma_1} \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right) \end{aligned}$$

Virtual control law:

$$z_3 = u_2(\tilde{z}_2) = -\kappa_{\gamma_2} \tilde{z}_2, \quad \kappa_{\gamma_2} > 0$$

Lyapunov function:

$$\begin{aligned} W_2 &= c_1 W_1 + \frac{1}{2}\tilde{z}_2^2 + c_2 \int_0^{-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2} \eta(s) ds, c_1, c_2 \geq 0 \\ \dot{W}_2 &= \left(c_1 z_1 + \kappa_{\gamma_1} \tilde{z}_2 - c_2 \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right)\right) \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right) \\ &\quad + \left(\tilde{z}_2 + c_2 \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right)\right) z_3 \end{aligned}$$

Calling $\xi = -\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2$ and by selecting $c_1 = -(1+\kappa_{\gamma_1})(\kappa_{\gamma_1} - \kappa_{\gamma_2} c_2)$, with $\kappa_{\gamma_2} c_2 > \kappa_{\gamma_1}$:

$$\dot{W}_2 = -(\kappa_{\gamma_2} c_2 - \kappa_{\gamma_1}) \xi \eta(\xi) - c_2 \eta^2(\xi) - \kappa_{\gamma_2} \tilde{z}_2^2 \leq 0$$

Flight path angle controller III

Step 2: Define $\tilde{z}_2 = z_2 - u_1(z_1) = z_2 + \kappa_{\gamma_1} z_1$

$$\begin{aligned} \dot{z}_1 &= \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right) \\ \dot{\tilde{z}}_2 &= z_3 + \kappa_{\gamma_1} \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right) \end{aligned}$$

Virtual control law:

$$z_3 = u_2(\tilde{z}_2) = -\kappa_{\gamma_2} \tilde{z}_2, \quad \kappa_{\gamma_2} > 0$$

Lyapunov function:

$$\begin{aligned} W_2 &= c_1 W_1 + \frac{1}{2}\tilde{z}_2^2 + c_2 \int_0^{-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2} \eta(s) ds, c_1, c_2 \geq 0 \\ \dot{W}_2 &= \left(c_1 z_1 + \kappa_{\gamma_1} \tilde{z}_2 - c_2 \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right)\right) \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right) \\ &\quad + \left(\tilde{z}_2 + c_2 \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right)\right) z_3 \end{aligned}$$

Calling $\xi = -\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2$ and by selecting $c_1 = -(1+\kappa_{\gamma_1})(\kappa_{\gamma_1} - \kappa_{\gamma_2} c_2)$, with $\kappa_{\gamma_2} c_2 > \kappa_{\gamma_1}$:

$$\boxed{\dot{W}_2 = -(\kappa_{\gamma_2} c_2 - \kappa_{\gamma_1}) \xi \eta(\xi) - c_2 \eta^2(\xi) - \kappa_{\gamma_2} \tilde{z}_2^2 \leq 0}$$

Step 3: Define $\tilde{z}_3 = z_3 - u_2(\tilde{z}_2) = z_3 + \kappa_{\gamma_2}\tilde{z}_2$

$$
\begin{aligned}
\dot{z}_1 &= \eta\left(-\left(1+\kappa_{\gamma_1}\right)z_1 + \tilde{z}_2\right)\\
\dot{\tilde{z}}_2 &= \tilde{z}_3 - \kappa_{\gamma_2}\tilde{z}_2 + \kappa_{\gamma_1}\eta\left(\xi\right)\\
\dot{\tilde{z}}_3 &= \beta_2\boldsymbol{\varphi}^T\cdot\boldsymbol{\theta} + \beta_{\delta_e}\delta_e\\
&\quad + \kappa_{\gamma_2}\left(\tilde{z}_3 - \kappa_{\gamma_2}\tilde{z}_2 + \kappa_{\gamma_1}\eta\left(\xi\right)\right)
\end{aligned}
$$

- $\beta_2 = \dfrac{\rho V_a^2 S\bar{c}}{2I_y}$, $\beta_{\delta_e} = \dfrac{\rho V_a^2 S\bar{c}}{2I_y}C_{m_{\delta_e}}$
- $\boldsymbol{\varphi} = \begin{bmatrix} 1 & \xi+\alpha_0 & \tilde{z}_3 - \kappa_{\gamma_2}\tilde{z}_2 \end{bmatrix}^T$
- $\boldsymbol{\theta} = \begin{bmatrix} C_{m_0} & C_{m_\alpha} & C_{m_q} \end{bmatrix}^T$ (unknown parameters)

Lyapunov function:

$$W_3 = c_3 W_2 + \frac{1}{2}\tilde{z}_3^2 + \frac{1}{2}\tilde{\boldsymbol{\theta}}_\gamma^T\boldsymbol{\Gamma}_\gamma^{-1}\tilde{\boldsymbol{\theta}}_\gamma$$

Then

$$
\begin{aligned}
\dot{W}_3 &= c_3\left(-(\kappa_{\gamma_2}c_2 - \kappa_{\gamma_1})\xi\eta\left(\xi\right) - c_2\eta^2\left(\xi\right) - \kappa_{\gamma_2}\tilde{z}_2^2\right) - \tilde{\boldsymbol{\theta}}_\gamma^T\boldsymbol{\Gamma}_\gamma^{-1}\dot{\hat{\boldsymbol{\theta}}}_\gamma\\
&\quad + \left(c_3\tilde{z}_2 + c_2\eta\left(\xi\right)\right)\tilde{z}_3 + \tilde{z}_3\left(\beta_2\boldsymbol{\varphi}^T\cdot\boldsymbol{\theta} + \beta_{\delta_e}\delta_e + \kappa_{\gamma_2}\left(\tilde{z}_3 - \kappa_{\gamma_2}\tilde{z}_2 + \kappa_{\gamma_1}\eta\left(\xi\right)\right)\right)
\end{aligned}
$$

Step 3: Define $\tilde{z}_3 = z_3 - u_2(\tilde{z}_2) = z_3 + \kappa_{\gamma_2}\tilde{z}_2$

$$\begin{aligned}
\dot{z}_1 &= \eta\left(-\left(1+\kappa_{\gamma_1}\right)z_1 + \tilde{z}_2\right) \\
\dot{\tilde{z}}_2 &= \tilde{z}_3 - \kappa_{\gamma_2}\tilde{z}_2 + \kappa_{\gamma_1}\eta\left(\xi\right) \\
\dot{\tilde{z}}_3 &= \beta_2\boldsymbol{\varphi}^T\cdot\boldsymbol{\theta} + \beta_{\delta_e}\delta_e \\
&\quad +\kappa_{\gamma_2}\left(\tilde{z}_3 - \kappa_{\gamma_2}\tilde{z}_2 + \kappa_{\gamma_1}\eta\left(\xi\right)\right)
\end{aligned}$$

- $\beta_2 = \frac{\rho V_a^2 S\bar{c}}{2I_y}$, $\beta_{\delta_e} = \frac{\rho V_a^2 S\bar{c}}{2I_y}C_{m_{\delta_e}}$
- $\boldsymbol{\varphi} = \left[\begin{array}{ccc} 1 & \xi+\alpha_0 & \tilde{z}_3 - \kappa_{\gamma_2}\tilde{z}_2 \end{array}\right]^T$
- $\boldsymbol{\theta} = \left[\begin{array}{ccc} C_{m_0} & C_{m_\alpha} & C_{m_q} \end{array}\right]^T$ (unknown parameters)

Lyapunov function:

$$W_3 = c_3W_2 + \frac{1}{2}\tilde{z}_3^2 + \frac{1}{2}\tilde{\boldsymbol{\theta}}_\gamma^T\boldsymbol{\Gamma_\gamma}^{-1}\tilde{\boldsymbol{\theta}}_\gamma$$

Then

$$\begin{aligned}
\dot{W}_3 &= c_3\left(-(\kappa_{\gamma_2}c_2 - \kappa_{\gamma_1})\xi\eta\left(\xi\right) - c_2\eta^2\left(\xi\right) - \kappa_{\gamma_2}\tilde{z}_2^2\right) - \tilde{\boldsymbol{\theta}}_\gamma^T\boldsymbol{\Gamma_\gamma}^{-1}\dot{\hat{\boldsymbol{\theta}}}_\gamma \\
&\quad + \left(c_3\tilde{z}_2 + c_2\eta\left(\xi\right)\right)\tilde{z}_3 + \tilde{z}_3\left(\beta_2\boldsymbol{\varphi}^T\cdot\boldsymbol{\theta} + \beta_{\delta_e}\delta_e + \kappa_{\gamma_2}\left(\tilde{z}_3 - \kappa_{\gamma_2}\tilde{z}_2 + \kappa_{\gamma_1}\eta\left(\xi\right)\right)\right)
\end{aligned}$$

Flight path angle controller V

Choose:

Control law

$$\delta_e = \frac{1}{\beta_{\delta_e}}\left(-\kappa_{\gamma_3}\tilde{z}_3 - \beta_2 \boldsymbol{\varphi}_\gamma^T \cdot \hat{\boldsymbol{\theta}}_\gamma\right)$$

Adaptation law

$$\dot{\hat{\boldsymbol{\theta}}}_\gamma = -\dot{\tilde{\boldsymbol{\theta}}}_\gamma = \beta_2 \tilde{z}_3 \boldsymbol{\Gamma}_{\boldsymbol{\gamma}} \boldsymbol{\varphi}_\gamma$$

Then:

$$\begin{aligned}\dot{W}_3 \quad = \quad & c_3\left(-(\kappa_{\gamma_2}c_2 - \kappa_{\gamma_1})\xi\eta\left(\xi\right) - c_2\eta^2\left(\xi\right) - \kappa_{\gamma_2}\tilde{z}_2^2\right) - (\kappa_{\gamma_3} - \kappa_{\gamma_2})\tilde{z}_3^2 \\ & + \tilde{z}_3\tilde{z}_2(c_3 + \kappa_{\gamma_2}^2) + \tilde{z}_3\eta\left(\xi\right)\left(c_2 + \kappa_{\gamma_1}\kappa_{\gamma_2}\right)\end{aligned}$$

Using $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ we get:

$$\begin{aligned}\dot{W}_3 \quad \leq \quad & c_3\left(-(\kappa_{\gamma_2}c_2 - \kappa_{\gamma_1})\xi\eta\left(\xi\right) - \left(c_2 - \frac{c_2 + \kappa_{\gamma_1}\kappa_{\gamma_2}}{2c_3}\right)\eta^2\left(\xi\right) - \left(\kappa_{\gamma_2} - \frac{c_3 + \kappa_{\gamma_2}^2}{2c_3}\right)\tilde{z}_2^2\right) \\ & - \left(\kappa_{\gamma_3} - \kappa_{\gamma_2} - \frac{c_3 + \kappa_{\gamma_2}^2 + c_2 + \kappa_{\gamma_1}\kappa_{\gamma_2}}{2}\right)\tilde{z}_3^2\end{aligned}$$

and choosing c_3, κ_{γ_3} and κ_{γ_2} properly, the origin is GAS.

Final considerations (IFAC WC Paper)

Control law

$$\delta_e = \frac{1}{\beta_{\delta_e}} \left(-\kappa_{\gamma_3} \left(q + \kappa_{\gamma_2} \left(\theta - \gamma_{ref} - \alpha_0 + \kappa_{\gamma_1} \left(\gamma - \gamma_{ref} \right) \right) \right) - \beta_2 \boldsymbol{\varphi}_\gamma^T \cdot \hat{\boldsymbol{\theta}}_\gamma \right)$$

Adaptation law:

$$\dot{\hat{\boldsymbol{\theta}}}_\gamma = \beta_2 \left(q + \kappa_{\gamma_2} \left(\theta - \gamma_{ref} - \alpha_0 + \kappa_{\gamma_1} \left(\gamma - \gamma_{ref} \right) \right) \cdot \boldsymbol{\Gamma}_{\boldsymbol{\gamma}} \boldsymbol{\varphi}_\gamma \right)$$

- System stable in any operating point.
- No need of C_L model or aerodynamic moment coefficients.
- Control law applicable to any conventional airplane.
- However, an estimation of α_0 and $C_{m_{\delta_e}}$ is needed.

Change in Step 2:

$$\begin{aligned}\dot{z}_1 &= \eta\left(-\left(1+\kappa_{\gamma_1}\right)z_1+\tilde{z}_2\right)\\ \dot{\tilde{z}}_2 &= z_3+\kappa_{\gamma_1}\eta\left(-\left(1+\kappa_{\gamma_1}\right)z_1+\tilde{z}_2\right)\end{aligned}$$

New virtual control law:

$$z_3 = u_2(\tilde{z}_1) = -c_1\tilde{z}_1$$

New Lyapunov function:

$$\begin{aligned}W_2 &= c_1W_1+\int_0^{-\left(1+\kappa_{\gamma_1}\right)z_1+\tilde{z}_2}\eta(s)ds, c_1 \geq 0\\ \dot{W}_2 &= c_1z_1\eta\left(\xi\right)+\eta\left(\xi\right)\left(-\left(1+\kappa_{\gamma_1}\right)\eta\left(\xi\right)+z_3+\kappa_{\gamma_1}\eta(\xi)\right)\end{aligned}$$

Reaching $\dot{W}_2 = -\eta^2(\xi)$.
By LaSalle's theorem, z_1 and $\tilde{z}_2$ thus tend to the largest invariant set contained in $\eta = 0$, which also implies $-\left(1+\kappa_{\gamma_1}\right)z_1+\tilde{z}_2 = 0$. The dynamics in the set $\eta = 0$ are:

$$\begin{aligned}\dot{z}_1 &= 0\\ \dot{\tilde{z}}_2 &= -c_1z_1\end{aligned}$$

Thus $0 = -\left(1+\kappa_{\gamma_1}\right)\dot{z}_1+\dot{\tilde{z}}_2 = -c_1z_1$, implying finally $z_1 = \tilde{z}_2 = 0$.

Change in Step 2:

$$\begin{aligned}\dot{z}_1 &= \eta\left(-\left(1+\kappa_{\gamma_1}\right)z_1+\tilde{z}_2\right)\\ \dot{\tilde{z}}_2 &= z_3+\kappa_{\gamma_1}\eta\left(-\left(1+\kappa_{\gamma_1}\right)z_1+\tilde{z}_2\right)\end{aligned}$$

New virtual control law:

$$z_3 = u_2(\tilde{z}_1) = -c_1\tilde{z}_1$$

New Lyapunov function:

$$\begin{aligned}W_2 &= c_1 W_1 + \int_0^{-\left(1+\kappa_{\gamma_1}\right)z_1+\tilde{z}_2}\eta(s)ds, c_1 \geq 0\\ \dot{W}_2 &= c_1 z_1 \eta\left(\xi\right)+\eta\left(\xi\right)\left(-\left(1+\kappa_{\gamma_1}\right)\eta\left(\xi\right)+z_3+\kappa_{\gamma_1}\eta(\xi)\right)\end{aligned}$$

Reaching $\dot{W}_2 = -\eta^2(\xi)$.
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Change in Step 2:

$$\begin{aligned} \dot{z}_1 &= \eta\left(-\left(1+\kappa_{\gamma_1}\right)z_1+\tilde{z}_2\right) \\ \dot{\tilde{z}}_2 &= z_3+\kappa_{\gamma_1}\eta\left(-\left(1+\kappa_{\gamma_1}\right)z_1+\tilde{z}_2\right) \end{aligned}$$

New virtual control law:

$$z_3 = u_2(\tilde{z}_1) = -c_1\tilde{z}_1$$

New Lyapunov function:

$$\begin{aligned} W_2 &= c_1 W_1 + \int_0^{-\left(1+\kappa_{\gamma_1}\right)z_1+\tilde{z}_2} \eta(s)ds, c_1 \geq 0 \\ \dot{W}_2 &= c_1 z_1 \eta\left(\xi\right)+\eta\left(\xi\right)\left(-\left(1+\kappa_{\gamma_1}\right)\eta\left(\xi\right)+z_3+\kappa_{\gamma_1}\eta(\xi)\right) \end{aligned}$$

Reaching $\dot{W}_2 = -\eta^2(\xi)$.
By LaSalle's theorem, z_1 and $\tilde{z}_2$ thus tend to the largest invariant set contained in $\eta = 0$, which also implies $-\left(1+\kappa_{\gamma_1}\right)z_1+\tilde{z}_2 = 0$. The dynamics in the set $\eta = 0$ are:

$$\begin{aligned} \dot{z}_1 &= 0 \\ \dot{\tilde{z}}_2 &= -c_1 z_1 \end{aligned}$$

Thus $0 = -\left(1+\kappa_{\gamma_1}\right)\dot{z}_1+\dot{\tilde{z}}_2 = -c_1 z_1$, implying finally $z_1 = \tilde{z}_2 = 0$.

Step 3: Define $\tilde{z}_3 = z_3 - u_2(\tilde{z}_1) = z_3 + c_1 z_1$

$$\begin{aligned}\dot{z}_1 &= \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right)\\ \dot{\tilde{z}}_2 &= \tilde{z}_3 - c_1 z_1 + \kappa_{\gamma_1}\eta\left(\xi\right)\\ \dot{\tilde{z}}_3 &= \beta_2 C_{m_{\delta_e}}\left(\boldsymbol{\varphi}^T\cdot\boldsymbol{\theta}+\delta_e\right)\\ &\quad -\beta_2\kappa_{\gamma_3}\tilde{z}_3 + c_1\eta\left(\xi\right)\end{aligned}$$

- $\beta_2 = \frac{\rho V_a^2 S\bar{c}}{2I_y}$; it is known $C_{m_{\delta_e}} < 0$.
- $\boldsymbol{\varphi} = \begin{bmatrix} 1 & \alpha & z_3 & \kappa_{\gamma_3}\tilde{z}_3 \end{bmatrix}^T$
- $\boldsymbol{\theta} = \begin{bmatrix} \frac{C_{m_0}}{C_{m_{\delta_e}}} & \frac{C_{m_\alpha}}{C_{m_{\delta_e}}} & \frac{C_{m_q}}{C_{m_{\delta_e}}} & \frac{1}{C_{m_{\delta_e}}} \end{bmatrix}^T$

Lyapunov function:

$$W_3 = W_2 + \frac{c_3}{2}\tilde{z}_3^2 + \frac{|C_{m_{\delta_e}}|}{2}\tilde{\boldsymbol{\theta}}_\gamma^T\boldsymbol{\Gamma}_\gamma{}^{-1}\tilde{\boldsymbol{\theta}}_\gamma$$

Then

$$\begin{aligned}\dot{W}_3 &= -\eta^2(\xi) + c_3\tilde{z}_3\beta_2 C_{m_{\delta_e}}\left(\boldsymbol{\varphi}^T\cdot\boldsymbol{\theta}+\delta_e\right) - c_3\beta_2\kappa_{\gamma_3}\tilde{z}_3^2 + \left(1 + c_1c_3\tilde{z}_3\eta\left(\xi\right)\right)\\ &\quad -|C_{m_{\delta_e}}|\tilde{\boldsymbol{\theta}}_\gamma^T\boldsymbol{\Gamma}_\gamma{}^{-1}\dot{\hat{\boldsymbol{\theta}}}_\gamma\end{aligned}$$

Step 3: Define $\tilde{z}_3 = z_3 - u_2(\tilde{z}_1) = z_3 + c_1 z_1$

$$\begin{aligned}
\dot{\tilde{z}}_1 &= \eta\left(-\left(1+\kappa_{\gamma_1}\right) z_1 + \tilde{z}_2\right) \\
\dot{\tilde{z}}_2 &= \tilde{z}_3 - c_1 z_1 + \kappa_{\gamma_1}\eta\left(\xi\right) \\
\dot{\tilde{z}}_3 &= \beta_2 C_{m_{\delta_e}}\left(\boldsymbol{\varphi}^T \cdot \boldsymbol{\theta} + \delta_e\right) \\
&\quad -\beta_2\kappa_{\gamma_3}\tilde{z}_3 + c_1\eta\left(\xi\right)
\end{aligned}$$

- $\beta_2 = \frac{\rho V_a^2 S\bar{c}}{2I_y}$; it is known $C_{m_{\delta_e}} < 0$.
- $\boldsymbol{\varphi} = \begin{bmatrix} 1 & \alpha & z_3 & \kappa_{\gamma_3}\tilde{z}_3 \end{bmatrix}^T$
- $\boldsymbol{\theta} = \begin{bmatrix} \frac{C_{m_0}}{C_{m_{\delta_e}}} & \frac{C_{m_\alpha}}{C_{m_{\delta_e}}} & \frac{C_{m_q}}{C_{m_{\delta_e}}} & \frac{1}{C_{m_{\delta_e}}} \end{bmatrix}^T$

Lyapunov function:

$$W_3 = W_2 + \frac{c_3}{2}\tilde{z}_3^2 + \frac{|C_{m_{\delta_e}}|}{2}\tilde{\boldsymbol{\theta}}_\gamma^T \boldsymbol{\Gamma_\gamma}^{-1}\tilde{\boldsymbol{\theta}}_\gamma$$

Then

$$\begin{aligned}
\dot{W}_3 &= -\eta^2(\xi) + c_3\tilde{z}_3\beta_2 C_{m_{\delta_e}}\left(\boldsymbol{\varphi}^T \cdot \boldsymbol{\theta} + \delta_e\right) - c_3\beta_2\kappa_{\gamma_3}\tilde{z}_3^2 + \left(1 + c_1 c_3\tilde{z}_3\eta\left(\xi\right)\right) \\
&\quad -|C_{m_{\delta_e}}|\tilde{\boldsymbol{\theta}}_\gamma^T \boldsymbol{\Gamma_\gamma}^{-1}\dot{\hat{\boldsymbol{\theta}}}_\gamma
\end{aligned}$$

Flight path angle controller V

Choose $c_3 = 1/c_1$ and

Control law

$$\delta_e = -\boldsymbol{\varphi}_\gamma^T \cdot \hat{\boldsymbol{\theta}}_\gamma$$

Adaptation law

$$\dot{\hat{\boldsymbol{\theta}}}_\gamma = -\dot{\tilde{\boldsymbol{\theta}}}_\gamma = -c_3 \beta_2 \tilde{z}_3 \boldsymbol{\Gamma}_{\boldsymbol{\gamma}} \boldsymbol{\varphi}_\gamma$$

Then:

$$\dot{W}_3 = -\eta^2(\xi) + c_3 \tilde{z}_3 \beta_2 \left(C_{m_{\delta_e}} + |C_{m_{\delta_e}}| \right) \boldsymbol{\varphi}^T \cdot \tilde{\boldsymbol{\theta}} - c_3 \beta_2 \kappa_{\gamma_3} \tilde{z}_3^2 + 2 \tilde{z}_3 \eta(\xi)$$

Using $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ and $C_{m_{\delta_e}} < 0$ we get:

$$\dot{W}_3 \leq -\frac{1}{2}\eta^2(\xi) - \left(c_3 \beta_2 \kappa_{\gamma_3} - \frac{1}{2} \right) \tilde{z}_3^2$$

and using LaSalle's Theorem as in the previous step, the origin is GAS.

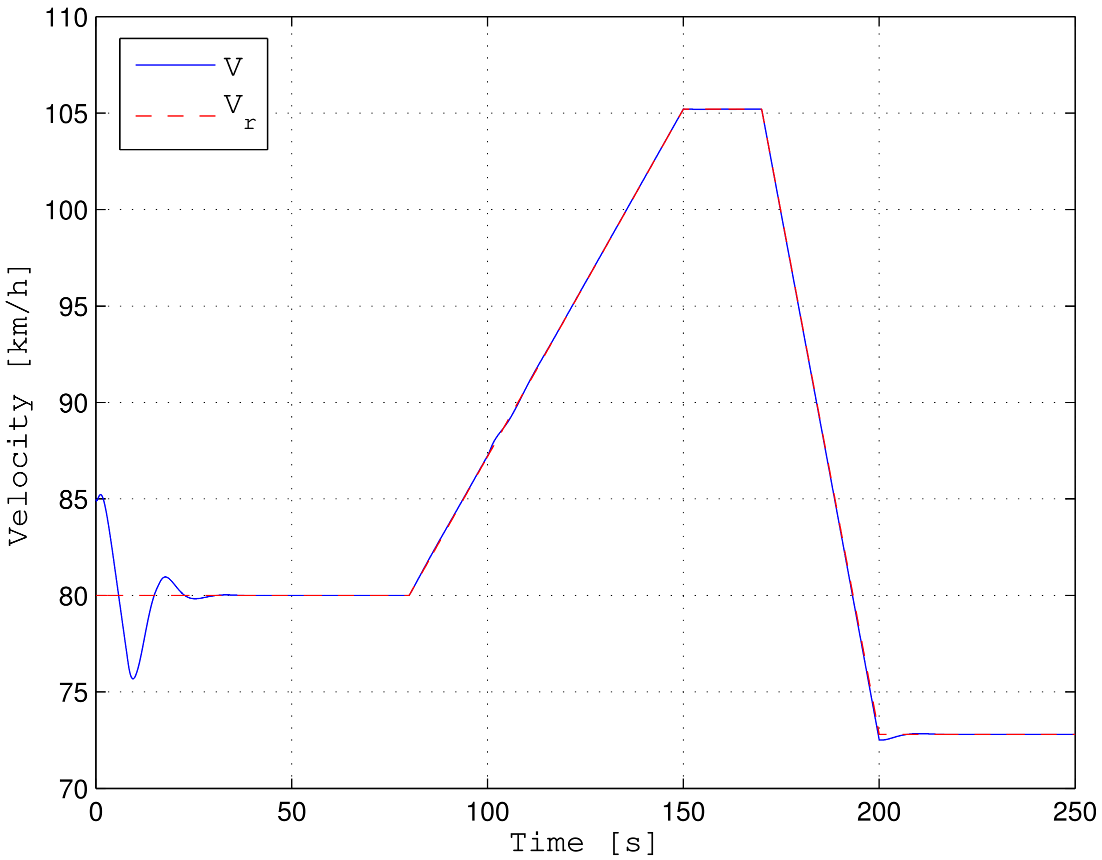
Final considerations (JGCD Paper)

- System stable in any operating point.
- No need of C_L model or aerodynamic moment coefficients.
- Control law applicable to any conventional airplane.
- No need to estimate α_0 or $C_{m_{\delta_e}}$.

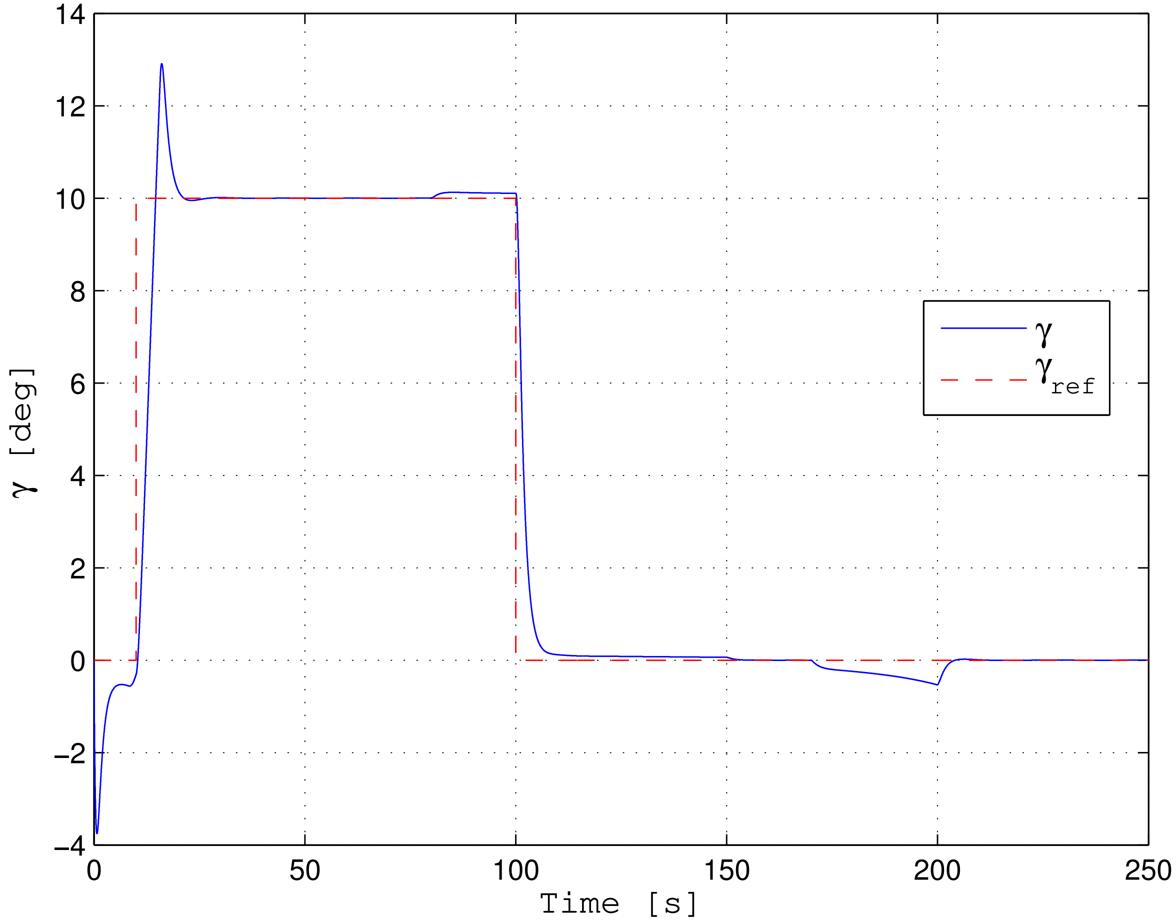
Outline

- Nonlinear model of Cefiro UAV.
- Aerodynamic model estimated using DATCOM and VL methods.
- Saturations in control signals included.

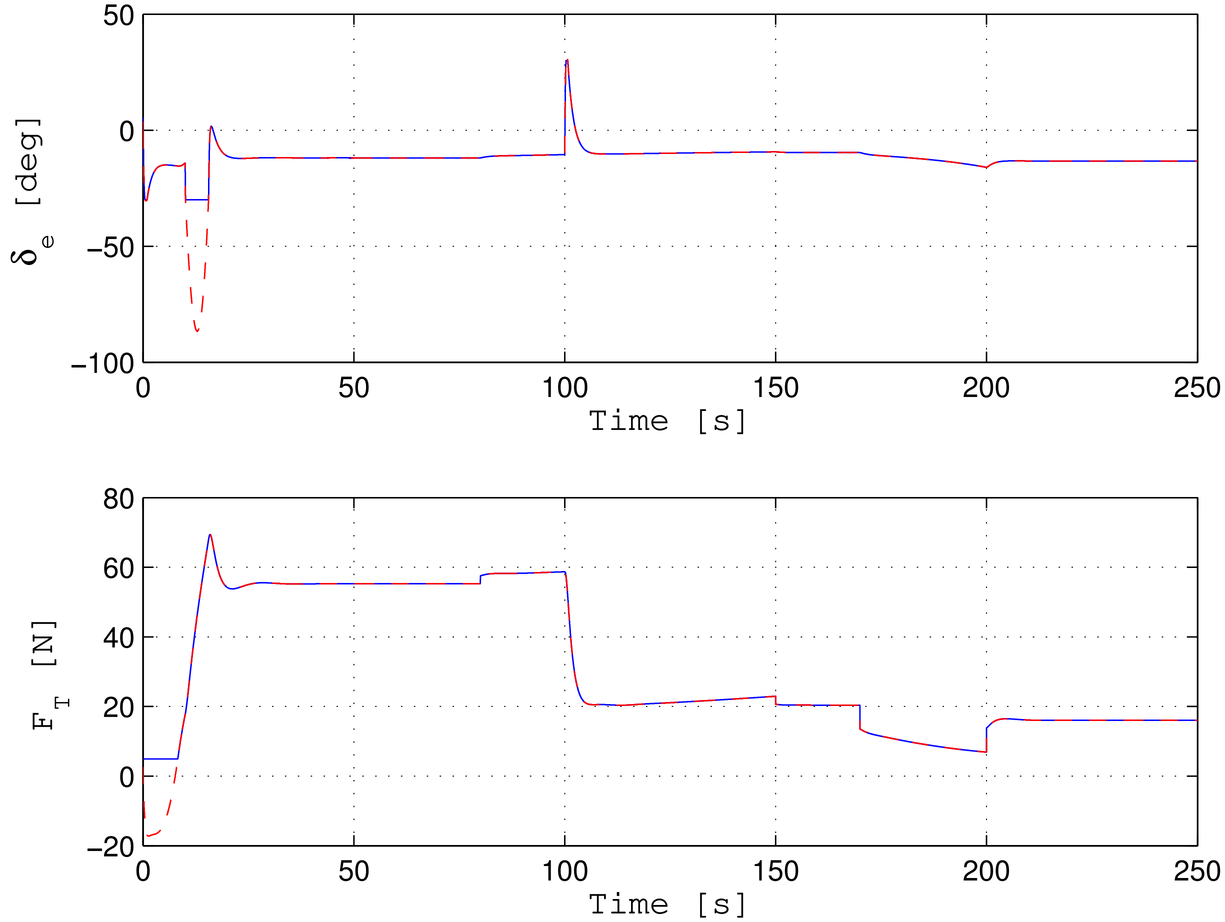
Airspeed seeking



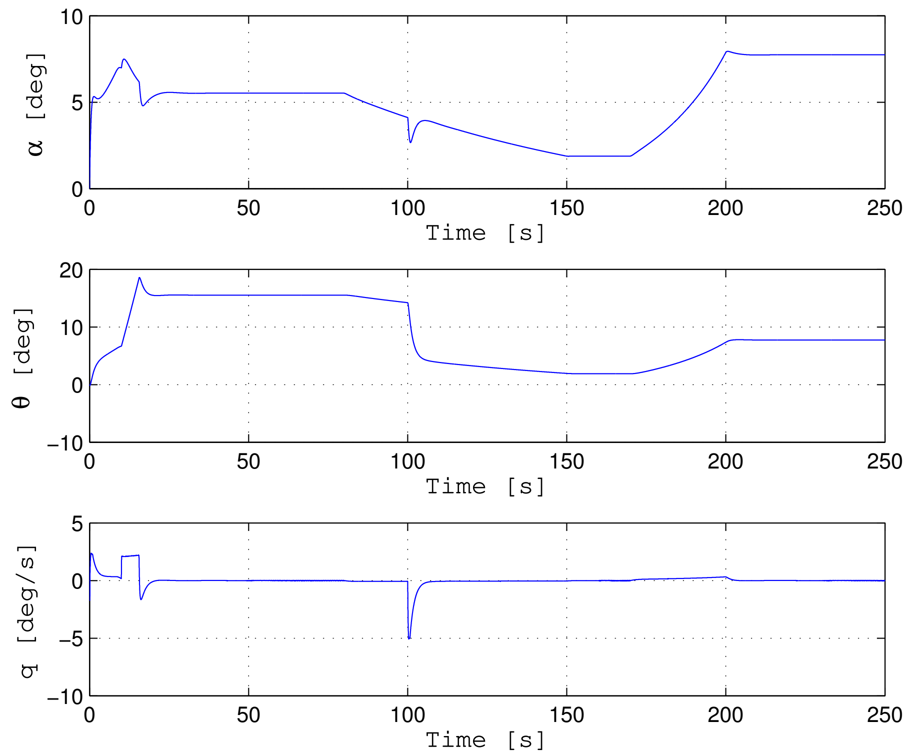
Flight path angle seeking

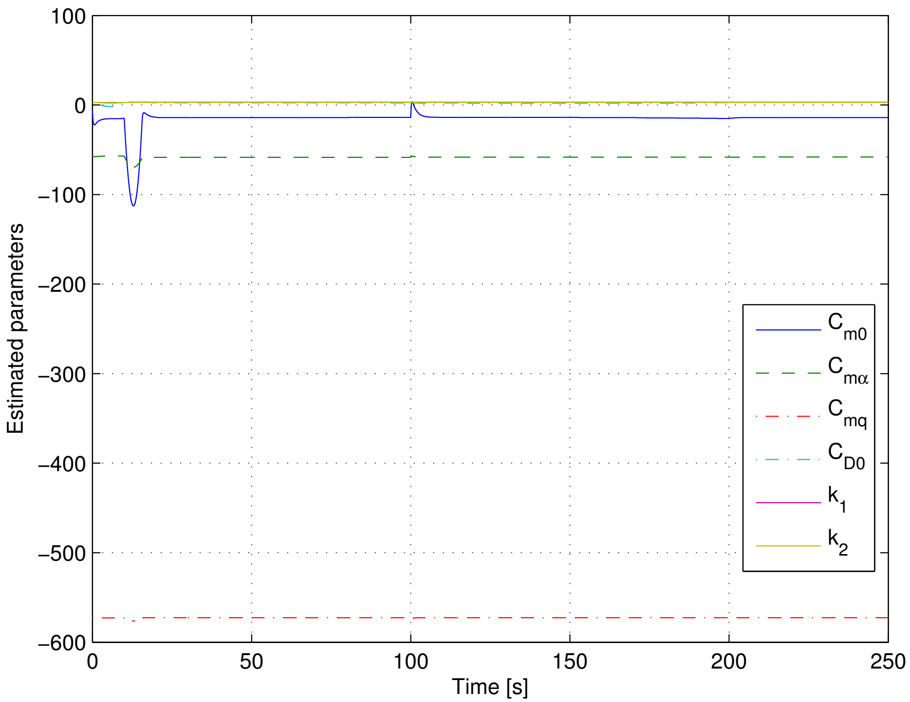


Control signals

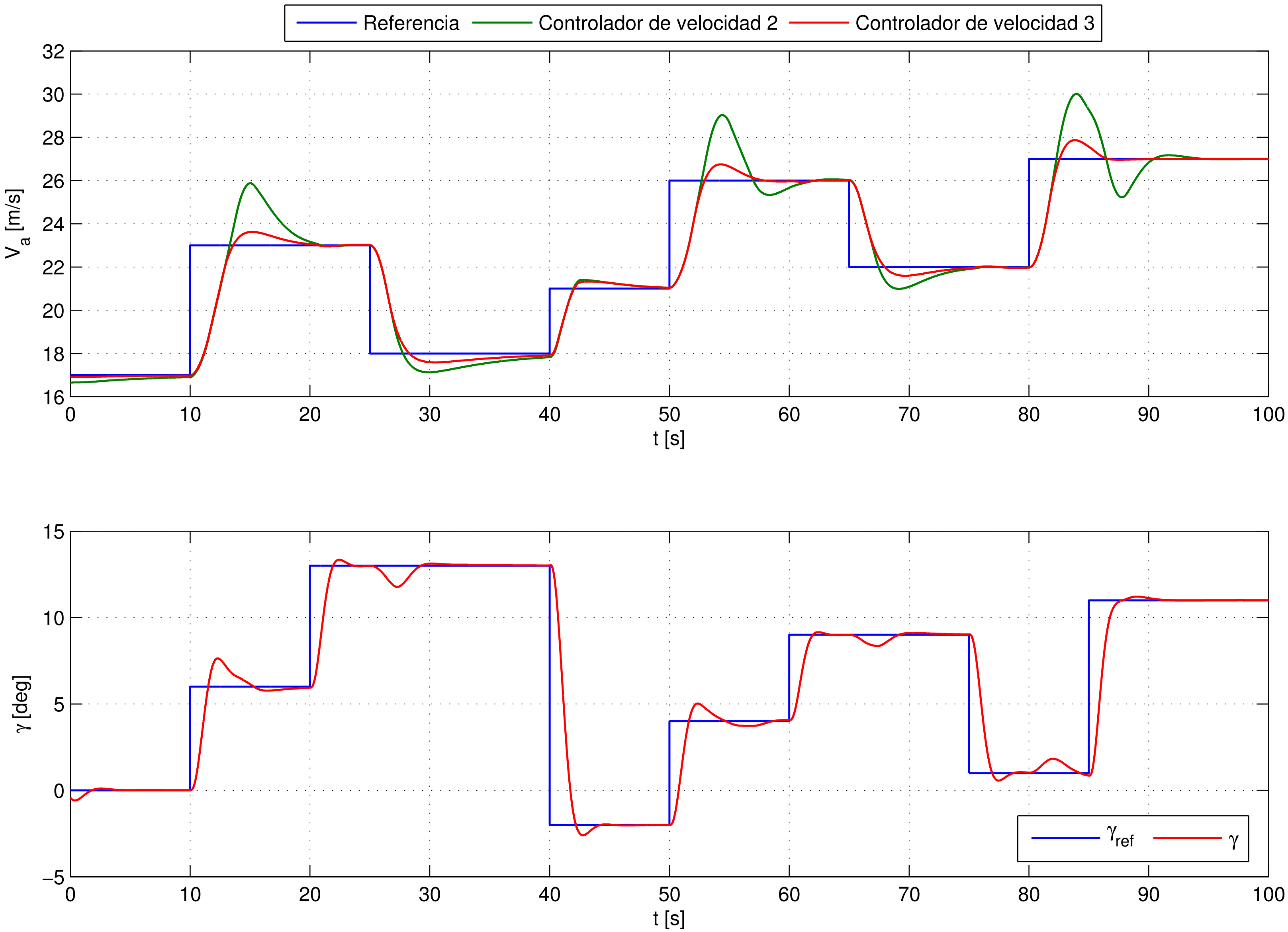


States and estimated parameters





Simulations - JGCD paper



Outline

Conclusions and Future Work

- A controller for longitudinal aircraft dynamics has been designed.
- Nonlinear approach. Control law valid for any operating point.
- Minimum aerodynamic model required. The controller can be adapted to any airplane.
- The control law obtained is simple and can be easily implemented.
- Saturations are taken into account.

Possible Extensions

- Include a propulsive model to use the throttle as control signal.
- Saturations for elevator deflection.

Conclusions and Future Work

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