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Higher Order Stability Properties of a 2D Navier Stokes System with an Explicit Boundary Controller

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Abstract— **In a previous work, we presented formulae for boundary control laws which stabilized the parabolic profile of an infinite channel flow, linearly unstable for high Reynolds number. Also know as the Poiseuille flow, this problem is frequently cited as a paradigm for transition to turbulence, whose stabilization for arbitrary Reynolds number, without using discretization, had so far been an open problem. L_2 stability was proved for the closed loop system. In this work, we extend the stability result to exponential stability in the H_1 and H_2 norms, and we state and prove some properties of the stabilizing controller, guaranteeing that the control law is well behaved.**

I. Introduction

In [11], an explicit boundary control law which stabilized a benchmark 2D linearized Navier-Stokes system was presented. For the resulting infinite dimensional closed loop system, a result guaranteeing L_2 exponential stability was proved. We complement this previous result by adding statements and proofs of exponential H_1 and H_2 stability, stronger forms of stability seldom found in flow control designs. The explicitness of the design allows as well to show some regularity results for the control laws, which are proved to be well defined and behaved. We do not prove well-posedness, however, with the high order Sobolev estimates that we derive it is certainly possible, though lengthy and far from trivial.

Most of the previous controllers for Navier-Stokes equations used their discretized version and employed high-dimensional algebraic Riccati equations for computation of gains [5]. This is the first result that provides an explicit control law (with symbolically computed gains) for stabilization in L_2, H_1 and H_2 norms, at an arbitrarily high Reynolds number in non-discretized Navier-Stokes equations. The only prior control design that was explicit, proved stability in the same norms, and did not employ discretization, was restricted to low Reynolds numbers [1], [2].

The results are applicable to both infinite and periodic channel flow with arbitrary periodic box size, and also extend to 3D [4]. Our control laws are written as state feedback, however, we have developed a dual observer design methodology [8] which we used to design an observer [10].

We start the paper by stating, in Section II, the mathematical model of the problem, which are the linearized Navier-Stokes equations for the velocity fluctuation around the (Poiseuille) equilibrium profile. In Section III, we review the control law that stabilizes the equilibrium profile, and state the main results of the paper. Section IV briefly reviews the L^2 proof of stability in [11], since it contains some ingredients required in subsequent sections. The proofs of H^1 and H^2 stability are presented, respectively, in Sections V, and VI. Section VII is devoted to study and prove some properties of the control laws.

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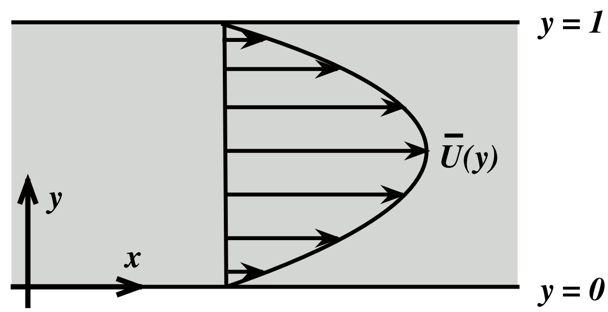

Fig. 1. 2D channel flow and equilibrium profile. Actuation is on the top wall.

II. Model

Consider a 2D incompressible channel flow evolving in a semi-infinite rectangle $(x,y) \in (-\infty,\infty) \times [0,1]$ as in Fig. 1. The (linearized) plant equations written in fluctuation variables are

$$u_t = \frac{1}{Re}(u_{xx}+u_{yy}) + 4y(y-1)u_x + 4(2y-1)V - p_x, \tag{1}$$

$$V_t = \frac{1}{Re}(V_{xx}+V_{yy}) + 4y(y-1)V_x - p_y, \tag{2}$$

where V is the wall-normal velocity, and u and p are the fluctuation streamwise velocity and pressure (see [11] for derivations). The boundary conditions are

$$u(x,0) = V(x,0) = 0, \tag{3}$$

$$u(x,1) = U_c(x), \tag{4}$$

$$V(x,1) = V_c(x). \tag{5}$$

The variables u and V also verify the continuity equation

$$u_x + V_y = 0. \tag{6}$$

Note the actuation variables $U_c(x)$ and $V_c(x)$ in (4) and (5), resp. for streamwise and normal velocity boundary control.

III. Controller

The expressions for the control laws are

$$U_c(t,x) = \int_0^1 \int_{-\infty}^{\infty} Q_u(x-\xi,\eta)u(t,\xi,\eta)d\xi d\eta, \tag{7}$$

$$V_c(t,x) = h(t,x), \tag{8}$$

where h verifies the equation

$$h_t = h_{xx} + g(t,x), \quad (9)$$

where

$$g = \int_0^1 \int_{-\infty}^{\infty} Q_V(x-\xi,\eta)V(t,\xi,\eta)d\xi d\eta + \int_{-\infty}^{\infty} Q_0(x-\xi)\left(u_y(t,\xi,0)-u_y(t,\xi,1)\right)d\xi, \quad (10)$$

and the kernels Q_u, Q_V and Q_0 are defined as

$$Q_u = \int_{-\infty}^{\infty} \chi(k)K(k,1,\eta)\mathrm{e}^{2\pi ik(x-\xi)}dk, \quad (11)$$

$$Q_V = \int_{-\infty}^{\infty} \chi(k)16\pi ki(2\eta-1)\cosh\left(2\pi k(1-\eta)\right) \times \mathrm{e}^{2\pi ik(x-\xi)}dk, \quad (12)$$

$$Q_0 = \int_{-\infty}^{\infty} \chi(k)\frac{2\pi ki}{Re}\mathrm{e}^{2\pi ik(x-\xi)}dk. \quad (13)$$

In expressions (11)–(13), $\chi(k)$ is a truncating function in the wave number space whose definition is

$$\chi(k) = \begin{cases} 1, & m < |k| < M \\ 0, & \text{otherwise} \end{cases} \quad (14)$$

where m and M are respectively the low and high cut-off wave numbers, two design parameters which can be conservatively chosen as $m \leq \frac{1}{32\pi Re}$ and $M \geq \frac{1}{\pi}\sqrt{\frac{Re}{2}}$. The function $K(k,y,\eta)$ appearing in (11) is a (complex valued) gain kernel defined as

$$K(k,y,\eta) = \lim_{n\to\infty} K_n(k,y,\eta), \quad (15)$$

where K_n is recursively defined as [1]

$$\begin{aligned} K_0 = & -\frac{Re}{3}\pi ik\eta\left(21y^2-6y(3+4\eta)+\eta(12+7\eta)\right) \\ & -2\pi k\frac{\cosh\left(2\pi k(1-y+\eta)\right)-\cosh\left(2\pi k(y-\eta)\right)}{\sinh\left(2\pi k\right)} \\ & +4iRe\eta(\eta-1)\sinh\left(2\pi k(y-\eta)\right) \\ & -6\eta i\frac{Re}{\pi k}(1-\cosh\left(2\pi k(y-\eta)\right)), \end{aligned} \quad (16)$$

$$\begin{aligned} K_n = & K_{n-1} \\ & -4\pi kiRe\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}\int_{-\delta}^{\delta}\left\{\frac{\sinh\left(\pi k(\xi+\delta)\right)}{\pi k}\right. \\ & \left.-(2\xi-1)+2(\gamma-\delta-1)\cosh\left(\pi k(\xi+\delta)\right)\right\} \\ & \times K_{n-1}\left(k,\frac{\gamma+\delta}{2},\frac{\gamma+\xi}{2}\right)d\xi d\delta d\gamma \\ & +\frac{Re}{2}\pi ik\int_{y-\eta}^{y+\eta}\int_0^{y-\eta}(\gamma-\delta)(\gamma-\delta-2) \\ & \times K_{n-1}\left(k,\frac{\gamma+\delta}{2},\frac{\gamma-\delta}{2}\right)d\delta d\gamma \\ & +2\pi k\int_0^{y-\eta}\frac{\cosh\left(2\pi k(1-\delta)\right)-\cosh\left(2\pi k\delta\right)}{\sinh\left(2\pi k\right)} \\ & \times K_{n-1}\left(k,y-\eta,\delta\right)d\delta. \end{aligned} \quad (17)$$

[1] This infinite sequence is convergent, smooth, and uniformly bounded over $(y,\eta) \in [0,1]^2$, and analytic in k, see [11].

Remark 3.1: Control kernels (12) and (13) can be explicitly expressed as

$$Q_V(\xi,\eta) = 8(2\eta-1)\frac{R_V(\xi,\eta,M)-R_V(\xi,\eta,m)}{\xi^2+(1-\eta)^2}, \quad (18)$$

$$Q_0(\xi,\eta) = \frac{R_0(\xi,\eta,M)-R_0(\xi,\eta,m)}{Re\,\xi}, \quad (19)$$

where

$$\begin{aligned} R_V(\xi,\eta,k) = & \frac{\left((1-\eta)^2-\xi^2\right)\sin(2\pi k\xi)\cosh(2\pi k(1-\eta))}{2\pi(\xi^2+(1-\eta)^2)} \\ & -\frac{\xi(1-\eta)\cos\left(2\pi k\xi\right)\sinh\left(2\pi k(1-\eta)\right)}{\pi(\xi^2+(1-\eta)^2)} \\ & -k(1-\eta)\sin(2\pi k\xi)\sinh(2\pi k(1-\eta)) \\ & +k\xi\cos\left(2\pi k\xi\right)\cosh\left(2\pi k(1-\eta)\right), \end{aligned} \quad (20)$$

$$R_0(\xi,\eta,k) = k\cos\left(2\pi k\xi\right)-\frac{\sin\left(2\pi k\xi\right)}{2\pi\xi}. \quad (21)$$

Control laws (7)–(17) guarantee the following results.

Theorem 3.1: The equilibrium $u(x,y) \equiv V(x,y) \equiv 0$ of system (1)–(5), (7)–(17) is exponentially stable in the L^2, H^1 and H^2 norms.

Theorem 3.2: Control laws U_c, V_c and kernels Q_u, Q_V, Q_0, as defined by (7)–(17), have the following properties:

i) U_c and V_c are spatially invariant in x.
ii) $\int_{-\infty}^{\infty} V_c(t,\xi)d\xi = 0$ (zero net flux).
iii) $|Q| \leq C/|x-\xi|$, for $Q = Q_u, Q_V, Q_0$.
iv) U_c and V_c are smooth functions of x.
v) Q_u, Q_V, Q_0 are real valued.
vi) Q_u, Q_V, Q_0 are smooth in their arguments.
vii) U_c and V_c are L^2 functions of x.
viii) All spatial derivatives of U_c and V_c are L^2 function of x.

Remark 3.2: Theorem 3.1, stated for the linearized equations (1)–(2), is valid for the nonlinear Navier-Stokes equations in a *local* sense, i.e., provided that the initial data are sufficiently close (in the appropiate norm) to the equilibrium.

Remark 3.3: H^2 stability suffices to establish continuity of the velocity field for a bounded domain, by Sobolev's Embedding Theorem [9]. The argument is not applicable to the infinite channel, but it holds if the channel is periodic, a setting for which our results extend trivially.

Remark 3.4: Theorem 3.2 ensures that the control laws are well behaved. Property i, spatial invariance, means that the feedback operators commute with translations in the x direction [3], which is crucial for implementation. Property ii ensures that we do not violate the physical restriction of zero net flux, which is derived from mass conservation. Property iii allows to truncate the integrals with respect to ξ to the vicinity of x, allowing sensing to be restricted just to a neighborhood (in the x direction) of the actuator. Properties iv to vi ensure that the control laws are well defined. Properties vii and viii prove finiteness of energy of the controllers and their spatial derivatives.

In the next sections we present a sketch of the proof of the theorems. We skip some intermediate steps due to space limitation; full details will be provided in a future publication. We begin by reviewing some results in [11].

IV. L^2 STABILITY

As common for infinite channels, we use a Fourier transform in x. The transform pair (direct and inverse transform) has the following definition:

$$f(k,y) = \int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{-2\pi ikx}dx, \quad (22)$$

$$f(x,y) = \int_{-\infty}^{\infty} f(k,y)\mathrm{e}^{2\pi ikx}dx. \quad (23)$$

Note that we use the same symbol f for both the original $f(x,y)$ and the image $f(k,y)$. In hydrodynamics, k is referred to as the "wave number."

One property of Fourier transform is Parseval's formula

$$||f||^2_{L^2}=\int_0^1\int_{-\infty}^{\infty} f^2(k,y)dkdy=\int_0^1\int_{-\infty}^{\infty} f^2(x,y)dxdy, \quad (24)$$

which allows to derive L^2 exponential stability in physical space from the same property in Fourier space.

We define the L^2 norm of $f(k,y)$ with respect to y:

$$||f(k)||^2_{\hat{L}^2} = \int_0^1 |f(k,y)|^2 dy. \quad (25)$$

The $\hat{L}^2$ norm as a function of k is related to the L^2 norm as

$$||f||^2_{L^2} = \int_{-\infty}^{\infty} ||f(k)||^2_{\hat{L}^2}dk. \quad (26)$$

Equations (1)–(2) written in the Fourier domain are

$$\begin{aligned} u_t &= \frac{-4\pi^2k^2u+u_{yy}}{Re}+8k\pi iy(y-1)u \\ &\quad +4(2y-1)V-2\pi ikp, \quad (27)\\ V_t &= \frac{-4\pi^2k^2V+V_{yy}}{Re}+8\pi kiy(y-1)V-p_y, \quad (28)\end{aligned}$$

and we can write a Poisson equation for the pressure [11],

$$-4\pi^2k^2p+p_{yy} = 16\pi ki(2y-1)V. \quad (29)$$

The boundary conditions of (27)–(29) are

$$\begin{aligned} u(k,0) &= V(k,0)=0, \quad (30)\\ u(k,1) &= U_c(k), \quad (31)\\ V(k,1) &= V_c(k), \quad (32)\\ p_y(k,0) &= \frac{V_{yy}(k,0)}{Re}, \quad (33)\\ p_y(k,1) &= \frac{V_{yy}(k,1)-4\pi^2k^2V_c(k)}{Re}-(V_c)_t(k). \quad (34)\end{aligned}$$

The continuity equation (6) expressed in Fourier space is

$$2\pi kiu(k,y)+V_y(k,y)=0. \quad (35)$$

Thanks to linearity and spatial invariance, there is no coupling between different wave numbers in (27)–(35). This allows us to consider these equations for each wave number independently. The main idea behind the design of the controller is to consider two different cases depending on the wave number k. For wave numbers $m<|k|<M$, which we refer to as *controlled* wave numbers, we design a backstepping controller that achieves stabilization, whereas for wave numbers in the range $|k|\geq M$ or in the range $|k|\leq m$, which we call *uncontrolled* wave numbers, the system is left without control but is exponentially stable [6].

A. Controlled wave numbers

In what follows, let the letters D and d with subscript denote some positive constant.

Control laws (7)–(17) in Fourier space are

$$\begin{aligned} U_c &= \int_0^1 K(k,1,\eta)u(t,k,\eta)d\eta, \quad (36)\\ (V_c)_t &= \frac{2\pi ki\left(u_y(k,0)-u_y(k,1)\right)-4\pi^2k^2V_c}{Re} \\ &\quad -16\pi ki\int_0^1 (2\eta-1) \\ &\quad \times V(k,\eta)\cosh\left(2\pi k(1-\eta)\right)d\eta. \quad (37)\end{aligned}$$

In [11] we showed that, with control laws (36)–(37), (27)–(28) are mapped into the family of heat equations

$$\begin{aligned} \alpha_t &= \frac{1}{Re}\left(-4\pi^2k^2\alpha+\alpha_{yy}\right), \quad (38)\\ \alpha(k,0) &= \alpha(k,1)=0\,, \quad (39)\end{aligned}$$

where

$$\begin{aligned} \alpha &= u-\int_0^y K(k,y,\eta)u(t,k,\eta)d\eta\,, \quad (40)\\ u &= \alpha+\int_0^y L(k,y,\eta)\alpha(t,k,\eta)d\eta\,, \quad (41)\\ \alpha &= i\frac{V_y-\int_0^y K(k,y,\eta)V_y(t,k,\eta)d\eta}{2\pi k} \quad (42)\\ V &= -2\pi ki\int_0^y\left[1+\int_\eta^y L(k,\eta,\sigma)d\sigma\right] \\ &\quad \times\alpha(t,k,\eta)d\eta\,. \quad (43)\end{aligned}$$

are respectively the direct and inverse transformation [7] for u and V, with K defined in (16)–(17) and L similarly. Using (38)–(39) and (40)–(43) the following results holds.

Proposition 4.1: For any k in the range $m<|k|<M$, the equilibrium $u(t,k,y)\equiv V(t,k,y)\equiv 0$ of (27)–(34) with control laws (37), (36) is exp. stable in the L^2 sense, i.e.,

$$\begin{aligned} &||V(t,k)||^2_{\hat{L}^2}+||u(t,k)||^2_{\hat{L}^2} \\ &\leq D_0\mathrm{e}^{\frac{1}{2Re}t}\left(||V(0,k)||^2_{\hat{L}^2}+||u(0,k)||^2_{\hat{L}^2}\right). \quad (44)\end{aligned}$$

Defining then

$$\begin{aligned} V^*(t,x,y) &= \int_{-\infty}^{\infty}\chi(k)V(t,k,y)\mathrm{e}^{2\pi ikx}dk, \quad (45)\\ u^*(t,x,y) &= \int_{-\infty}^{\infty}\chi(k)u(t,k,y)\mathrm{e}^{2\pi ikx}dk, \quad (46)\end{aligned}$$

the following result holds.

Proposition 4.2: Consider equations (1)–(5) with control laws (7)–(8). Then u^* and V^* decay exponentially:

$$\begin{aligned} &||V^*(t)||^2_{L^2}+||u^*(t)||^2_{L^2} \\ &\leq D_0\mathrm{e}^{\frac{1}{2Re}t}\left(||V^*(0)||^2_{L^2}+||u^*(0)||^2_{L^2}\right). \quad (47)\end{aligned}$$

B. Uncontrolled wave number analysis

In [11] we proved the following result.

Proposition 4.3: If $m=\frac{1}{32\pi Re}$ and $M=\frac{1}{\pi}\sqrt{\frac{Re}{2}}$, then for both $|k|\leq m$ and $|k|\geq M$ the equilibrium $u(t,k,y)\equiv$

$V(t,k,y) \equiv 0$ of the uncontrolled system (27)–(34) is exponentially stable in the L^2 sense:

$$\begin{aligned} &||V(t,k)||_{\hat{L}^2}^2 + ||u(t,k)||_{\hat{L}^2}^2 \\ \leq \quad & \mathrm{e}^{\frac{1}{4Re}t}\left(||V(0,k)||_{\hat{L}^2}^2 + ||u(0,k)||_{\hat{L}^2}^2\right). \end{aligned} \quad (48)$$

In the proof, we defined

$$\Lambda(k,t) = \frac{1}{2}\left(||V(t,k)||_{\hat{L}^2}^2 + ||u(t,k)||_{\hat{L}^2}^2\right), \quad (49)$$

and showed that, for $|k| < m$ and $|k| > M$,

$$\dot{\Lambda} \leq -\frac{1}{4Re}\Lambda. \quad (50)$$

Using Proposition 4.3, the following result holds.

Proposition 4.4: The variables ϵ_u and ϵ_V defined as

$$\epsilon_u(t,x,y) = \int_{-\infty}^{\infty} (1-\chi(k))\, u(t,k,y)\mathrm{e}^{2\pi i k x} dk, \quad (51)$$

$$\epsilon_V(t,x,y) = \int_{-\infty}^{\infty} (1-\chi(k))\, V(t,k,y)\mathrm{e}^{2\pi i k x} dk, \quad (52)$$

decay exponentially

$$\begin{aligned} &||\epsilon_V(t)||_{L^2}^2 + ||\epsilon_u(t)||_{L^2}^2 \\ \leq \quad & \mathrm{e}^{\frac{1}{4Re}t}\left(||\epsilon_V(0)||_{L^2}^2 + ||\epsilon_u(0)||_{L^2}^2\right). \end{aligned} \quad (53)$$

C. Analysis for the entire wave number range

Since

$$u(t,x,y) = u^*(t,x,y) + \epsilon_u(t,x,y), \quad (54)$$

$$V(t,x,y) = V^*(t,x,y) + \epsilon_V(t,x,y), \quad (55)$$

and since the L^2 norm of V is the sum of the L^2 norms of $V^*(t,k,y)$ and $\epsilon_V(t,k,y)$ (and similarly for u), the L^2 part of Theorem 3.1 follows from Propositions 4.2 and 4.4.

V. H^1 STABILITY

We define the H^1 norm of $f(x,y)$ as

$$||f||_{H^1}^2 = ||f||_{L^2}^2 + ||f_x||_{L^2}^2 + ||f_y||_{L^2}^2. \quad (56)$$

We also define the H^1 norm of $f(k,y)$ with respect to y

$$||f(k)||_{\hat{H}^1}^2 = (1+4\pi^2k^2)||f(k)||_{\hat{L}^2}^2 + ||f_y(k)||_{\hat{L}^2}^2. \quad (57)$$

The $\hat{H}^1$ norm as a function of k is related to the H^1 norm

$$||f||_{H^1}^2 = \int_{-\infty}^{\infty} ||f(k)||_{\hat{H}^1}^2 dk. \quad (58)$$

A. H^1 stability for controlled wave numbers

For each k, one has that

$$||f(k)||_{\hat{H}^1}^2 \leq (5+16\pi^2M^2)||f_y(k)||_{\hat{H}^1}^2, \quad (59)$$

where we have used (57) and Poincare's inequality. This proves the equivalence, for any k, of the $\hat{H}^1$ norm of $f(k,y)$ and the $\hat{L}^2$ norm of just $f_y(k,y)$. Therefore, we only have to show exponential decay for u_y and V_y.

Due to the backstepping transformations (40), (41) and (42), (43), the following bounds are derived from simple estimates on α and α_y from (38)

$$||u_y(t,k)||_{\hat{L}^2}^2 \leq D_1\mathrm{e}^{-\frac{2}{5Re}t}||u_y(0,k)||_{\hat{L}^2}^2, \quad (60)$$

$$||V_y(t,k)||_{\hat{L}^2}^2 \leq D_0\mathrm{e}^{-\frac{1}{2Re}t}||V_y(0,k)||_{\hat{L}^2}^2, \quad (61)$$

Using estimates (60)–(61) the following proposition can be stated at each k in the controlled range.

Proposition 5.1: For any k in the range $m < |k| < M$, the equilibrium $u(t,k,y) \equiv V(t,k,y) \equiv 0$ of the system (27)–(34) with feedback control laws (37), (36) is exponentially stable in the H^1 sense

$$\begin{aligned} &||V(t,k)||_{\hat{H}^1}^2 + ||u(t,k)||_{\hat{H}^1}^2 \\ \leq \quad & D_2\mathrm{e}^{\frac{2}{5Re}t}\left(||V(0,k)||_{\hat{H}^1}^2 + ||u(0,k)||_{\hat{H}^1}^2\right). \end{aligned} \quad (62)$$

Integrating (62) in the *controlled* wave number range $m < |k| < M$, and using (58), the following result holds.

Proposition 5.2: Consider equations (1)–(5) with control laws (7)–(8). Then the variables $u^*(t,x,y)$ and $V^*(t,x,y)$ defined in (45)–(46) decay exponentially in the H^1 norm:

$$\begin{aligned} &||u^*(t)||_{H^1}^2 + ||V^*(t)||_{H^1}^2 \\ \leq \quad & D_2\mathrm{e}^{\frac{2}{5Re}t}\left(||u^*(0)||_{H^1}^2 + ||V^*(0)||_{H^1}^2\right). \end{aligned} \quad (63)$$

B. H^1 stability for uncontrolled wave numbers

Following the same argument [11], that produced (49)–(50), the following bound holds

$$\dot{\Lambda} \leq -\frac{\Lambda}{8Re} - \frac{\Lambda_H}{2Re}, \quad (64)$$

where

$$\Lambda_H(k,t) = \frac{1}{2}\left(||u_y(t,k)||_{\hat{L}^2}^2 + ||V_y(t,k)||_{\hat{L}^2}^2\right). \quad (65)$$

The time derivative of Λ_H can be bounded as

$$\begin{aligned} \frac{d\Lambda_H}{dt} = & -\frac{1}{Re}\left(||u_{yy}||_{\hat{L}^2}^2 + ||V_{yy}||_{\hat{L}^2}^2\right) \\ & +4k^2\pi^2\int_0^1 \frac{u_{yy}\bar{u} + \bar{u}_{yy}u + \bar{V}_{yy}V + V_{yy}\bar{V}}{2Re} dy \\ & +2\pi k i \int_0^1 \frac{u_{yy}\bar{p} - \bar{u}_{yy}p}{2} dy \\ & -\int_0^1 \frac{\bar{V}_{yy}p_y + V_{yy}\bar{p}_y}{2} dy \\ & +8\pi k i \int_0^1 y(y-1) \\ & \times \frac{u_{yy}\bar{u} - \bar{u}_{yy}u - \bar{V}_{yy}V + V_{yy}\bar{V}}{2} dy \\ & -4\int_0^1 (2y-1)\frac{u_{yy}\bar{V} + \bar{u}_{yy}V}{2} dy, \end{aligned} \quad (66)$$

where we have used integration by parts and the Dirichlet boundary conditions of the uncontrolled wave number range. Doing further integration by parts and using (35), we obtain

$$\begin{aligned} \frac{d\Lambda_H}{dt} = & -\frac{1}{Re}\left(||u_{yy}||_{\hat{L}^2}^2 + ||V_{yy}||_{\hat{L}^2}^2\right) - \frac{8k^2\pi^2}{Re}\Lambda_h \\ & -16\pi^2k^2\int_0^1 (2y-1)\frac{\bar{u}V - u\bar{V}}{2} dy \\ & -\left.\frac{\bar{V}_{yy}p + V_{yy}\bar{p}}{2}\right|_0^1. \end{aligned} \quad (67)$$

Only the last term remains to be estimated. Using (33)–(34) with V_c being zero for uncontrolled wave number range, the last term in (67) can be expresssed as

$$\frac{\bar V_{yy}p+V_{yy}\bar p}{2}\bigg|_0^1 = Re\frac{\bar p_y p + p_y\bar p}{2}\bigg|_0^1. \quad (68)$$

This quantity can be estimated using the following lemma.

Lemma 5.1: If the pressure p verifies the Poisson equation (29) with boundary conditions (33)–(34), then

$$-\frac{\bar p_y p + p_y\bar p}{2}\bigg|_0^1 \leq 16||V(t,k)||^2_{\hat L^2}. \quad (69)$$

Using the lemma, the time derivative of Λ_H can be estimated as follows:

$$\frac{d\Lambda_H}{dt} \leq -\frac{8k^2\pi^2}{Re}\Lambda_H + 16\pi^2k^2\Lambda + 16Re\Lambda. \quad (70)$$

We take the following Lyapunov functional

$$\Lambda_T = \Lambda_H + (1+64Re^2+4\pi^2k^2+64Re\pi^2k^2)\Lambda, \quad (71)$$

which is *equivalent* to the H^1 norm,

$$||u(t,k)||^2_{\hat H^1} + ||V(t,k)||^2_{\hat H^1} = 2(1+4\pi^2k^2)\Lambda + 2\Lambda_H. \quad (72)$$

Computing the derivative of (71)

$$\frac{d\Lambda_T}{dt} \leq -\frac{\Lambda_H}{2Re} - \frac{1+4\pi^2k^2}{8Re}\Lambda \leq -d_1\Lambda_T. \quad (73)$$

Deriving an estimate of the H^1 norm from this estimate for Λ_T, one reaches the following result.

Proposition 5.3: If $m=\frac{1}{32\pi Re}$ and $M=\frac{1}{\pi}\sqrt{\frac{Re}{2}}$, then for both $|k|\leq m$ and $|k|\geq M$ the equilibrium $u(t,k,y)\equiv V(t,k,y)\equiv 0$ of the uncontrolled system (27)–(34) is exponentially stable in the H^1 sense:

$$\begin{aligned}&||V(t,k)||^2_{\hat H^1}+||u(t,k)||^2_{\hat H^1}\\ \leq\ & D_3 e^{-d_1 t}\left(||V(0,k)||^2_{\hat H^1}+||u(0,k)||^2_{\hat H^1}\right). \quad (74)\end{aligned}$$

Since the decay rate in (74) is independent of k, integrating (74) and using (58) the following result holds.

Proposition 5.4: The variables $\epsilon_u(t,x,y)$ and $\epsilon_V(t,x,y)$ defined in (51)–(52) decay exponentially in the H^1 norm as

$$\begin{aligned}&||\epsilon_u(t)||^2_{H^1}+||\epsilon_V(t)||^2_{H^1}\\ \leq\ & D_3 e^{-d_1 t}\left(||\epsilon_u(0)||^2_{H^1}+||\epsilon_V(0)||^2_{H^1}\right). \quad (75)\end{aligned}$$

C. Analysis for all wave numbers

From Propositions 5.2 and 5.4, as in Section IV-C, H^1 stability is proved.

$$\begin{aligned}&||u(t)||^2_{H^1}+||V(t)||^2_{H^1}\\ \leq\ & D_4 e^{-d_1 t}\left(||u(0)||^2_{H^1}+||V(0)||^2_{H^1}\right). \quad (76)\end{aligned}$$

VI. H^2 STABILITY

The H^2 norm of $f(x,y)$ is defined as

$$||f||^2_{H^2} = ||f||^2_{H^1} + ||f_{xx}||^2_{L^2} + ||f_{xy}||^2_{L^2} + ||f_{yy}||^2_{L^2}. \quad (77)$$

We also define the H^2 norm of $f(k,y)$ with respect to y as

$$\begin{aligned}||f(k)||^2_{\hat H^2} =\ & ||f(k)||^2_{\hat H^1} + 16\pi^4k^4||f(k)||^2_{\hat L^2}\\ & +4\pi^2k^2||f_y(k)||^2_{\hat L^2} + ||f_{yy}(k)||^2_{\hat L^2}. \quad (78)\end{aligned}$$

The $\hat H^2$ norm as a k function is related to the H^2 norm as

$$||f||^2_{H^2} = \int_{-\infty}^{\infty} ||f(k)||^2_{\hat H^2}dk. \quad (79)$$

A. H^2 stability for controlled wave numbers

Thanks to the backstepping transformations (40), (41) and (42), (43), we can write the following estimates, which are derived from simple estimates on α, α_y and α_{yy} from (38):

$$||u(t,k)||^2_{\hat H^2} \leq D_5 e^{-\frac{2}{5Re}t}||u(0,k)||^2_{\hat H^2}, \quad (80)$$

$$||V(t,k)||^2_{\hat H^2} \leq D_6 e^{-\frac{2}{5Re}t}||V(0,k)||^2_{\hat H^2}. \quad (81)$$

Using estimates (80)–(81) the following proposition holds at each k in the controlled range.

Proposition 6.1: For any k in the range $m<|k|<M$, the equilibrium $u(t,k,y)\equiv V(t,k,y)\equiv 0$ of (27)–(34) with feedback laws (37), (36) is exp. stable in the H^2 sense

$$\begin{aligned}&||V(t,k)||^2_{\hat H^2}+||u(t,k)||^2_{\hat H^2}\\ \leq\ & D_7 e^{\frac{2}{5Re}t}\left(||V(0,k)||^2_{\hat H^2}+||u(0,k)||^2_{\hat H^2}\right). \quad (82)\end{aligned}$$

Integrating (82) in the *controlled* wave number range $m<|k|<M$, and using (79), the following result holds.

Proposition 6.2: Consider equations (1)–(5) with control laws (8)–(7). Then the variables $u^*(t,x,y)$ and $V^*(t,x,y)$ defined in (45)–(46) decay exponentially in the H^2 norm:

$$\begin{aligned}&||u^*(t)||^2_{H^2}+||V^*(t)||^2_{H^2}\\ \leq\ & D_8 e^{\frac{2}{5Re}t}\left(||u^*(0)||^2_{H^2}+||V^*(0)||^2_{H^2}\right). \quad (83)\end{aligned}$$

B. H^2 stability for uncontrolled wave numbers

For the uncontrolled range, thanks to boundary conditions, the $\hat H^2$ norm $||u(t,k)||_{\hat H^2}$ is equivalent to the norm

$$||u(t,k)||^2_{\hat H^1} + \int_0^1 \left|u_{yy}(t,k,y) - 4\pi^2k^2u(t,k,y)\right|^2 dy, \quad (84)$$

i.e., to the $\hat H^1$ norm plus the $\hat L^2$ norm of the Laplacian, denoted $||\triangle_k u(k)||^2_{\hat L^2}$. This can be shown integrating (84) by parts. We state another norm equivalence as a lemma.

Lemma 6.1: Consider u and V verifying equations (27)–(28). Then, for the uncontrolled wave number range, the norm $||u||^2_{\hat H^2}+||V||^2_{\hat H^2}$ is equivalent to the norm

$$||u||^2_{\hat H^1}+||V||^2_{\hat H^1}+||u_t||^2_{\hat L^2}+||V_t||^2_{\hat L^2}. \quad (85)$$

This means that the Laplacian operator in norm (84) can be replaced by a time derivative, when considering the joint H^2 norms of u and V.

From Lemma 6.1 one gets $\hat H^2$ stability for the uncontrolled wave numbers. This is obtained by considering the norm $||u_t||^2_{\hat L^2}+||V_t||^2_{\hat L^2}$ as a Lyapunov functional whose derivative can be bounded as

$$\frac{d}{dt}\frac{||u_t||^2_{\hat L^2}+||V_t||^2_{\hat L^2}}{2} \leq -\frac{1}{4Re}\left(||u_t||^2_{\hat L^2}+||V_t||^2_{\hat L^2}\right), \quad (86)$$

which follows by taking the time derivative of (27)–(28) and applying the same argument as for L^2 stability. Thus,

$$\begin{aligned}&||u_t(t,k)||^2_{\hat L^2}+||V_t(t,k)||^2_{\hat L^2}\\ \leq\ & e^{-\frac{1}{2Re}t}\left(||u_t(0,k)||^2_{\hat L^2}+||V_t(0,k)||^2_{\hat L^2}\right). \quad (87)\end{aligned}$$

Using (87), (74), and Lemma 6.1, the following result holds.

Proposition 6.3: If $m=\frac{1}{32\pi Re}$ and $M=\frac{1}{\pi}\sqrt{\frac{Re}{2}}$, then for both $|k|\leq m$ and $|k|\geq M$ the equilibrium $u(t,k,y)\equiv$

$V(t,k,y) \equiv 0$ of the uncontrolled system (27)–(34) is exponentially stable in the H^2 sense:

$$\begin{aligned} &||V(t,k)||^2_{\hat{H}^2} + ||u(t,k)||^2_{\hat{H}^2} \\ \leq\ & D_8^2 D_3 \mathrm{e}^{-d_1 t}\left(||V(0,k)||^2_{\hat{H}^2} + ||u(0,k)||^2_{\hat{H}^2}\right). \quad (88)\end{aligned}$$

Since the decay rate in (88) is independent of k, using (79) we can integrate (88) for *all* uncontrolled wave numbers.

Proposition 6.4: The variables $\epsilon_u(t,x,y)$ and $\epsilon_V(t,x,y)$ defined as in (51)–(52) decay exponentially in the H^2 norm

$$\begin{aligned} &||\epsilon_u(t)||^2_{H^2} + ||\epsilon_V(t)||^2_{H^2} \\ \leq\ & D_8^2 D_3 \mathrm{e}^{-d_1 t}\left(||\epsilon_u(0)||^2_{H^2} + ||\epsilon_V(0)||^2_{H^2}\right). \quad (89)\end{aligned}$$

C. Analysis for all wave numbers

From Propositions 6.2 and 6.4, as in Section IV-C, H^2 stability is proved. One gets that

$$\begin{aligned} &||u(t)||^2_{H^2} + ||V(t)||^2_{H^2} \\ \leq\ & D_9 \mathrm{e}^{-d_1 t}\left(||u(0)||^2_{H^2} + ||V(0)||^2_{H^2}\right). \quad (90)\end{aligned}$$

VII. Proof of Theorem 3.2

Consider expressions (7)–(17).

Points i and iv are deduced trivially from the fact that (7) and (10) are defined as convolutions, and properties of the heat equation (9).

Points ii and iii were proved in the Appendix of [11].

From the definition of the inverse Fourier transform (23), it is straightforward to show that if the real part of $f(k,y)$ is even and the imaginary part of $f(k,y)$ is odd, then the resulting $f(x,y)$ will always be real. Then, Point v can be proved showing that the functions under the integrals in (11)–(13) have this property. This is immediate for (12) and (13). For (11), the property must be shown for the kernel K, defined by the sequence (16)–(17). This can be proved by induction. For K_0, the property is evident. For K_n, if the property is assumed for K_{n-1}, then from (17) and since even times even or odd times odd is even, and the product of odd times even is odd, K_n has the property. Hence the limit K has a real inverse transform, and kernel Q_u is real.

Point vi is deduced from the definition of the kernels (11)–(13) as Fourier inverse integrals.

For Point vii, consider expression (7) and (11). Then,

$$\begin{aligned} ||U_c||^2_{L^2} &= \int_{-\infty}^{\infty} U_c(t,x)^2 dx \\ &= \int_{-\infty}^{\infty} \chi(k)\left|\int_0^1 K(k,1,\eta)u(t,y,k)d\eta\right|^2 dk \\ &\leq 2(M-m)\max_{m\leq|k|\leq M}\{||K||_\infty\}||u(t)||^2_{L^2}, \quad (91)\end{aligned}$$

and the result follows from Theorem 3.1.

On the other hand, for V_c one has to use its dynamic equation (9)–(10), and a Lyapunov functional consisting in half its L^2 norm. One then has, using Young's inequality

$$\begin{aligned} \frac{d}{dt}\frac{|V_c(k)|^2}{2} \leq\ & \frac{-\pi^2 k^2}{Re}|V_c(k)|^2 \\ &+\frac{|u_y|^2(t,k,0)+|u_y|^2(t,k,1)}{Re} \\ &+64\cosh(2\pi M)\,||V(t,k)||^2_{\hat{L}^2}, \quad (92)\end{aligned}$$

and supposing the control law is initialized at zero , and using the $\hat{H}^2$ norm to bound the second line of (92),

$$\begin{aligned} |V_c(t,k)|^2 \leq\ & \int_0^t \mathrm{e}^{-\frac{\pi^2 m^2}{Re}(t-\tau)}\left[10\frac{||u(\tau,k)||^2_{\hat{H}^2}}{Re}\right. \\ &\left.+64\cosh(2\pi M)\,||V(\tau,k)||^2_{\hat{L}^2}\right]d\tau. \quad (93)\end{aligned}$$

Integrating in k

$$\begin{aligned} ||V_c(t)||^2_{L^2} \leq\ & \int_0^t \mathrm{e}^{-\frac{\pi^2 m^2}{Re}(t-\tau)}\left[10\frac{||u(\tau)||^2_{H^2}}{Re}\right. \\ &\left.+64\cosh(2\pi M)\,||V(\tau)||^2_{L^2}\right]d\tau, \quad (94)\end{aligned}$$

and then the result follows from Theorem 3.1.

For Point viii, consider the jth spatial derivative of U_c and calculate its L_2 spatial norm

$$\begin{aligned} \left|\left|\frac{d^j}{dx^j}U_c\right|\right|^2_{L^2} &= \int_{-\infty}^{\infty}\left(\frac{d^j}{dx^j}U_c(t,x)\right)^2 dx \\ &\leq (2\pi M)^{2j}||U_c||^2_{L^2}, \quad (95)\end{aligned}$$

so the result for U_c follows from Point vii. We proceed similarly for V_c, thus proving Point viii.

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