Spacecraft Dynamics

Lesson 8: Active Attitude Control

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Active control systems

- Passive control systems can allow for some perturbation rejection and give stability enough for some applications.
- However, particularly at the beginning of a mission, all spacecraft need to perform:
  - Slew maneuvers
  - Adjustments of spin speed
  - Stationkeeping maneuvers
- Thus, in many cases, one needs an active control systems (active in the sense of requiring additional energy to work as well as some kind of logic).
- In missions requiring high accuracies, that active control system will be the primary system. Then, one speaks about three-axis stabilized attitude control.
- In other cases, it may be a secondary system, which only requires occasional use.

Actuators

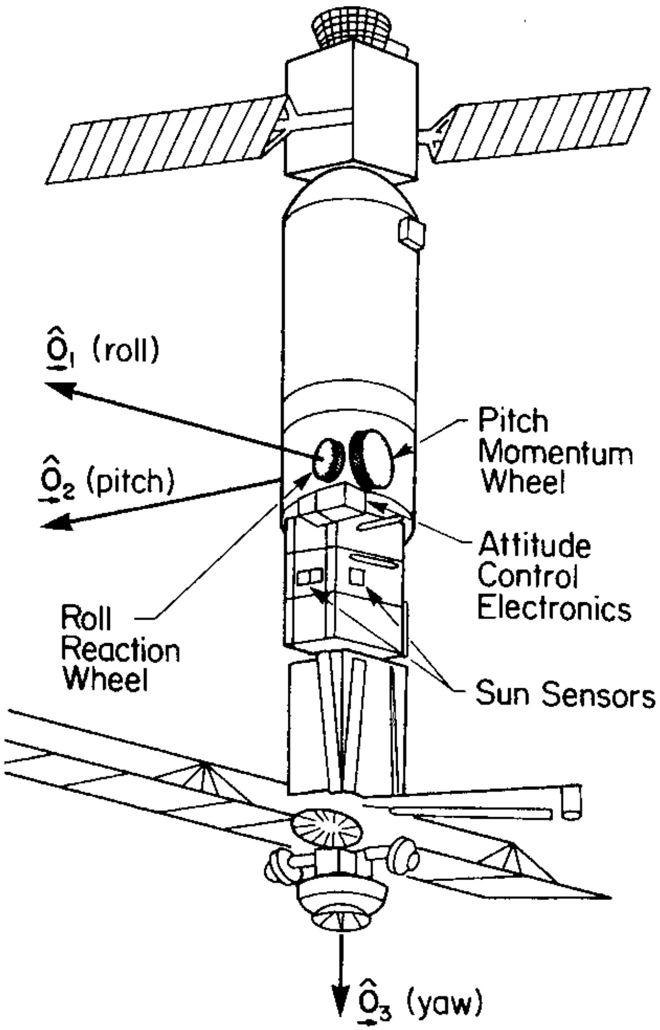
- Before explaining the algorithms for attitude control, it is important to quickly review the actuators that are used to modify the attitude of a spacecraft (through some term in Euler's equations). The different types of actuators are:
  - Thrusters: based on expelling mass. Since mass is finite these devices have limited use. Known as Reaction Control Systems.
  - Reaction wheels and inertia wheels, with changing angular speeds, as seen in Lesson 5.
  - Control Moment Gyroscopes (CMG): they are as inertia wheels (a disc-like device spinning at large speeds), which, instead of modifying their angular speeds, tilt their axis of rotation through motorized gimbals, thus quickly modifying their angular momentum.
  - Magnetorquers, which use the magnetic field to produce a torque.
  - Structural elements for passive control: booms, yo-yo devices, nutation dampers... not covered here.
- It is normal to have several kind of actuators in a spacecraft for redundancy and given that they have different properties.

Three-axis stabilized attitude control

- Satellites with three-axis stabilized attitude control can have any kind of pointing (inertial, orbital, some ground target...)
- Objectives may be two: either to keep the satellite (in the presence of perturbations) in a fixed attitude (a simple regulation/stabilization problem) or to perform a slew maneuver (which maybe to track a target or just modifying the attitude).
- There are two main families of actuators to achieve these goals: reaction/inertia wheels /CMGs (also known as momentum exchange systems) and RCS. Magnetorquers can also partially perform this but it is a bit more difficult due to a direction without actuation: we will not consider them.
- We will start with the first goal, since the second is more difficult, for both reaction/inertia wheels and RCS.
- How to perform slew maneuvers will also be consider but only for reaction/inertia wheels.

Momentum exchange systems

- For the highest degree of precision in attitude, manoeuvrability and stabilization, and for any orientation independent of the inertia tensor, one can use momentum exchange systems which use reaction wheels, inertia wheels and/or CMGs, based on conservation of angular momentum.
- Nevertheless these are expensive system, with low tolerance to failures, and require an auxiliary system (thruster or magnetorquers) to unload momentum and thus avoid saturation.



Spacraft with three reaction wheels

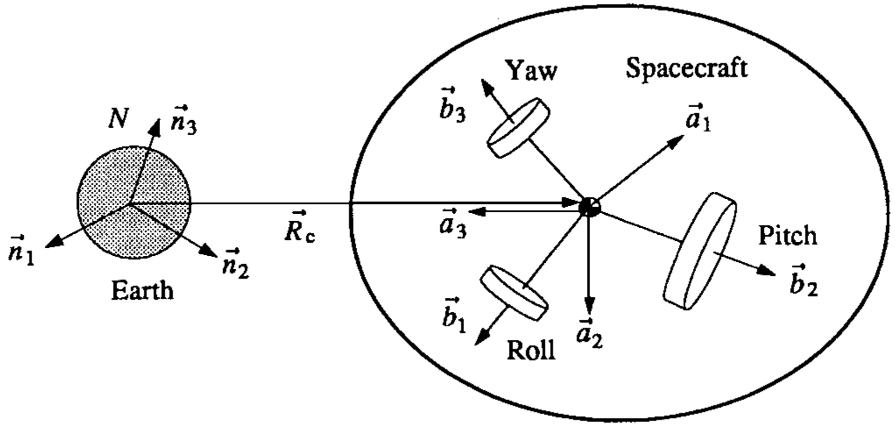


Fig. 6.10 Gyrostat in a circular orbit.

- Assume the situation in the figure, with three reaction wheels in the three principal axes:

$$\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + \dot{h}_1 + \omega_2 h_3 - \omega_3 h_2 &= M_1 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 + \dot{h}_2 + \omega_3 h_1 - \omega_1 h_3 &= M_2 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 + \dot{h}_3 - \omega_2 h_1 + \omega_1 h_2 &= M_3
\end{aligned}$$

- The angular momentum of wheels is denoted as $h_i = \omega_{R_i} I_{R_i}$. These are control variables!

Spacraft with three reaction wheels

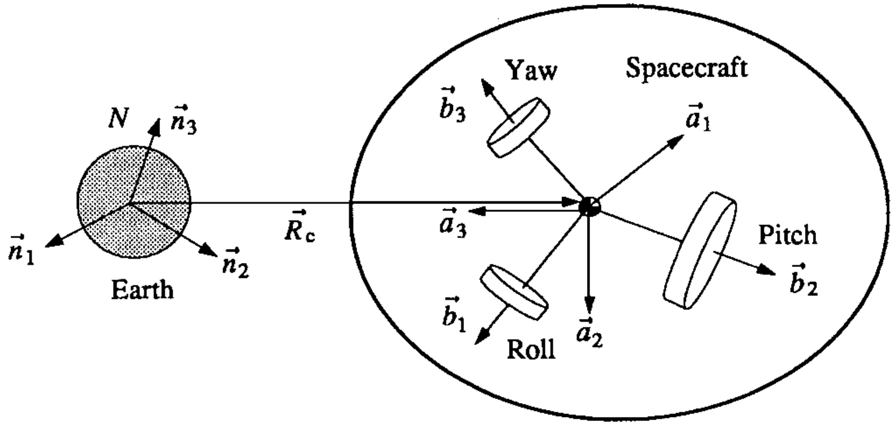


Fig. 6.10 **Gyrostat in a circular orbit.**

- Remember also from Lesson 5 that once we know the speed we need for the wheels, it can be achieved by using the wheels' internal electrical motors.
- The model from Lesson 5 was:

$$\begin{aligned} I_{R1}\dot{\omega}_1 + \dot{h}_1 &= J_1 \\ I_{R2}\dot{\omega}_2 + \dot{h}_2 &= J_3 \\ I_{R3}\dot{\omega}_3 + \dot{h}_3 &= J_3 \end{aligned}$$

where J_i is the torque of the electrical motors. This is in the end what we can really actuate directly.

Spacecraft with three reaction wheels

- Let us now use a nomenclature in which we denote the effect of the wheels with the letter u by following the classical control nomenclature:

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= u_1 + M_1 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= u_2 + M_2 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= u_3 + M_3
\end{aligned}
$$

where

$$
\begin{aligned}
u_1 &= -\dot{h}_1 - \omega_2 h_3 + \omega_3 h_2 \\
u_2 &= -\dot{h}_2 - \omega_3 h_1 + \omega_1 h_3 \\
u_3 &= -\dot{h}_3 - \omega_1 h_2 + \omega_2 h_1
\end{aligned}
$$

This is, $\vec{u} = -\dot{\vec{h}} + \vec{h}^{\times}\vec{\omega}$

- In addition we have the kinematic differential equation

$$\dot{q} = \frac{1}{2} q \star q_{\vec{\omega}}$$

Regulation: Stabilizing a given attitude

- For regulation of a fixed attitude, the problem is stabilizing the values $q(t) = q_{ref}$ and $\omega(t) = 0$. In addition, we assume that we initially start close to that value of the state.
- Thus, we linearize Euler's equations around $\omega(t) = 0$. Ignoring perturbing torques (Question: what could we try to do to mitigate perturbing torques?):

$$\frac{d}{dt}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\end{bmatrix} + \begin{bmatrix}1/I_1&0&0\\0&1/I_2&0\\0&0&1/I_3\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

where $\vec{u} = -\dot{\vec{h}} + \vec{h}^{\times}\delta\vec{\omega}$

- Notice that if we find $\vec{u}$ solving the control problem, we could find the corresponding values of $\vec{h}$ by solving the differential equation (however: physical limitations, such as saturations or rate limits could pose a problem).

Stabilization

- On the other hand, the attitude quaternion should be close to the reference attitude (if we start close to the attitude q_{ref}).
- By following Lesson 2, then we can write $q = q_{ref} \star \delta q$, where q_{ref} is the desired attitude and δq the attitude quaternion:

$$\delta q(\vec{a}) = \frac{1}{\sqrt{4 + \|\vec{a}\|^2}} \begin{bmatrix} 2 \\ \vec{a} \end{bmatrix}$$

- From Lesson 4 the relationship between $\vec{a}$ and the angular velocity is $\dot{\vec{a}} \approx \delta\vec{\omega} + \vec{a} \times \vec{\omega}_{ref}$, since $\vec{\omega}_{ref} = \vec{0} \rightarrow \dot{\vec{a}} \approx \delta\vec{\omega}$.
- Thus:

$$\frac{d}{dt} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\omega_1 \\ \delta\omega_2 \\ \delta\omega_3 \end{bmatrix}$$

- Combining the equations for the error in angular velocity and attitude we find a full description of the error of the system, in the next slide.

Stabilization

- System description:

$$\frac{d}{dt}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\\ a_1\\ a_2\\ a_3\end{bmatrix} = \begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{bmatrix}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\\ a_1\\ a_2\\ a_3\end{bmatrix} + \begin{bmatrix}1/I_1&0&0\\0&1/I_2&0\\0&0&1/I_3\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

- Call $\vec{x}$ to the variables describing the state, this is a classical way to write a linear system

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

- We can use "our favorite linear method" to find a (linear) control law $\vec{u} = K\vec{x}$, which then later one needs to transform in required velocities for the wheels by solving the angular speed that relates $\vec{u}$ with the angular momentum of the wheels, and then later transform that into commands for the wheels' motors.
- A possible method is LQR (linear quadratic regulator) with "infinite horizon". Another is pole placement.

The LQR method

- Given
$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$
find a control law $\vec{u}(t)$ (with feedback: $\vec{u} = K\vec{x}$) minimizing:
$$J = \int_0^\infty (\vec{x}^T(t)Q\vec{x}(t) + \vec{u}^T(t)R\vec{u}(t))dt$$
- Problem posed and solved first by Rudolph Kalman!
- Assumptions: Q, R symmetrical and $Q > 0$ (definite semidefinite positive, which is equivalent to all eigenvalues positive) and $R \geq 0$ (semidefinite positive, which is equivalent to all eigenvalues non-negative).
- Additional assumption: The system is "controlable". Meaning that "is is possible to solve the problem" (it is easy to solve control problems that cannot be solved. For instance $\dot{x}_1 = u_1, \ \dot{x}_2 = x_2$.) Mathematically a problem is controllable if $\mathcal{C} = [B \ AB \ A^2B \ A^{n-1}B]$ is full row rank, where n is the number of states. Is this verified in our case?

The LQR method

- The control law that solves the problem is
$$\vec{u} = K\vec{x}$$
where the gain K is found as follows
  1. Find the matrix P that solve the so-called "algebraic Riccati equation":
  $$Q + A^T P + PA - PBR^{-1}B^T P = 0$$
  for instance with the Matlab command "are" (which requires the Control Systems Toolbox) `P=are(A,B*inv(R)*B',Q);`
  2. The gain is then $K = -R^{-1}B^T P$
- The Riccati equation is solvable only if the system is controllable.
- Optimal control should guarantee a good behavior of the system, but does not take into account the actuator's saturation or other nonlinear behavior. The choice of Q and R greatly influences the quality of the controller (more conservative or more aggresive).

The LQR method

- To implement a control law

$$\vec{u} = K\vec{x}$$

let us first remember the definition of $\vec{x}$.
- As $\vec{\omega}_{ref} = \vec{0}$, the first three components are the real value of angular speed.
- The next three components are $\vec{a}$, from which one extracts the quaternion error. It is easy to see that

$$\vec{a} = 2\frac{\delta\vec{q}}{\delta q_0}$$

which comes from $\delta q = q^*_{ref} \star q(t)$.
- Once the control $\vec{u}$ is computed, one needs to solve $\dot{\vec{h}} = -\vec{u} + \vec{h}^{\times}\delta\vec{\omega}$ to find out how to solve the angular momentum of the wheels.

Slew maneuvers and tracking

- We have studied in Lessons 2 and 4 how to compute a given angular velocity to maneuver from a given attitude to another.
- Remember that, given q_i and q_f and a certain time T it was required to find $q_R = q_i^* \star q_f$, extract Euler's axis $\vec{e}$ and angle θ, and then $\vec{\omega} = \vec{e}\omega(t)$, where ω needs to verify $\int_0^T \omega(\tau) d\tau$.
- In addition, we can impose additional conditions such as starting and finishing at rest, for instance by imposing a shape to $\omega(t)$ of the form $\omega(t) = At(t - T)$ (Exercise: find A). Other conditions could be imposed.
- Once we find the required angular velocity, if we substitute it in Euler's equation we can find the control. This is sometimes called "open loop control" or feedforward control, and does not use feedback.

Slew maneuvers and tracking

- If we call the found angular velocity $\vec{\omega}_{ref}(t)$, and the quaternion generated by that angular speed the reference quaternion $q_{ref}(t)$, we can also find a "reference control" (feedforward control) $\vec{u}_{ref}$ as:

$$
\begin{aligned}
u_{ref1} &= I_1\dot{\omega}_{ref1} + (I_3 - I_2)\omega_{ref2}\omega_{ref3} \\
u_{ref2} &= I_2\dot{\omega}_{ref2} + (I_1 - I_3)\omega_{ref3}\omega_{ref1} \\
u_{ref3} &= I_3\dot{\omega}_{ref3} + (I_2 - I_1)\omega_{ref1}\omega_{ref2}
\end{aligned}
$$

- As before from this $\vec{u}_{ref}$ we can find the required speed of the wheels and from that speed of the wheels, the internal electrical motors' torque that would be needed to perform the maneuver.
- What would happen if we try just to apply this feedforward control without any feedback mechanism?
- The problem of following the reference profile $\vec{\omega}_{ref}(t)$,$q_{ref}(t)$ is sometimes called the tracking problem.

Tracking

- One possible idea to solve tracking is as follows: linearize around the reference profile. Compute an *additional* feedback controller around the reference profile that is added to the feedforward control (so we have feedforward+feedback) so we close the loop and guarantee stability (at least with respect to small errors and perturbations) so that the system is kept on the desired reference profile.
- Thus let $\delta\vec{\omega} = \vec{\omega} - \vec{\omega}_{ref}$, $\delta\vec{u} = \vec{u} - \vec{u}_{ref}$, and use the quaternion error as previously defined. The linearized equations are:

$$\begin{aligned}
I_1\delta\dot{\omega}_1 + (I_3 - I_2)(\omega_{ref2}\delta\omega_3 + \delta\omega_2\omega_{ref3}) &= \delta u_1 + M_1 \\
I_2\delta\dot{\omega}_2 + (I_1 - I_3)(\omega_{ref3}\delta\omega_1 + \delta\omega_3\omega_{ref1}) &= \delta u_2 + M_2 \\
I_3\delta\dot{\omega}_3 + (I_2 - I_1)(\omega_{ref1}\delta\omega_2 + \delta\omega_1\omega_{ref2}) &= \delta u_3 + M_3
\end{aligned}$$

and for the attitude error:

$$\dot{\vec{a}} \approx \delta\vec{\omega} - \vec{\omega}_{ref}^{\times}\vec{a}$$

Tracking

- System description ignoring perturbing torques:

$$\frac{d}{dt}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\\ a_1\\ a_2\\ a_3\end{bmatrix} = \begin{bmatrix} 0 & \frac{I_2-I_3}{I_1}\omega_{ref3} & \frac{I_2-I_3}{I_1}\omega_{ref2} & 0 & 0 & 0\\ \frac{I_3-I_1}{I_2}\omega_{ref3} & 0 & \frac{I_3-I_1}{I_2}\omega_{ref1} & 0 & 0 & 0\\ \frac{I_1-I_2}{I_3}\omega_{ref2} & \frac{I_1-I_2}{I_3}\omega_{ref1} & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & \omega_{ref3} & -\omega_{ref2}\\ 0 & 1 & 0 & -\omega_{ref3} & 0 & \omega_{ref1}\\ 0 & 0 & 1 & \omega_{ref2} & -\omega_{ref1} & 0 \end{bmatrix}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\\ a_1\\ a_2\\ a_3\end{bmatrix}$$

$$+\begin{bmatrix} 1/I_1 & 0 & 0\\ 0 & 1/I_2 & 0\\ 0 & 0 & 1/I_3\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\delta u_1\\ \delta u_2\\ \delta u_3\end{bmatrix}$$

- Classical description as before

$$\dot{\vec{x}} = A(t)\vec{x} + B(t)\delta\vec{u}$$

- Now A and B are time-varying: cannot use the LQR method as before.
- We need more advanced methods, such as LQR (linear quadratic regulator) with “finite horizon”.

Tracking with finite horizon LQR

- Given

$$\dot{\vec{x}} = A(t)\vec{x} + B(t)\delta\vec{u}$$

  find $\delta\vec{u}(t)$ with feedback $(\delta\vec{u}(t) = K(t)\vec{x})$ minimizing

$$J = \int_0^T (\vec{x}^T(t)Q(t)\vec{x}(t) + \delta\vec{u}^T(t)R(t)\delta\vec{u}(t))dt + \vec{x}^T(T)Q_{end}\vec{x}(T)$$

- Assumptions: Q, R, Q_{end} symmetric and $Q_{end}, Q > 0, R \geq 0$.
- Since it is a finite horizon controller, the controllability hypothesis is not required, but there could be problems if there is a loss of controllability of the system.

Tracking with finite horizon LQR

- The control law that minimizes J is as follows:

$$\delta\vec{u} = K(t)\vec{x}$$

  where the gain $K(t)$ is found as follows:

  1. Find $P(t)$ that solved the so-called "Riccati differential equation":

  $$-\dot{P} = A^T P + PA - PBR^{-1}B^T P + Q, \quad P(T) = Q_{end}$$

     for instance using ode45 in Matlab.
  2. The gain is then $K(t) = -R^{-1}B^T P(t)$
- Riccati's differential equation is always solvable! However, it cannot be solved in real time, because it needs to be solved backwards in time (there is a final condition instead of an initial condition). Thus one solves it in advance and stores the values of $K(t)$.
- As before: Choices of Q and R (also Q_{end}) determines the quality of the controller (more conservative or more aggresive).

Tracking with finite horizon LQR

- To implement the control law

$$\delta\vec{u} = K(t)\vec{x}$$

  one needs to remember the definition of $\vec{x}$.
- As $\vec{\omega}_{ref} \neq \vec{0}$, the first three components are $\vec{\omega} - \vec{\omega}_{ref}$.
- The second three components correspond to $\vec{a}$, that need to be extracted from the quaternion error. Remember that

$$\vec{a} = 2\frac{\delta\vec{q}}{\delta q_0}$$

  for which we need to compute $\delta q = q^*_{ref} \star q(t)$.
- The final control is $\vec{u} = \vec{u}_{ref} + \delta\vec{u}$.
- Remember that once $\vec{u}$ is known, at each instant is required to solve $\dot{\vec{h}} = -\vec{u} + \vec{h}^{\times}\delta\vec{\omega}$ to know how to modify the angular momentum of the wheels and therefore their internal torque J_i.

Nonlinear control

- "Nonlinear control" comprises a wide range of techniques that do not require the use of linearization.
- Consider the following problem. Starting from $\vec{\omega}(0)$ and $q(0)$ we want to reach the identity attitude at rest. It is enough for us if the system "tends" to that state, this is, our goal is that $\vec{\omega}(t) \to \vec{0}$ y $q_0(t) \to 1$, $\vec{q}(t) \to \vec{0}$ when $t \to \infty$.
- This is, we make "asymptotically stable" the equilibrium $\vec{\omega} = \vec{0}$, $q_0 = 1$, $\vec{q} = \vec{0}$.
- If this is true, for any initial condition, then one says that the equilibrium is globally asymptotically stable.
- Notice that the target attitude could be any, just by making a rotation of the inertial frame as $q' = q_{ref}^* \star q$.
- We solve this problem with the so-called "Lyapunov function technique".

Nonlinear control

- Let us start by remembering than since we don't linearize, now our system is the original one, writing as before the control terms in the equations.
- First, the angular velocity equations:

$$
\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{u_1}{I_1} \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \frac{u_2}{I_2} \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{u_3}{I_3}
\end{aligned}
$$

Nonlinear control: Lyapunov functions

- Can we find u_1, u_2 and u_3 such that the equilibirum $\vec{\omega} = \vec{0}$ is globally asymptotically stable?
- The technique of Lyapunov functions is as follows. Let V be a regular function (continuous, differentiable) that depends on the state (in this case, the angular velocity and quaternions) such that :
  - It is always positive for any value of the states, except when the state is zero; and for zero, it is zero (this is, positive definite).
  - The time derivative of V is definite negative (this is, negative for any value of the state except zero).
- Then it follows that the origin (zero value of the state) is asymptotically stable (this method can be understood by looking at the level curves of V).
- If in addition the limit of V when the state goes to infinity also tends to infinity, the result is global.

Nonlinear control: Lyapunov functions

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  - The time derivative of V is definite negative (this is, negative for any value of the state except zero).
- Then it follows that the origin (zero value of the state) is asymptotically stable (this method can be understood by looking at the level curves of V).
- If in addition the limit of V when the state goes to infinity also tends to infinity, the result is global.

Nonlinear control: Lyapunov functions

- Let us see how this works out for our first case with only angular velocity. Consider:

$$V = I_1 \frac{\omega_1^2}{2k} + I_2 \frac{\omega_2^2}{2k} + I_3 \frac{\omega_3^2}{2k}$$

- We see that the first conditions is fulfilled if k is a positive constant (we will define it later).
- Taking derivative:

$$V_t = I_1 \frac{\omega_1 \dot{\omega}_1}{k} + I_2 \frac{\omega_2 \dot{\omega}_2}{k} + I_3 \frac{\omega_3 \dot{\omega}_3}{k}$$

- Substituting the derivatives:

$$V_t = \frac{\omega_1((I_2 - I_3)\omega_2\omega_3 + u_1)}{k} + \frac{\omega_2((I_3 - I_1)\omega_3\omega_1 + u_2)}{k} + \frac{\omega_3((I_1 - I_2)\omega_1\omega_2 + u_3)}{k}$$

Nonlinear control: finding the control

- Simplifying

$$V_t = \frac{\omega_1 u_1}{k} + \frac{\omega_2 u_2}{k} + \frac{\omega_3 u_3}{k}$$

- Let us choose now: $u_1 = -c_1\omega_1$, $u_2 = -c_2\omega_2$, $u_3 = -c_3\omega_3$, where c_i is a positive constant. Replacing this in V_t:

$$V_t = -\frac{c_1\omega_1^2 + c_2\omega_2^2 + c_3\omega_3^2}{k}$$

- Thus by the technique of Lyapunov, it is proven that $\vec{\omega} = 0$ is globally asymptotically stable. Note that the value of C_i and k does not matter as long as they are positive, but the value of C_i will influence the performance of the control law.

Nonlinear control: including quaternions

- Let us consider now the full system including the quaternions

$$
\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{u_1}{I_1} \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \frac{u_2}{I_2} \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{u_3}{I_3} \\
\dot{q}_0 &= -\frac{1}{2}\left(q_1\omega_1 + q_2\omega_2 + q_3\omega_3\right) \\
\dot{q}_1 &= \frac{1}{2}\left(q_0\omega_1 - q_3\omega_2 + q_2\omega_3\right) \\
\dot{q}_2 &= \frac{1}{2}\left(q_3\omega_1 + q_0\omega_2 - q_1\omega_3\right) \\
\dot{q}_3 &= \frac{1}{2}\left(-q_2\omega_1 + q_1\omega_2 + q_0\omega_3\right)
\end{aligned}
$$

Nonlinear control: La Salle's Theorem

- Can we find values of u_1, u_2 and u_3 guaranteeing that the equilibrium $\vec{\omega} = \vec{q} = \vec{0}$, $q_0 = 1$ is asymptotically stable?
- Unfortunately Lyapunov is not enough!
- We also need "La Salle's Theorem":
  - Let V be a Lyapunov function such that its derivative is **semidefinite negative** (this is negative or zero). Let us call E the set of states verifying $\dot{V} = 0$.
  - Let M be the largest **invariant set** of the system contained in E.
- Then the state goes to M when time goes to infinity.
- What is the invariant set of a system? Is a set such that if the initial condition starts in the set, the state stays in the set for all t.

Nonlinear control: finding the control (again)

- Use the Lyapunov function

$$V = I_1\frac{\omega_1^2}{2k} + I_2\frac{\omega_2^2}{2k} + I_3\frac{\omega_3^2}{2k} + (q_0-1)^2 + q_1^2 + q_2^2 + q_3^2$$

- We see that the first condition of being a Lyapunov function is verified (q_0 has been displaced so that $q_0 = 1$ is at the origin).
- Taking a derivative:

$$V_t = I_1\frac{\omega_1\dot{\omega}_1}{k} + I_2\frac{\omega_2\dot{\omega}_2}{k} + I_3\frac{\omega_3\dot{\omega}_3}{k} + 2(q_0-1)\dot{q}_0 + 2q_1\dot{q}_1 + 2q_2\dot{q}_2 + 2q_3\dot{q}_3$$

- Substituting:

$$\begin{aligned} V_t = &\frac{\omega_1((I_2-I_3)\omega_2\omega_3+u_1)}{k} + \frac{\omega_2((I_3-I_1)\omega_3\omega_1+u_2)}{k} + \frac{\omega_3((I_1-I_2)\omega_1\omega_2+u_3)}{k} \\ &-(q_0-1)\left(q_1\omega_1+q_2\omega_2+q_3\omega_3\right) + q_1\left(q_0\omega_1-q_3\omega_2+q_2\omega_3\right) \\ &+q_2\left(q_3\omega_1+q_0\omega_2-q_1\omega_3\right) + q_3\left(-q_2\omega_1+q_1\omega_2+q_0\omega_3\right) \end{aligned}$$

Nonlinear control: finding the control (again)

- Simplifying

$$V_t = \frac{\omega_1 u_1}{k} + \frac{\omega_2 u_2}{k} + \frac{\omega_3 u_3}{k} + (q_1\omega_1 + q_2\omega_2 + q_3\omega_3)$$

- Let us choose now: $u_1 = -(kq_1 + c_1\omega_1)$, $u_2 = -(kq_2 + c_2\omega_2)$, $u_3 = -(kq_3 + c_3\omega_3)$, where c_i is a positive constant.
Substituting:

$$\begin{aligned} V_t &= -\frac{\omega_1(kq_1 + c_1\omega_1)}{k} - \frac{\omega_2(kq_2 + c_2\omega_2)}{k} - \frac{\omega_3(kq_3 + c_3\omega_3)}{k} \\ &\quad + (q_1\omega_1 + q_2\omega_2 + q_3\omega_3) \\ &= -\frac{c_1\omega_1^2 + c_2\omega_2^2 + c_3\omega_3^2}{k} \end{aligned}$$

- We cannot apply Lyapunov directly, we need La Salle!
- First of all, the set E is just $\omega_1 = \omega_2 = \omega_3 = 0$ for all t.

Finding the invariant set M

- Replace $\omega_1 = \omega_2 = \omega_3 = 0$ in the syste for all t (in particular this implies that the derivatives are zero):

$$
\begin{aligned}
0 &= 0 + u_1 \\
0 &= 0 + u_2 \\
0 &= 0 + u_3 \\
\dot{q}_0 &= 0 \\
\dot{q}_1 &= 0 \\
\dot{q}_2 &= 0 \\
\dot{q}_3 &= 0
\end{aligned}
$$

- Thus the invariant set verifies $u_1 = u_2 = u_3 = 0$, and q constant.
- Since $u_1 = -(kq_1 + c_1\omega_1)$, $u_2 = -(kq_2 + c_2\omega_2)$, $u_3 = -(kq_3 + c_3\omega_3)$, we obtain $q_1 = q_2 = q_3 = 0$.

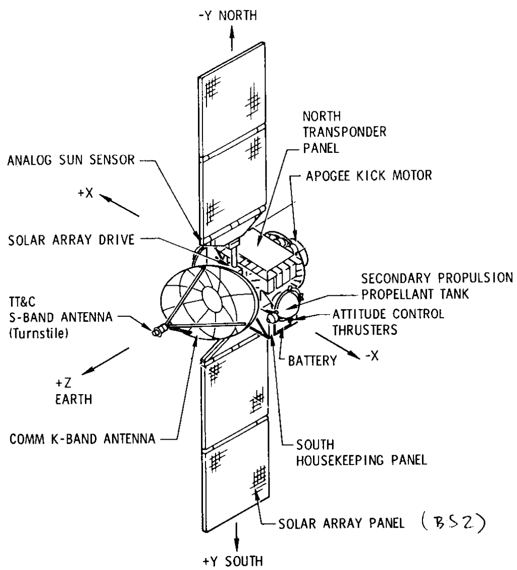
Final stability result. Winding phenomenon.

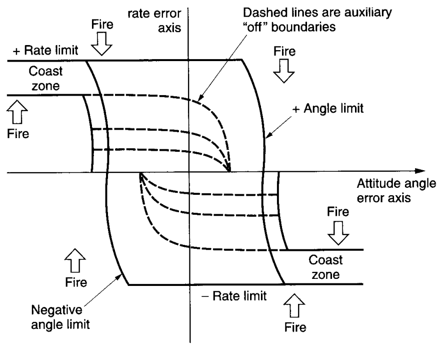
- Finally, since the quaternion must be unity, we get $q_0 = \pm 1$. Since $q_0 = 1$ is the origin of the Lyapunov function, it becomes stable (in fact $q_0 = -1$ becomes unstable; which is a problem since it is the same point, this is called the winding phenomenon).
- If one uses negative k in the control law then it can be similarly shown that $q_0 = -1$ becomes stable and $q_0 = 1$ unstable. This can be verified by switching the Lyapunov function to

$$V = -I_1\frac{\omega_1^2}{2k} - I_2\frac{\omega_2^2}{2k} - I_3\frac{\omega_3^2}{2k} + (q_0+1)^2 + q_1^2 + q_2^2 + q_3^2$$

- If one fixes $k = k_0 \cdot \mathrm{sgn}(q_0)$ then one stabilizes the "closest" equilibirum.
- Very interestingly: in the control law there are no inertias in the formulas, thus we don't need knowledge of them. This is an universal control law. However one needs to know the state ($\vec{\omega}$ and q) to be able to apply the control law.

Reaction Control Systems (RCS)





- In situations that require high/fast manoeuvrability one can use a Reaction Control Systems or RCS, using a set of thruster distributed on the vehicle to quickly and efficiently modify attitude.
- The so-called "propulsion logic" establishes when thrusters are fired and if a small tolerance of attitude/angular velocity can be accepted.
- Normally it is a combination of "dead zones" (no actuations) and hysteresis (to avoid the repetitive firing of thrusters exhausting all fuel).
- Thrusters usually are actuators "all or nothing", thus always acting in saturation.
- This means that RCS are intrinsically nonlinear, but discontinuous as well.

Reaction Control Systems

- For RCS, we can model the effect of the thrusters as torques in Euler's Equation.
- We are only going to consider the regulation problem (stabilization of an attitude to which we are already close). Linearizing and taking Euler angles in the sequence 1-2-3 with small angles, and combining the linearized kinematic and dynamics, the system to be controlled becomes:

$$\begin{aligned} I_1 \ddot{\theta}_1 &\approx u_1 \,, \\ I_2 \ddot{\theta}_2 &\approx u_2 \,, \\ I_3 \ddot{\theta}_3 &\approx u_3 \,, \end{aligned}$$

- Next we design u_1, u_2 and u_3 to stabilize the system; each axis is independent of one another. Classical methods of control (or Lyapunov) cannot be used for thrusters since they cannot give a variable value (a control law such as $u = Kx$ does not work). This is the only options are $u = 0, u_{MAX}, u_{MIN}$, where u_{MIN} should be negative (we can assume to simplify $u_{MIN} = -u_{MAX}$). We will use more explicit/geometrical ideas.

Control with thrusters

- Consider a single axis, then $\ddot{\alpha} = u$ (where u is redefined by dividing by inertia), with initial conditions $\dot{\alpha}_0$ and α_0. Integrating the differential equation:

$$\dot{\alpha} - \dot{\alpha}_0 = tu, \quad \alpha - \alpha_0 - t\dot{\alpha}_0 = \frac{t^2}{2}u$$

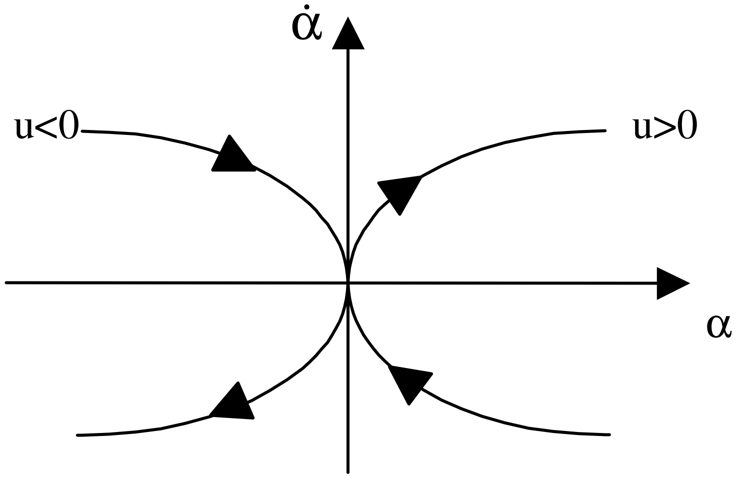
- If one removes time from the system:

$$\alpha - \alpha_0 = \frac{\dot{\alpha}_0(\dot{\alpha} - \dot{\alpha}_0)}{u} + \frac{(\dot{\alpha} - \dot{\alpha}_0)^2}{2u}$$

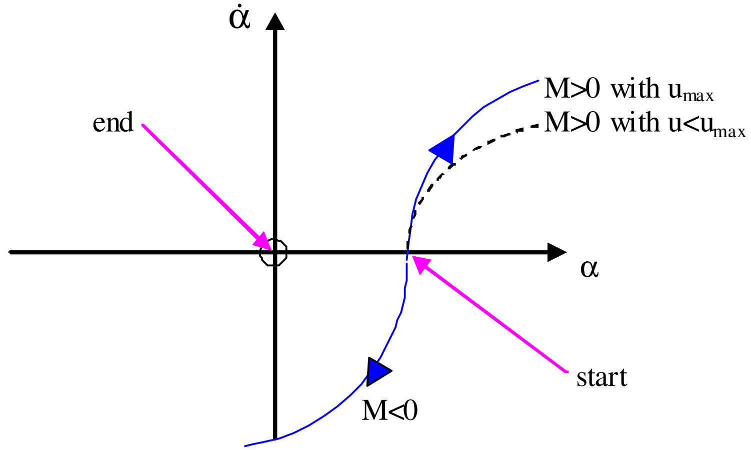
- This is the equation of a parabola in the phase plane (θ-$\dot{\theta}$), whose shape will depend from initial conditions and the choices of control ($u = 0, u_{MAX}, -u_{MAX}$). If $u = 0$ time cannot be removed and the system's behavior is reduced to moving along the segment $\alpha - \alpha_0 = t\dot{\alpha}_0$.

Control with thrusters

- Example of parabolas with zero initial condition (arrows indicate how the system behaves):

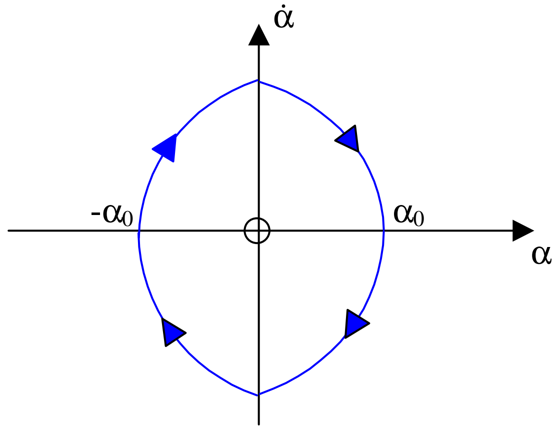


- To move we need to use the parabolas:

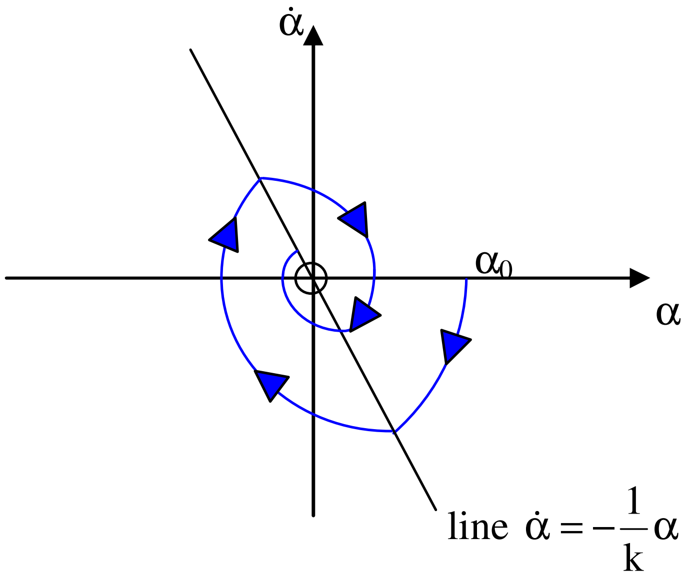


Control with thrusters

- First idea: $u = -u_{MAX}\mathrm{sign}(\alpha)$. The result is a limit cycle:

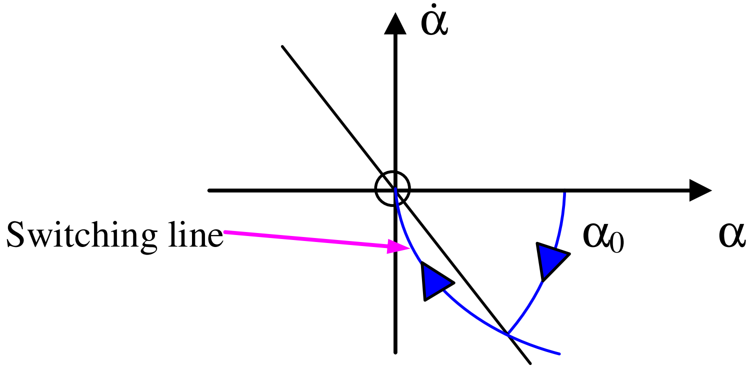


- To avoid oscillation: $u = -u_{MAX}\mathrm{sign}(\alpha + k\dot{\alpha})$, with $k > 0$. The result:

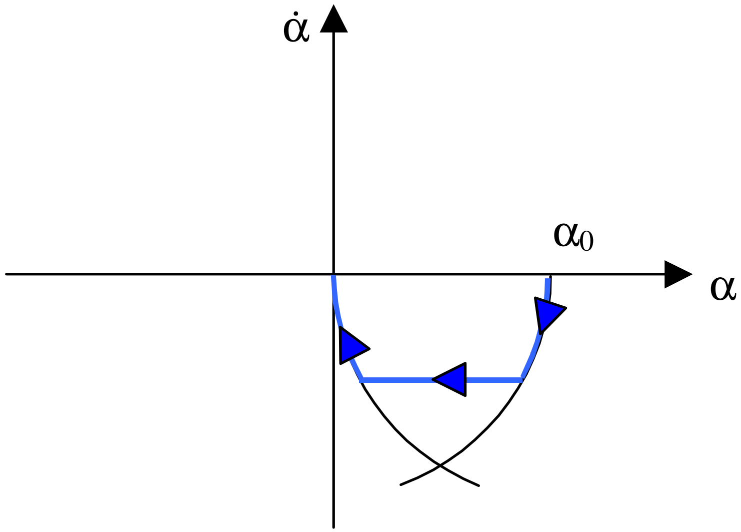


Control with thrusters

- To arrive in a finite time: $u = -u_{MAX}\mathrm{sign}(\alpha - \frac{1}{2u_{MAX}}\dot{\alpha}|\dot{\alpha}|)$ (exercise). The result:



- If one fixes a minimum time and wants to minimize fuel (exercise):



Control with thrusters: additional considerations

- The procedure just explained cannot be applied if one cannot neglect nonlinearities (gyroscopic couplings make necessary the use of all the axis simultaneously). Then one needs to use the full theory of optimal control.
- In practice it is enough that the solutions converge close enough to the origin (to avoid switching on the thurster too often). This requires the use of dead zones and hysteresis.