Spacecraft Dynamics

Lesson 5: Attitude Dynamics

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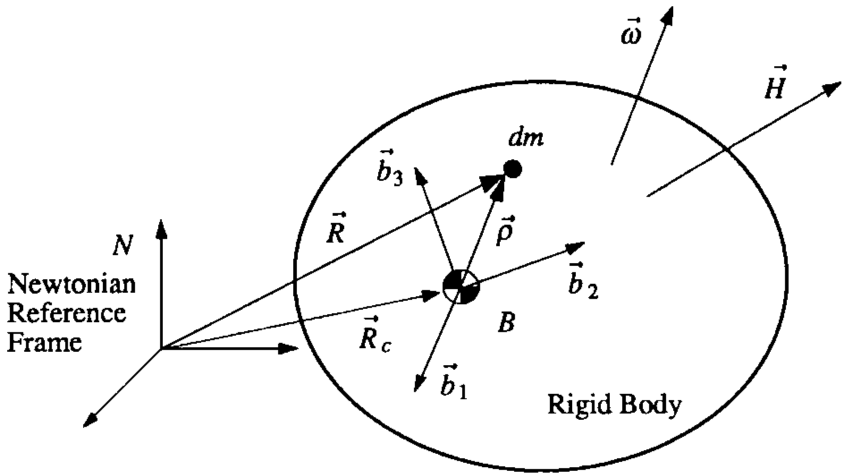
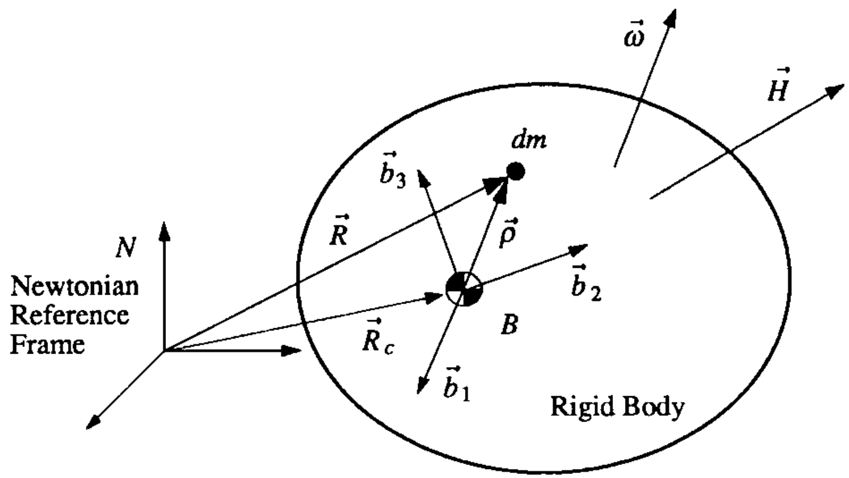
Spacecraft attitude dynamics

- Spacecraft attitude dynamics are given by the equations of rotational dynamics. These describe the relation between causes (torques exerted on the vehicle) and effects (angular velocity). Solved together with kinematics.
- Main hypothesis: The vehicle is a rigid body (rigid-body hypothesis). If there are flexible/mobile parts, the model needs to be extended to include them. Thus we can define the rotation of the body frame (fixed at the center of mass of the body) w.r.t. the inertial frame, as in previous lessons.

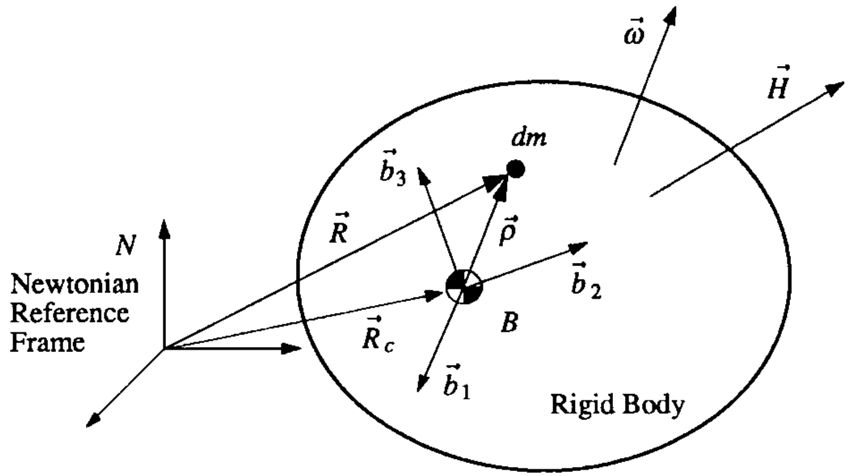


Angular momentum and Torque I



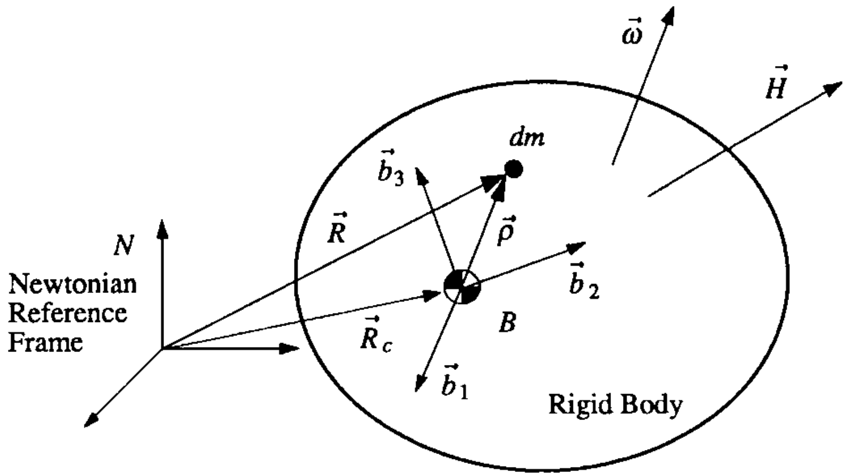
- For each point of the body with mass dm, one has $\ddot{\vec{R}}dm = d\vec{F}$. Taking moment with respect to the center of mass B, we get $\vec{\rho} \times \ddot{\vec{R}}dm = \vec{\rho} \times d\vec{F} = d\vec{M}_B$, and integrating over the volume V, we get a relation involving the total moment of the forces with respect to B (the total *torque*): $\int_V \vec{\rho} \times \ddot{\vec{R}}dm = \vec{M}_B$.
- Notice that these time-derivatives are considered w.r.t. the inertial frame.

Angular momentum and Torque II



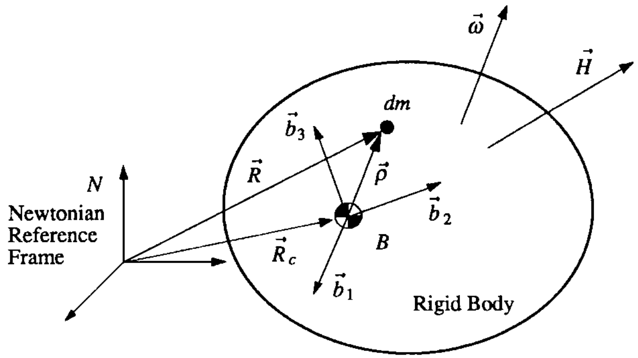
- The absolute angular momentum with respect to B, $\vec{\Gamma}_B$, is defined as: $\vec{\Gamma}_B = \int_V \vec{\rho} \times \dot{\vec{R}} dm$.
- Note $\dot{\vec{\Gamma}}_B = \int_V \dot{\vec{\rho}} \times \dot{\vec{R}} dm + \int_V \vec{\rho} \times \ddot{\vec{R}} dm$.
- Since $\vec{R} = \vec{R}_C + \vec{\rho}$, replacing it in the first term we get: $\dot{\vec{\Gamma}}_B = \int_V \dot{\vec{\rho}} \times \dot{\vec{\rho}} dm + \int_V \dot{\vec{\rho}} \times \dot{\vec{R}}_c dm + \vec{M}_B$
- The first term is zero. The second verifies $\int_V \dot{\vec{\rho}} \times \dot{\vec{R}}_c dm = \left(\frac{d}{dt} \int_V \vec{\rho} dm\right) \times \dot{\vec{R}}_c = \vec{0}$.
- Therefore $\dot{\vec{\Gamma}}_B = \vec{M}_B$

Angular momentum and Inertia I



- The angular momentum $\vec{\Gamma}_B$ verifies $\vec{\Gamma}_B = \int_V \vec{\rho} \times \dot{\vec{R}} dm = \int_V \vec{\rho} \times \dot{\vec{R}}_c dm + \int_V \vec{\rho} \times \dot{\vec{\rho}} dm = \int_V \vec{\rho} \times \dot{\vec{\rho}} dm.$
- Remember Coriolis' equation $\left(\frac{d}{dt}\vec{\rho}\right)_N = \left(\frac{d}{dt}\vec{\rho}\right)_B + \vec{\omega}_{B/N} \times \vec{\rho}$, where N is an inertial frame and B the body axes. Then, $\left(\frac{d}{dt}\vec{\rho}\right)_N = \vec{\omega}_{B/N} \times \vec{\rho}$.
- Therefore: $\vec{\Gamma}_B = \int_V \vec{\rho} \times (\vec{\omega}_{B/N} \times \vec{\rho}) dm = \left(-\int_V \vec{\rho}^{\times} \vec{\rho}^{\times} dm\right) \vec{\omega}_{B/N}$
- Define the inertia tensor $\mathcal{I} = -\int_V \vec{\rho}^{\times} \vec{\rho}^{\times} dm = \int_V \left[(\rho^T \vec{\rho})\mathrm{Id} - \rho \vec{\rho}^T\right] dm$

Angular momentum and Inertia II



- Thus $\vec{\Gamma}_B = \mathcal{I} \cdot \vec{\omega}_{B/N}$. The explicit expression of the inertia tensor is $\mathcal{I} = \begin{bmatrix} \int_V (\rho_2^2 + \rho_3^2) dm & -\int_V \rho_1 \rho_2 dm & -\int_V \rho_1 \rho_3 dm \\ -\int_V \rho_1 \rho_2 dm & \int_V (\rho_1^2 + \rho_3^2) dm & -\int_V \rho_2 \rho_3 dm \\ -\int_V \rho_1 \rho_3 dm & -\int_V \rho_2 \rho_3 dm & \int_V (\rho_1^2 + \rho_2^2) dm \end{bmatrix}$
- Since the matrix is symmetric: it is diagonalizable. Thus one can find the *principal axes* where $\mathcal{I}$ is diagonal:

$$\mathcal{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

- The largest moment of inertia I_i is about an axis which is denoted as major axis; the smallest, about the minor axis. The remaining one is about the intermediate axis.

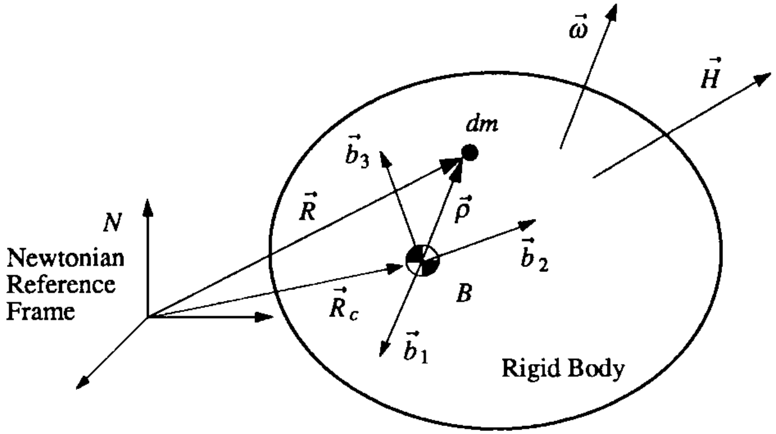
Angular momentum and Inertia III

- Assume we have a vehicle composed of n parts, each of them with known mass M_k, center of mass $\vec{r}_{ck}$ and inertia tensor $\mathcal{I}_k$. Then one can find the inertial tensor of the spacecraft as

$$\mathcal{I} = \sum_{k=1}^{n} \left[M_k \left(\|\vec{r}_{ck}\|^2 \mathrm{Id} - \vec{r}_{ck} \vec{r}_{ck}^{\,T} \right) + \mathcal{I}_k \right]$$

- Note that $\vec{r}_{ck}$ is the vector joining the center of mass of the k part with the whole spacecraft center of mass.
- Spacecraft are formed by a number of structural elements so this is a widely used formula. However, we will not need it in general for our lessons.

Kinetic energy



- Kinetic energy is defined as $T = \frac{1}{2}\int_V \dot{\vec{\rho}} \cdot \dot{\vec{\rho}} dm$.
- Using $\left(\frac{d}{dt}\vec{\rho}\right)_N = \vec{\omega}_{B/N} \times \vec{\rho}$, we get $T = \frac{1}{2}\int_V \dot{\vec{\rho}} \cdot (\vec{\omega}_{B/N} \times \vec{\rho}) dm = \frac{1}{2}\vec{\omega}_{B/N} \cdot \int_V (\vec{\rho} \times \dot{\vec{\rho}}) dm = \frac{1}{2}\vec{\omega}_{B/N} \cdot \vec{\Gamma}_B = \frac{1}{2}\vec{\omega}_{B/N} \cdot \mathcal{I} \cdot \vec{\omega}_{B/N}$.
- In principal axes, if $\vec{\omega}_{B/N} = [\omega_1 \; \omega_2 \; \omega_3]^T$, one gets:
$$\vec{\Gamma}_B = \begin{bmatrix} \omega_1 I_1 \\ \omega_2 I_2 \\ \omega_3 I_3 \end{bmatrix}$$
- Thus: $T = \dfrac{\omega_1^2 I_1 + \omega_2^2 I_2 + \omega_3^2 I_3}{2}$

Euler's Equations

- Start from $\dot{\vec{\Gamma}} = \vec{M}$. Since the time-derivative is in the inertial frame, taking it in body axes we get: $\left(\frac{d}{dt}\vec{\Gamma}\right)_N = \left(\frac{d}{dt}\vec{\Gamma}\right)_B + \vec{\omega}_{B/N} \times \vec{\Gamma} = \vec{M}$.
- Replacing the expression of angular momentum in terms of the inertia tensor: $\left(\frac{d}{dt}\mathcal{I} \cdot \vec{\omega}_{B/N}\right)_B + \vec{\omega}_{B/N} \times \left(\mathcal{I} \cdot \vec{\omega}_{B/N}\right) = \vec{M}$
- Using the rigid-body hypothesis $\left(\frac{d}{dt}\mathcal{I}\right)_B = 0$, we get: $\mathcal{I} \cdot \dot{\vec{\omega}}_{B/N} + \vec{\omega}^{\times}_{B/N}\mathcal{I} \cdot \vec{\omega}_{B/N} = \vec{M}$.
- Developing in principal axes and writing $\vec{M} = [M_1 \; M_2 \; M_3]^T$

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= M_1 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= M_2 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= M_3
\end{aligned}
$$

Torque-Free rotation

- Our first detailed study is of torque-free rotation, this is, when torque is zero: $\vec{M} = \vec{0}$. Under this assumption, the angular momentum of the system is preserved.
- This does not ever happen in reality since there are always some small perturbing torques (albeit they can be small).
- We will see some analytical solutions but the most interesting results are those concerning the stability of the rotation; in particular, we will find the major axis rule.
- We consider two cases: axisymmetric (two equal moments of inertia: the spinning top) and asymmetric (the three moments of inertia are different)
- The totally symmetric case ($I_1 = I_2 = I_3$) decouples Euler's equations and can be trivially solved (the resulting angular velocities are constant and independent from each other).

Axisymmetric case. Analytical solution.

- Consider $I_1 = I_2 = I$, $I_3 \neq I$.
- Euler's equations now read:

$$
\begin{aligned}
I\dot{\omega}_1 + (I_3 - I)\omega_2\omega_3 &= 0 \\
I\dot{\omega}_2 + (I - I_3)\omega_1\omega_3 &= 0 \\
I_3\dot{\omega}_3 &= 0
\end{aligned}
$$

- First, we obtain $\omega_3 = \mathrm{Cst} = n$ (spin rate of the spacecraft about it symmetry axes). Define $\lambda = \frac{I - I_3}{I} n$, denoted as the "relative spin rate". The first two equations result in

$$
\begin{aligned}
\dot{\omega}_1 - \lambda\omega_2 &= 0 \\
\dot{\omega}_2 + \lambda\omega_1 &= 0
\end{aligned}
$$

This is the ODE of a harmonic oscillator, whose solution is:

$$
\begin{aligned}
\omega_1 &= \omega_1(0)\cos\lambda t + \omega_2(0)\sin\lambda t \\
\omega_2 &= \omega_2(0)\cos\lambda t - \omega_1(0)\sin\lambda t
\end{aligned}
$$

Axisymmetric case. Analytical solution.

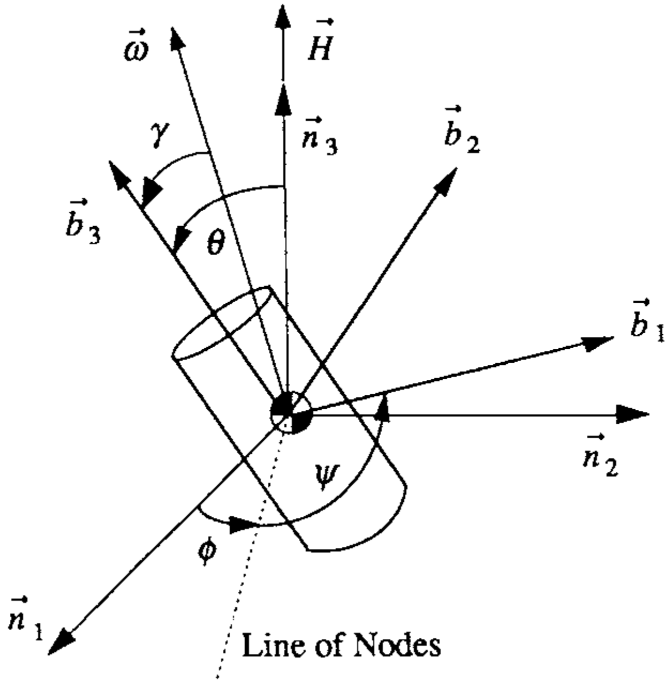
- It is easy to see that $\omega_1^2 + \omega_2^2 = \mathrm{Cst} = \omega_{12}^2$, the so-called transverse angular velocity. Thus, $\|\omega\| = \sqrt{\omega_{12}^2 + n^2} = \mathrm{Cst}$ and its third component is also constant. Therfore, $\vec{\omega}$ seen in the body frame describes a cone about the body symmetry axes, of angle $\gamma = \arctan\left(\frac{\omega_{12}}{n}\right)$.
- On the other hand $\vec{\Gamma} = \vec{\mathrm{Cst}}$ in the inertial frame by conservation of angular momentum. We choose the z axis of the inertial frame as pointing in the direction of $\vec{\Gamma}$ ($\vec{H}$ in the figure). In addition $\Gamma = \|\vec{\Gamma}\|$ must be constant as well.
- In body axes, $\vec{\Gamma} = [I\omega_1 \; I\omega_2 \; I_3 n]^T$, so that $\vec{\Gamma} \cdot \vec{e}_z^{\,b} = I_3 n = \cos\theta\Gamma$, this is, the angle between $\vec{\Gamma}$ and the body z axis is constant; this angle, θ, is the nutation angle. In addition:

$$\tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{\Gamma^2 - I_3^2 n^2}}{I_3 n} = \frac{I\omega_{12}}{I_3 n} = \frac{I}{I_3}\tan\gamma$$

- Exercise: prove that the angle between $\vec{\Gamma}$ y $\vec{\omega}$ is $\theta - \gamma = cst$.

Axisymmetric case. Analytical solution.

- Thus the situation is as in the figure (where $\vec{H} = \vec{\Gamma}$).



- This justifies introducing Euler angles to describe the movement, in the sequence (3,1,3), where one already knows that $\theta = \mathrm{Cst}$.

$$n \xrightarrow[z^n]{\phi} S \xrightarrow[x^S]{\theta} S' \xrightarrow[z^{S'}]{\psi} BFS$$

Axisymmetric case. Analytical solution.

- For the sequence

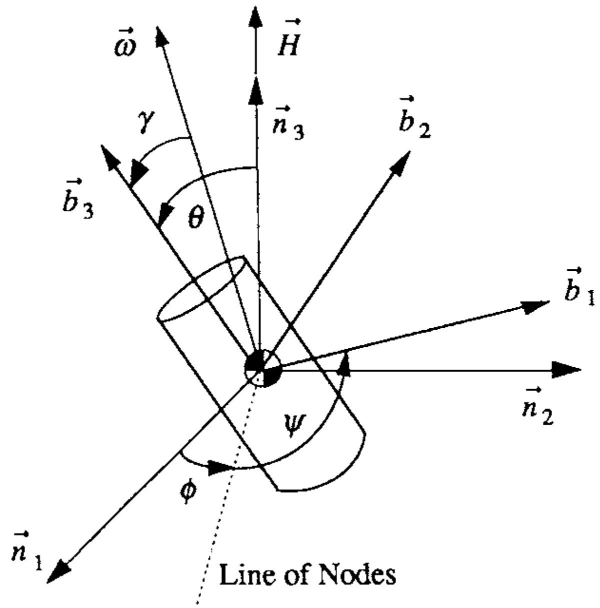
$$n \xrightarrow[z^n]{\phi} S \xrightarrow[x^S]{\theta} S' \xrightarrow[z^{S'}]{\psi} BFS$$

the kinematics are, replacing $\theta = \mathrm{Cst}$:

$$\begin{aligned}
\omega_1 &= \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi = \dot\phi \sin\theta \sin\psi \\
\omega_2 &= \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi = \dot\phi \sin\theta \cos\psi \\
\omega_3 &= \dot\psi + \dot\phi \cos\theta
\end{aligned}$$

- Applying $\omega_1^2 + \omega_2^2 = \omega_{12}^2$ we obtain: $\omega_{12} = \dot\phi \sin\theta$. Thus $\dot\phi = \frac{\omega_{12}}{\sin\theta} = \mathrm{Cst}$, the precession rate. Finally $\dot\psi = n - \dot\phi \cos\theta = n - \frac{\omega_{12}}{\tan\theta} = n - \frac{I_3 n}{I} = n\frac{I - I_3}{I} = \lambda = \mathrm{Cst}$.
- Similarly $\dot\phi = \frac{\omega_{12}}{\sin\theta} = \frac{I_3 n}{I \cos\theta} = \frac{I_3(\dot\psi + \dot\phi \cos\theta)}{I \cos\theta}$, from where $\dot\phi = \frac{I_3 \dot\psi}{(I - I_3)\cos\theta}$.

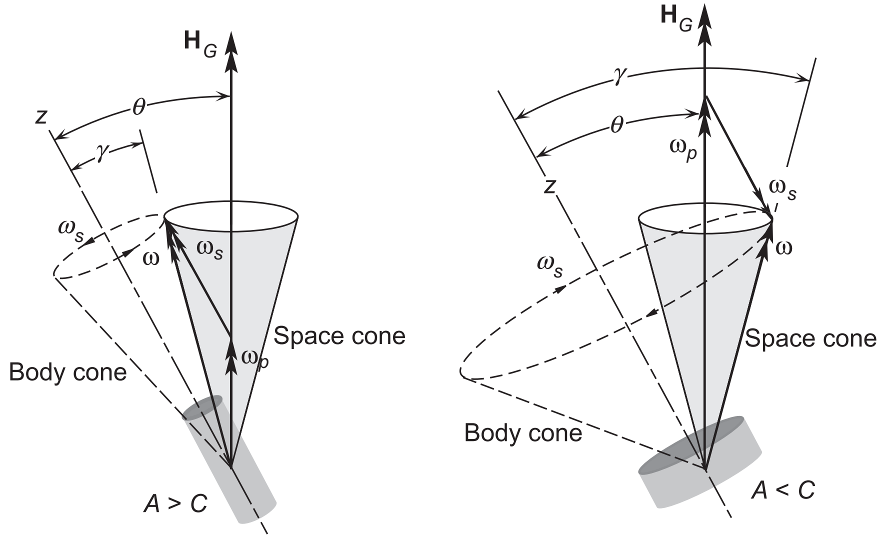
Axisymmetric case. Geometrical interpretation.



$$n \xrightarrow[z^n]{\phi} S \xrightarrow[x^S]{\theta} S' \xrightarrow[z^{S'}]{\psi} BFS$$

- Considering the sequence and taking into account the fact that the nutation angle is constant and the other two angles change uniformly, one can imagine the movement as the rolling of one cone over another without slipping (with constant angular speeds $\dot{\phi}$ and $\dot{\psi}$); the point of contact is where the angular velocity $\vec{\omega}$ lies.

Axisymmetric case. Geometrical interpretation.



(a) Prograde precession (b) Retrograde precession

- Remember $\tan\gamma = \tan\theta\frac{I_3}{I}$ y $\dot{\phi} = \frac{I_3\dot{\psi}}{(I-I_3)\cos\theta}$. Two cases arise:
  - Prolate body (thin symmetry axis, $I_3 < I$): this is case (a). Since $\gamma < \theta$ the cones roll one outside the other and since the signs of $\dot{\phi}$ and $\dot{\psi}$ are equal the rotation is in the same direction (prograde precession).
  - Oblate body (thick symmetry axis, $I_3 > I$): this is case (b). Since $\gamma > \theta$ the cones roll one inside the other and since the signs of $\dot{\phi}$ y $\dot{\psi}$ are opposite the rotation is in the opposite direction (retrograde precession).

Torque-free rotation of an asymmetrical body

- In the asymmetrical case, there exists a major, minor and intermediate axis. The equations cannot be solved in terms of conventional functions.

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= 0 \end{aligned}$$

- Some authors solve these equations by using Jacobi's "elliptical functions". However, it is not easy to understand/interpret these functions, so we take a more "geometric" path.
- Notice that, due to conservation of angular momentum, $\vec{\Gamma}$ is constant (in inertial axes). Therefore $\|\vec{\Gamma}\| = \Gamma$ is constant no matter what axes are used to write $\vec{\Gamma}$. In particular, in the body frame, $\vec{\Gamma} = [I_1\omega_1 \;\; I_2\omega_2 \;\; I_3\omega_3]^T$, therefore $\Gamma^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \mathrm{Cst.}$

Torque-free rotation of an asymmetrical body

- Similarly, in torque-free rotations the kinetic energy T is also preserved. Which impliese $2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = \mathrm{Cst}'$
- Therefore the components of the angular velocity, $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$, no matter their values, must satisfy

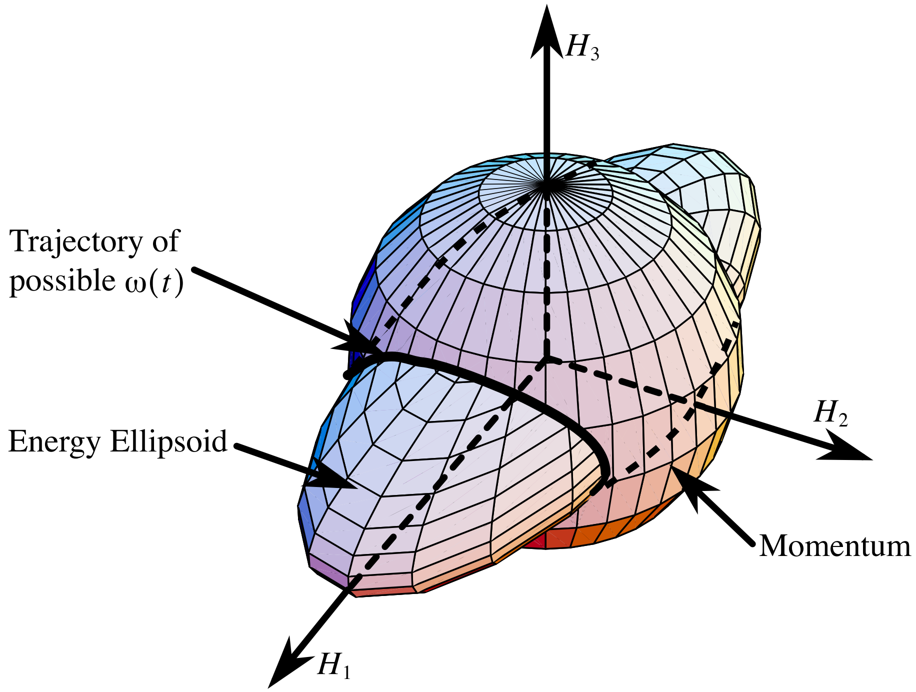
$$\frac{\omega_1^2}{\frac{\Gamma^2}{I_1^2}} + \frac{\omega_2^2}{\frac{\Gamma^2}{I_2^2}} + \frac{\omega_3^2}{\frac{\Gamma^2}{I_3^2}} = 1$$

$$\frac{\omega_1^2}{\frac{2T}{I_1}} + \frac{\omega_2^2}{\frac{2T}{I_2}} + \frac{\omega_3^2}{\frac{2T}{I_3}} = 1$$

- These are the equations of two ellipsoids: the angular momentum ellipsoid and the kinetic energy ellipsoid. Thus the angular velocity vector must always lie in the intersection of these two ellipsoids; these intersections are known as “polhode curves”.

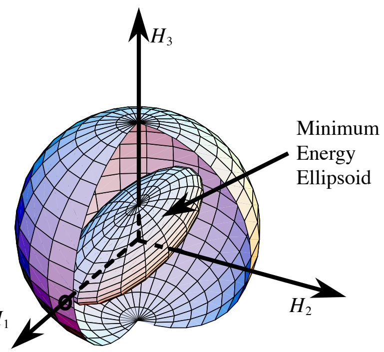
Polhode curves

- In general the curves, for given ellipsoids, are two disjoint, closed curves.

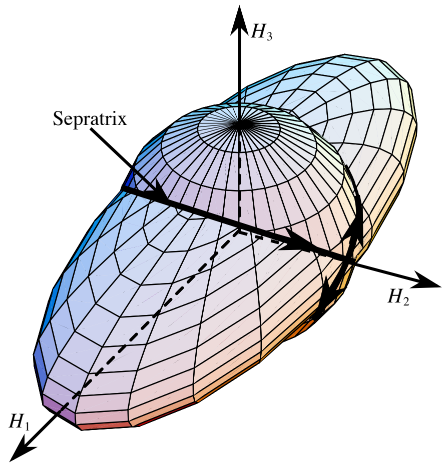


- In two cases the intersection reduces to two points: when the ellipsoids are tangent. These cases correspond to maxima or minima of the energy. In addition, there is a saddle point when the intermediate axes coincide, and the resulting curve is called the separatrix.

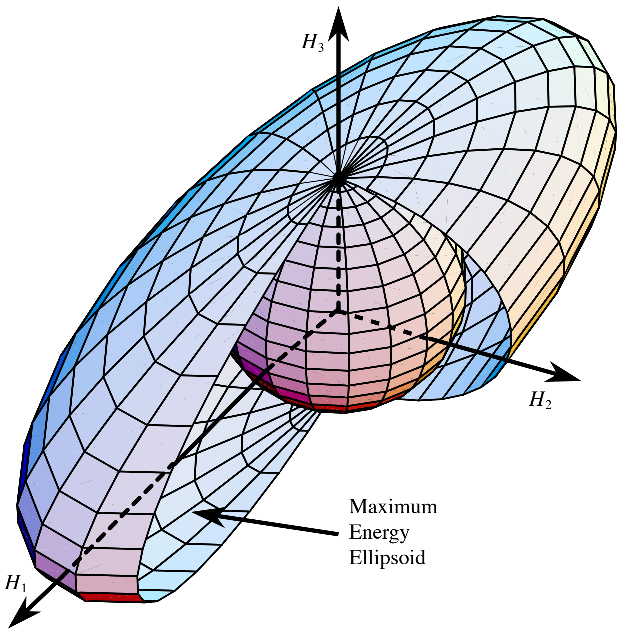
Polhode curves: special cases



(i) Minimum Energy Case



(ii) Intermediate Energy Case



(iii) Maximum Energy Case

Torque-free rotation of an asymmetrical body

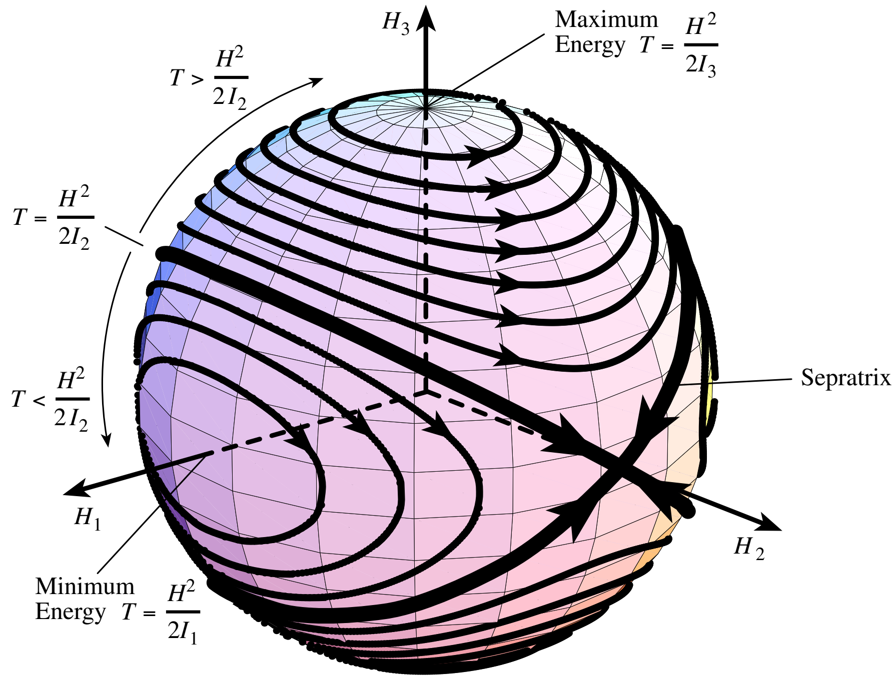
- Assume that $I_3 < I_2 < I_1$ (if not re-index the axes). Define $I^* = \frac{\Gamma^2}{2T}$. Subtracting the ellipsoid equations and multiplying by Γ^2, one gets:

$$I_1\omega_1^2\left(I_1 - I^*\right) + I_2\omega_2^2\left(I_2 - I^*\right) + I_3\omega_3^2\left(I_3 - I^*\right) = 0$$

- Note that if $I^* < I_3$ all terms are positive (for non-zero angular speed) so they cannot add to zero. Similarly if $I^* > I_1$ all terms are negative. Thus, $I^* \in [I_3, I_1]$. For fixed Γ, this implies that kinetic energy has to lie inside an interval. The extrema are $I^* = I_1$ (minimal energy, implies $\omega_2 = \omega_3 = 0$ and thus a rotation about the 1 axis, the major one) and $I^* = I_3$ (maximal energy, implies $\omega_1 = \omega_2 = 0$ and thus a rotation about the 3rd axis, the minor oner)
- The case $I^* = I_2$ has additional solutions besides pure rotations about the 2 axis ($\omega_1 = \omega_3 = 0$); these are called separatrices.

Polhode curves for fixed Γ

- If Γ (H in the figure) is fixed and we vary the energy, we obtain all possible polhode curves over the surface of the momentum ellipsoid, including the separatrices.



Stability of spinning spacecraft about a principal axis

- The simplest solutions of torque-free motion are pure rotations (spins) about a principal axis. Next, we start from the solution of equilibrium $\bar{\omega}_3 = n = \mathrm{Cst}$ and $\bar{\omega}_1 = \bar{\omega}_2 = 0$. We study the stability of this equilibrium as a function of whether the 3rd axis is major, minor or intermediate.
- Let us perturb the equilibrium, defining $\omega_1 = \delta\omega_1$, $\omega_2 = \delta\omega_2$ and $\omega_3 = n + \delta\omega_3$. Substituting in Euler's equations:

$$\begin{aligned} I_1\delta\dot{\omega}_1 + (I_3 - I_2)\delta\omega_2(n + \delta\omega_3) &= 0 \\ I_2\delta\dot{\omega}_2 + (I_1 - I_3)\delta\omega_1(n + \delta\omega_3) &= 0 \\ I_3\delta\dot{\omega}_3 + (I_2 - I_1)\delta\omega_2\delta\omega_1 &= 0 \end{aligned}$$

- Neglecting second-order terms:

$$\begin{aligned} I_1\delta\dot{\omega}_1 + n(I_3 - I_2)\delta\omega_2 &= 0 \\ I_2\delta\dot{\omega}_2 + n(I_1 - I_3)\delta\omega_1 &= 0 \\ I_3\delta\dot{\omega}_3 &= 0 \end{aligned}$$

Stability of spinning spacecraft about a principal axis

- The equation of $\delta\omega_3$ defines a marginally stable equilibrium: the perturbed solutions don't grow, but they don't dissipate either.
- The equations for $\delta\omega_1$ and $\delta\omega_2$ can be combined as

$$\delta\ddot{\omega}_1 + \frac{n^2(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\delta\omega_1 = 0$$

- The stability of the solution to this equation depends on the sign of $(I_3 - I_2)(I_3 - I_1)$. For a positive sign, solutions are oscillatory (again, they don't grow or dissipate: marginally stable). If the sign is negative, the solutions are exponential and one of the solutions grows in time (unstable)
- If 3 is the major axis: $(I_3 - I_2)(I_3 - I_1) = + \times + > 0$: stable.
- If 3 is the minor axis:$(I_3 - I_2)(I_3 - I_1) = - \times - > 0$: stable.
- If 3 is the intermediate axis:, $(I_3 - I_2)(I_3 - I_1) = + \times - < 0$: unstable.

Stability of spinning spacecraft with energy dissipation

- While the previous calculation is correct under a rigid-body assumption (Euler's Equations), real-life solids are not perfectly rigid.
- There is always some deviation from the rigid body that can cause some energy dissipation (flexibility effects, friction between mobile parts, fuel sloshing). This modifies the previous calculation as the system tends to go to an energy minima.
- Assume again $I_1 > I_2 > I_3$. One idea (energy sink model) is to, starting from physical principles (conservation of angular momentum), find a minima of energy given the angular momentum. This is, solve the mathematical minimization problem

$$\min I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$$
$$\text{subject to} \quad I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \Gamma^2$$

Stability of spinning spacecraft with energy dissipation

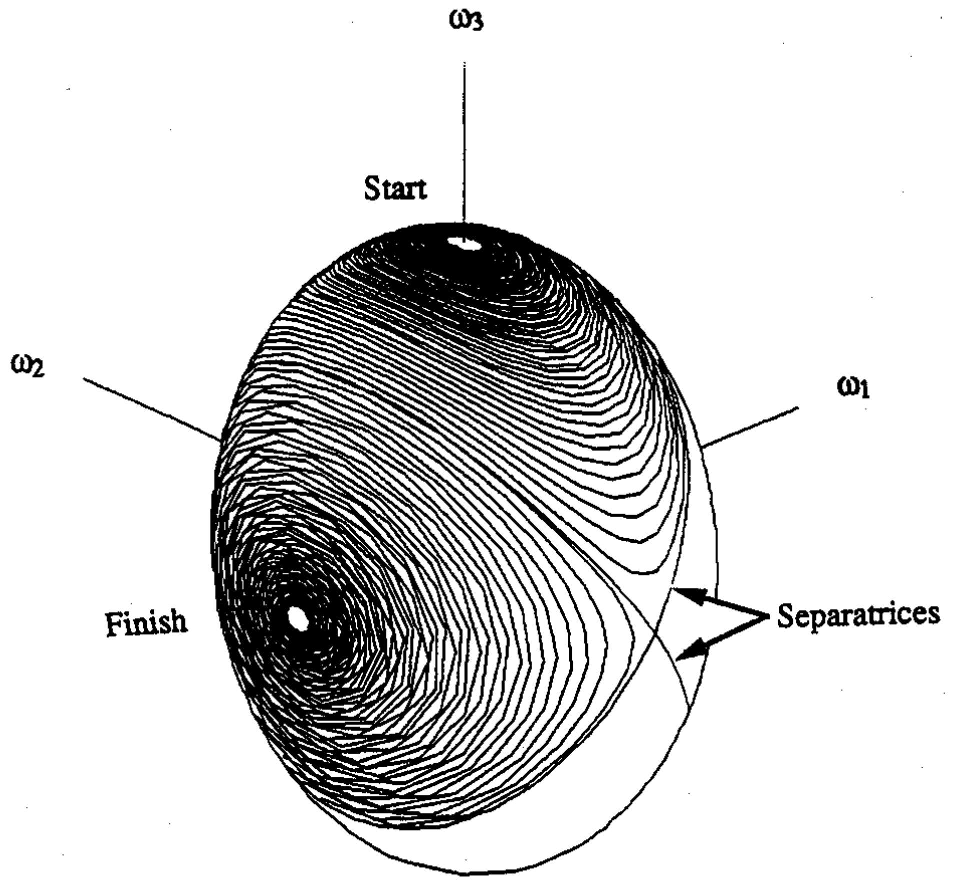
- Using Lagrange multipliers:

$$L(\omega_1, \omega_2, \omega_3, \lambda) = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + \lambda(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 - \Gamma^2)$$

- One has $0 = \frac{\partial L}{\partial \omega_i} = 2I_i\omega_i(1 + \lambda I_i), \quad i = 1, 2, 3$
- Therefore there are three solutions:
  - $\omega_2 = \omega_3 = 0$, $\lambda = -\frac{1}{I_1}$, $\omega_1 = \frac{\Gamma}{I_1}$. $T = \frac{\Gamma^2}{2I_1}$.
  - $\omega_1 = \omega_3 = 0$, $\lambda = -\frac{1}{I_2}$, $\omega_2 = \frac{\Gamma}{I_2}$. $T = \frac{\Gamma^2}{2I_2}$.
  - $\omega_1 = \omega_2 = 0$, $\lambda = -\frac{1}{I_3}$, $\omega_3 = \frac{\Gamma}{I_3}$. $T = \frac{\Gamma^2}{2I_3}$.
- Comparing the values of the objective function (the energy), clearly the minimum is given by the first solution (the second is a saddle point and third one is the maximum). Thus the only spin which is mathematically stable and at the same time a minimum for the energy are rotations about the major axis.
- Based on this argument, we can now state the major axis rule: "For spacecraft with dissipation of energy, the only stable spins are those about the major axis".

Stability of spinning spacecraft with energy dissipation

- The geometrical effect of the major axis rule is that polhodes become a single closed spiral curve that goes from the maximum of energy to the minimum of energy:



Example: fuel sloshing

- Consider a satellite with a spherical tank filled with viscous fuel, so that the fuel (with inertia J and friction coefficient Δ) can be modelled as a "solid bubble" with its own angular speed $\vec{\sigma} = [\sigma_1 \ \sigma_2 \ \sigma_3]^T$ relative to the satellite.
- ExtraCstd from C.D. Rahn, P.M. Barba, "Reorientation Maneuver for Spinning Spacecraft", AIAA Journal of Guidance, Dynamics and Control, Vol. 14, 1991.

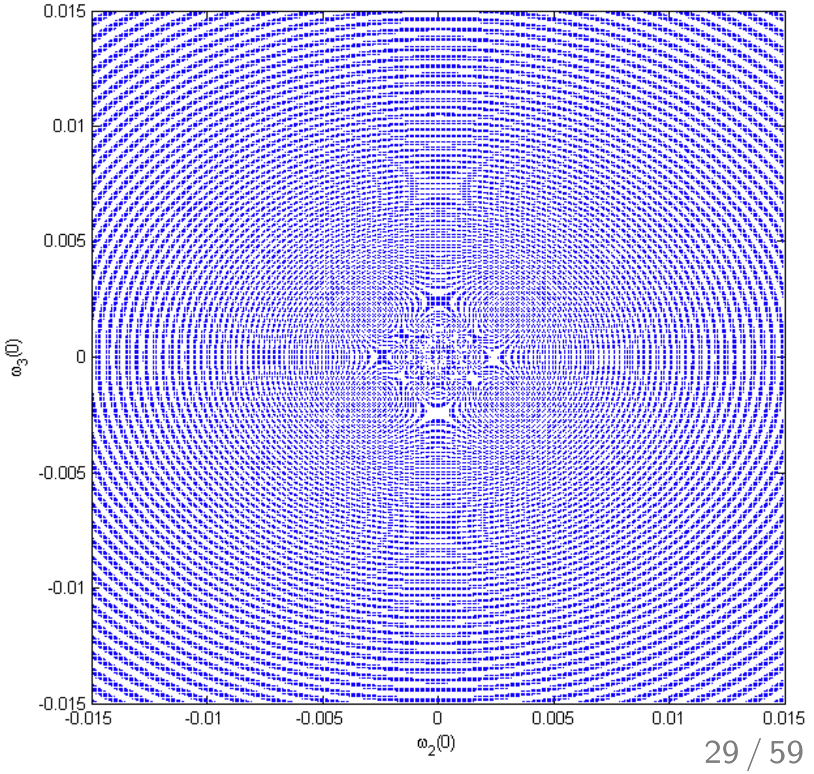
$$
\begin{aligned}
(I_1 - J)\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= \Delta\sigma_1 \\
(I_2 - J)\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= \Delta\sigma_2 \\
(I_3 - J)\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= \Delta\sigma_3 \\
\dot{\sigma}_1 + \dot{\omega}_1 + \omega_2\sigma_3 - \omega_3\sigma_2 &= -\frac{\Delta\sigma_1}{J} \\
\dot{\sigma}_2 + \dot{\omega}_2 + \omega_3\sigma_1 - \omega_1\sigma_3 &= -\frac{\Delta\sigma_2}{J} \\
\dot{\sigma}_3 + \dot{\omega}_3 + \omega_1\sigma_2 - \omega_2\sigma_1 &= -\frac{\Delta\sigma_3}{J}
\end{aligned}
$$

- By dissipation, any starting spin ends up a major axis spin; however, it is not possible to know a priori the orientation of the rotation, since the equations display strange (chaotics) dynamics.

Example: fuel sloshing

- The fact that the equations have chaotic dynamics means that the sense of rotation totally depends on the initial condition, to the point that any change on the initial condition, no matter how small, can produce a variation in the sense of rotation.

- Thus, to all practical effect, it is not possible to predict the final sense of the rotation.
- A plot in which one marks with the same color the initial conditions producing the same sense of rotation becomes enormously complex, due to this chaotic property of the equation. These kind of plots are known as fractals.



Major axis rule. Additional comments.

- The instability of minor axis spinners is, from the point of view of time-scales, much slower than the instability of intermediate axis spinners, depending on the rate of energy dissipation.
- If one desires a major axis spin one can amplify energy dissipation by adding dampers, such as nutation dampers (pendula with added friction).
- However, if for some reason one needs a minor axis spin this is no issue if it is only required for a short period of time and dissipation is not too large. Later the body will return to a major axis spin naturally.
- Important: the presence of mobile part such as inertia wheels may change these theoretical results.

Rotational dynamics with a wheel

- Let us start with how Euler's equations are modified by the presence of k wheels.
- For each wheel i, assumed axisymmetric, define I_{Ri} as its momentum of inertia in the rotation direction $\vec{e}_i$ and its relative (to the spacecraft) angular speed as ω_{Ri}.
- Since a wheel is symmetric, it does not change the distribution of mass: total spacecraft inertia does not change at all.
- The angular moment of the spacecraft + wheels is: $\vec{\Gamma} = \mathcal{I}\vec{\omega}_{B/N} + \sum_{i=0}^{k} \vec{e}_i I_{Ri} \omega_{Ri}$
- Expressing the derivative $\dot{\vec{\Gamma}} = \vec{M}$ in the body frame one can obtain the differential equations of motion.

Three wheels in principal axes

- If there is a wheel about each principal axis, the spacecraft angular momentum is $\vec{\Gamma} = \mathcal{I}\vec{\omega}_{B/N} + \begin{bmatrix} \omega_{R1} I_{R1} \\ \omega_{R2} I_{R2} \\ \omega_{R3} I_{R3} \end{bmatrix}$
- Thus the dynamics is given by

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + I_{R1}\dot{\omega}_{R1} + I_{R3}\omega_{R3}\omega_2 - I_{R2}\omega_{R2}\omega_3 &= M_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 + I_{R2}\dot{\omega}_{R2} + I_{R1}\omega_{R1}\omega_3 - I_{R3}\omega_{R3}\omega_1 &= M_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 + I_{R3}\dot{\omega}_{R3} + I_{R2}\omega_{R2}\omega_1 - I_{R1}\omega_{R1}\omega_2 &= M_3 \end{aligned}$$

- One needs to add the equations describen the wheels' spin. For instance, if for each axis an electric motor with (internal) torque J_{Ri} drives the wheels, these equations would be

$$\begin{aligned} I_{R1}(\dot{\omega}_1 + \dot{\omega}_{R1}) &= J_1 \\ I_{R2}(\dot{\omega}_2 + \dot{\omega}_{R2}) &= J_3 \\ I_{R3}(\dot{\omega}_3 + \dot{\omega}_{R3}) &= J_3 \end{aligned}$$

One wheel about the 3rd axis

- Assume that a spacecraft has an inertial wheel about the 3rd axis, with inertia I_R, and spinning at a velocity ω_R relative to the spacecraft. It could even be a part of the spacecraft (see dual spin-stabilization in lesson 7).
- Angular momentum is $\Gamma = [I_1\omega_1 \;\; I_2\omega_2 \;\; I_3\omega_3 + I_R\omega_R]^T$.
- Rotational dynamics become

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + I_R\omega_R\omega_2 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 - I_R\omega_R\omega_1 &= 0 \\ I_3\dot{\omega}_3 + I_R\dot{\omega}_R + (I_2 - I_1)\omega_2\omega_1 &= 0 \end{aligned}$$

- One needs to add $I_R(\dot{\omega}_3 + \dot{\omega}_R) = J$, where J is the torque driving the wheel (if any).

Spin stability with a wheel.

- One can use the motor to produce a torque that mantains ω_R constant. Then:

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + I_R\omega_R\omega_2 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 - I_R\omega_R\omega_1 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= 0 \end{aligned}$$

- New terms appear that modify the previous stability analysis. Even the intermediate axis can be made stable! Repeating the steps for mathematical stability:

$$\delta\ddot{\omega}_1 + \frac{\left(n(I_3 - I_2) + I_R\omega_R\right)\left(n(I_3 - I_1) + I_R\omega_R\right)}{I_1 I_2}\delta\omega_1 = 0$$

- Now if 1 is the minor axis and 2 the major, the condition for stability is $n(I_3 - I_2) + I_R\omega_R > 0$, this is, $\omega_R > \frac{I_2 - I_3}{I_R} n$.
- Next, we repeat the analysis in the case of energy dissipation by using the energy sink method.

Spin stability with a wheel and energy dissipation.

- Let us minimize the energy fixing the angular momentum (since it is a torque-free motion).
- Then

$$
\begin{aligned}
2T &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + I_R\omega_R^2, \\
\Gamma^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + (I_3\omega_3 + I_R\omega_R)^2
\end{aligned}
$$

- The last term of the energy can be ignored since it is a constant and does not influence the minimization process. The problem is posed as

$$
\begin{aligned}
&\min I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \\
&\text{subject to } \; I_1^2\omega_1^2 + I_2^2\omega_2^2 + (I_3\omega_3 + I_R\omega_R)^2 = \Gamma^2
\end{aligned}
$$

Spin stability with a wheel and energy dissipation.

- Using Lagrange multipliers

$$L(\omega_1,\omega_2,\omega_3,\lambda) = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + \lambda(I_1^2\omega_1^2 + I_2^2\omega_2^2 + (I_3\omega_3 + I_R\omega_R)^2 - \Gamma^2)$$

- One gets $0 = \frac{\partial L}{\partial \omega_i} = 2I_i\omega_i(1+\lambda I_i), \quad i = 1,2$ y $0 = \frac{\partial L}{\partial \omega_3} = 2I_3(\omega_3 + \lambda(I_3\omega_3 + I_R\omega_R))$
- Several solutions exist, we take

$$\omega_1 = \omega_2 = 0, \quad \omega_3 = n, \quad \lambda = -\frac{n}{I_3 n + I_R\omega_R}.$$

- To identify if it is a minimum or not, we use the following theorem: Let $L(x,y,z) = F(x,y,z) + \lambda G(x,y,z)$ be the Lagrangian of the system so that F is the function to minimize and $G(x,y,z) = 0$ the constraint. Then, construct the matrices:

$$H_3 = \begin{bmatrix} 0 & \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} \\ \frac{\partial G}{\partial y} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial y^2} \end{bmatrix}, \quad H_4 = \begin{bmatrix} 0 & \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial x \partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial y^2} & \frac{\partial^2 L}{\partial y \partial z} \\ \frac{\partial G}{\partial z} & \frac{\partial^2 L}{\partial x \partial z} & \frac{\partial^2 L}{\partial y \partial z} & \frac{\partial^2 L}{\partial z^2} \end{bmatrix},$$

Spin stability with a wheel and energy dissipation.

- If x^*, λ^* is the critical point under analysis (i.e., the point that makes the first derivatives of L zero), to determine if there is a minimum or not, it follows that if:
  1. $\frac{\partial G}{\partial x}(x^*, y^*, z^*) \neq 0$
  2. $\mathrm{Det}(H_3(x^*, y^*, z^*, \lambda^*)) < 0$
  3. $\mathrm{Det}(H_4(x^*, y^*, z^*, \lambda^*)) < 0$

  then there is a minimum at the critical point (sufficient condition, not necessary!).
- In our particular case, to verify the theorem, define $x = \omega_3$, $y = \omega_1$, $z = \omega_2$. Then:

$$
\begin{aligned}
H_3 &= \begin{bmatrix} 0 & 2I_3(I_3 n + I_r\omega_R) & 0 \\ 2I_3(I_3 n + I_r\omega_R) & 2I_3(1+\lambda I_3) & 0 \\ 0 & 0 & 2I_1(1+\lambda I_1) \end{bmatrix}, \\
H_4 &= \begin{bmatrix} 0 & 2I_3(I_3 n + I_r\omega_R) & 0 & 0 \\ 2I_3(I_3 n + I_r\omega_R) & 2I_3(1+\lambda I_3) & 0 & 0 \\ 0 & 0 & 2I_1(1+\lambda I_1) & 0 \\ 0 & 0 & 0 & 2I_2(1+\lambda I_2) \end{bmatrix}.
\end{aligned}
$$

Spin stability with a wheel and energy dissipation.

- The (sufficient) conditions for a minimum are:
  1. $\frac{\partial G}{\partial x}(x^*, y^*, z^*) = 2I_3(I_3 n + I_r \omega_R) \neq 0$ (since if the other two angular speeds are zero, one has $I_3 n + I_r \omega_R = \pm\Gamma \neq 0$).
  2. $\mathrm{Det}(H_3(x^*, y^*, z^*, \lambda^*)) = -8I_3^2(I_3 n + I_r \omega_R)^2 I_1(1 + \lambda I_1) < 0$
  3. $\mathrm{Det}(H_4(x^*, y^*, z^*, \lambda^*)) = \mathrm{Det}(H_3) 2I_2(1 + \lambda I_2) < 0$
- Two conditions are then reached

$$\begin{aligned} 1 + \lambda I_1 &> 0, \\ 1 + \lambda I_2 &> 0. \end{aligned}$$

- Using the value of λ that we derived before:

$$\begin{aligned} 1 - \frac{I_1 n}{I_3 n + I_R \omega_R} &> 0, \\ 1 - \frac{I_2 n}{I_3 n + I_R \omega_R} &> 0. \end{aligned}$$

- One has to be careful with the sign of $I_3 n + I_R \omega_R$ since when solving for ω_R the sign of the inequality can change.

Spin stability with a wheel and energy dissipation.

- Instead of solving for ω_R we can simplify the fraction, reaching:

$$\frac{(I_3 - I_1)n + I_R\omega_R}{I_3 n + I_R\omega_R} > 0,$$
$$\frac{(I_3 - I_2)n + I_R\omega_R}{I_3 n + I_R\omega_R} > 0,$$

- Two cases:
  1. If $I_3 n + I_R\omega_R > 0$, this is, $\omega_R > -\frac{I_3 n}{I_R}$, the conditions reduce to $\omega_R > \frac{(I_1 - I_3)n}{I_R}$, $\omega_R > \frac{(I_2 - I_3)n}{I_R}$.
  2. If $I_3 n + I_R\omega_R < 0$, this is, $\omega_R < -\frac{I_3 n}{I_R}$, the conditions reduce to$\omega_R < \frac{(I_1 - I_3)n}{I_R}$, $\omega_R < \frac{(I_2 - I_3)n}{I_R}$.
- Notice that these conditions are similar (but more restrictive) than the ones obtained without energy dissipation!

Spin stability with a wheel: Example.

- Consider a satellite with a wheel in the 3rd axis with:

$$\mathcal{I} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 20 \end{bmatrix} \mathrm{kg}\cdot\mathrm{m}^2, \qquad n = 60 \text{ r.p.m.}, \qquad I_R = 2 \text{ kg}\cdot\mathrm{m}^2.$$

- Need to study the required spinning speed for the weel for the 3rd axis (intermediate) to be stable.
- With the rigid-body hypothesis (no dissipation): $(n(I_2 - I_3) - I_R\omega_R)(n(I_3 - I_1) + I_R\omega_R) < 0$. Two cases
  1. First parenthesis is negative, second positive. Conditions become: $\omega_R > \frac{n(I_2-I_3)}{I_R} = 300$ r.p.m. and $\omega_R > \frac{n(I_3-I_1)}{I_R} = -300$ r.p.m.. Since the first condition is more stringent: $\omega_R > 300$ r.p.m..
  2. Second parenthesis is negative, first positive. Conditions become: $\omega_R < \frac{n(I_2-I_3)}{I_R} = 300$ r.p.m. and $\omega_R < \frac{n(I_3-I_1)}{I_R} = -300$ r.p.m.. Now the second condition is more restrictive, therefore $\omega_R < -300$ r.p.m..
- Thus, the spin is stable if $\omega_R > 300$ r.p.m. or if $\omega_R < -300$ r.p.m., but unstable if $\omega_R \in [-300, 300]$ r.p.m.

Spin stability with a wheel and energy dissipation: Example.

- With energy dissipation, two cases show up again:
  1. If $I_3 n + I_R\omega_R > 0$, this is $\omega_R > -\frac{I_3 n}{I_R} = -600$ r.p.m., then $\omega_R > \frac{(I_1-I_3)n}{I_R} = -300$ r.p.m., $\omega_R > \frac{(I_2-I_3)n}{I_R} = 300$ r.p.m.. The third condition is more restrictive so $\omega_R > 300$ r.p.m..
  2. If $I_3 n + I_R\omega_R < 0$, this $\omega_R < -\frac{I_3 n}{I_R} = -600$ r.p.m., then $\omega_R < \frac{(I_1-I_3)n}{I_R} = -300$ r.p.m., $\omega_R < \frac{(I_2-I_3)n}{I_R} = 300$ r.p.m.. The first condition is the more stringent, thus $\omega_R < -600$ r.p.m..
- Thus, the spin is stable if $\omega_R > 300$ r.p.m. or if $\omega_R < -600$ r.p.m., but unstable if $\omega_R \in [-600, 300]$ r.p.m..
- Notice in $\omega_R \in [-600, -300]$ r.p.m. the two models differ; however, the model with dissipation is more realistic, so the conclusion is that the rigid-body model is failing in that interval of ω_R!

Non-zero torque spins

- In practice there are always some perturbation torques. While typically of small magnitude, they might be persistent (such as gravity gradient which acts in the full orbit at all times). They might be large as well, for instance in the case of imperfectly aligned thrusters during manoeuvres.
- We analyze two cases:
  - Perturbation torque acting on a spinning solid (gyroscopic effect).
  - Gravity gradient stability.

Spinning body subject to a constant external torque.

- Hypothesis:
  - Axisymmetrical spacecraft: $I_1 = I_2 = I$.
  - Spinning spacecraft with speed n about axis 3, this is, $\omega_3 = n$.
  - Perturbation torque M_1 constant about the axis 1. No torque about the other axes.
- Example: spin-stabilized spacecraft making a propulsive manoeuvre with slight unalignment of the thruster axis with the center of mass. If there is no spin, the resulting torque causes an immediate rotation of the vehicle and failure of the manoeuvre.
- We will see that a spinner acquires the so-called "gyroscopic rigidity" and the perturbing torque produces a slight movement of precession and nutation of the spin axis.

Spinning body subject to a constant external torque.

- Euler's equations reduce to

$$\begin{aligned} I\dot{\omega}_1 + (I_3 - I)\omega_2\omega_3 &= M_1 \\ I\dot{\omega}_2 + (I - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 &= 0 \end{aligned}$$

- We find immediately $\omega_3 = \mathrm{Cst} = n$ and define $\lambda = \frac{I-I_3}{I} n$ y $\mu = \frac{M_1}{I}$. Two equations remain to be solved:

$$\begin{aligned} \dot{\omega}_1 - \lambda\omega_2 &= \mu \\ \dot{\omega}_2 + \lambda\omega_1 &= 0 \end{aligned}$$

- Taking time derivative in the first equation and substituting the second:

$$\ddot{\omega}_1 + \lambda^2\omega_1 = 0$$

- Harmonic oscillator: $\omega_1(t) = A \sin \lambda t + B \cos \lambda t$.

Spinning body subject to a constant external torque.

- Substituting the solution in the 1st equation $\omega_2(t) = A\cos\lambda t - B\sin\lambda t - \frac{\mu}{\lambda}$.
- Replacing initial conditions $\omega_1(0)$ and $\omega_2(0)$ we reach: $B = \omega_1(0)$, $A = \omega_2(0) + \frac{\mu}{\lambda}$. Thus:

$$\begin{aligned} \omega_1 &= \left(\omega_2(0) + \frac{\mu}{\lambda}\right)\sin\lambda t + \omega_1(0)\cos\lambda t = \frac{\mu}{\lambda}\sin\lambda t \\ \omega_2 &= \left(\omega_2(0) + \frac{\mu}{\lambda}\right)\cos\lambda t - \omega_1(0)\sin\lambda t - \frac{\mu}{\lambda} = \frac{\mu}{\lambda}(\cos\lambda t - 1) \end{aligned}$$

where finally we have replaced $\omega_1(0) = \omega_2(0) = 0$.

- Use now Euler angles

$$I \xrightarrow[x^n]{\theta_1} S \xrightarrow[y^S]{\theta_2} S' \xrightarrow[z^{S'}]{\theta_3} BFS$$

- Developing the kinematic equations we stop at:

$$\begin{aligned} \dot{\theta}_1 &= \frac{\omega_1\cos\theta_3 - \omega_2\sin\theta_3}{\cos\theta_2} \\ \dot{\theta}_2 &= \omega_1\sin\theta_3 + \omega_2\cos\theta_3 \\ \dot{\theta}_3 &= \omega_3 + (-\omega_1\cos\theta_3 + \omega_2\sin\theta_3)\tan\theta_2 \end{aligned}$$

Spinning body subject to a constant external torque.

- Take zero initial conditions for the angles.
- With the expectation that θ_1 and θ_2 should be rather small whereas θ_3 has to be large (it is the angle of the spin axis) we replace $\cos\theta_2 \approx 1$ y $\tan\theta_2 \approx \theta_2$ (verify later!). Reaching:

$$\begin{aligned}\dot{\theta}_1 &= \omega_1\cos\theta_3 - \omega_2\sin\theta_3\\ \dot{\theta}_2 &= \omega_1\sin\theta_3 + \omega_2\cos\theta_3\\ \dot{\theta}_3 &= \omega_3 + \theta_2\left(-\omega_1\cos\theta_3 + \omega_2\sin\theta_3\right) = \omega_3 - \theta_2\dot{\theta}_1\end{aligned}$$

- Assume as well $\omega_3 \gg \theta_2\dot{\theta}_1$, then we find $\theta_3 = \omega_3 t = nt$.
- The equations for θ_1 y θ_2 are:

$$\begin{aligned}\dot{\theta}_1 &= \omega_1\cos nt - \omega_2\sin nt\\ \dot{\theta}_2 &= \omega_1\sin nt + \omega_2\cos nt\end{aligned}$$

- Substituting the values of ω_1 and ω_2 previously found:

$$\begin{aligned}\dot{\theta}_1 &= \frac{\mu}{\lambda}\sin\lambda t\cos nt - \frac{\mu}{\lambda}(\cos\lambda t - 1)\sin nt = \frac{\mu}{\lambda}\left(\sin\left(\lambda - n\right)t + \sin nt\right)\\ \dot{\theta}_2 &= \frac{\mu}{\lambda}\sin\lambda t\sin nt + \frac{\mu}{\lambda}(\cos\lambda t - 1)\cos nt = \frac{\mu}{\lambda}\left(\cos\left(\lambda - n\right)t - \cos nt\right)\end{aligned}$$

Spinning body subject to a constant external torque.

- By simple integration and using the initial condition we reach

$$\begin{aligned}\theta_1 &= \frac{\mu}{\lambda}\left(\frac{1-\cos(\lambda-n)\,t}{\lambda-n}+\frac{1-\cos nt}{n}\right)\\ \theta_2 &= \frac{\mu}{\lambda}\left(\frac{\sin(\lambda-n)\,t}{\lambda-n}-\frac{\sin nt}{n}\right)\end{aligned}$$

- Defining $A_p = \frac{\mu}{\lambda(n-\lambda)}$ y $\omega_p = n - \lambda$, amplitude and frequency of precession, respectively, and $A_n = \frac{\mu}{\lambda n}$ y $\omega_n = n$, amplitude and frequency of nutation, respectively. The solution is then written as:

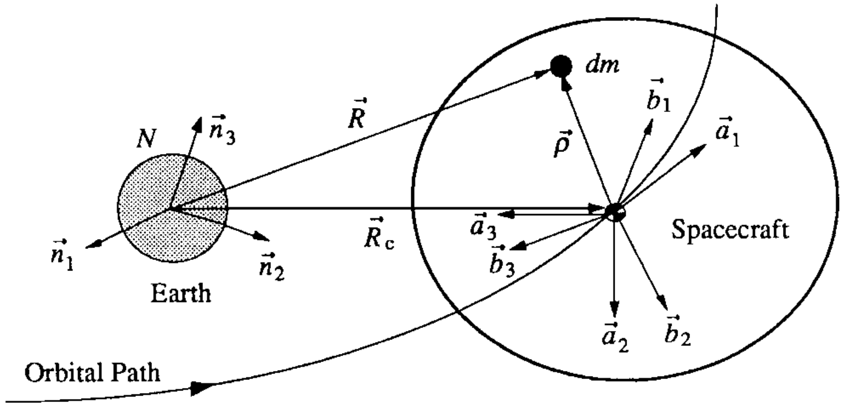
$$\begin{aligned}\theta_1 &= -A_p\left(1-\cos\omega_p t\right)+A_n\left(1-\cos\omega_n t\right)\\ \theta_2 &= A_p\sin\omega_p t - A_n\sin\omega_n t\end{aligned}$$

- Superposition of two circular movements: epicycloid.
- Amplitudes are given by $A_p = \frac{M_1}{(I-I_3)n^2}\frac{I}{I_3}$ y $A_n = \frac{M_1}{(I-I_3)n^2}$, and the gyroscopic effect increases as n, I_3/I, and the difference $I - I_3$ increases. The amplitudes should be small for the assumptions to be true: large n.

Gravity gradient.

- The most important perturbation torque is gravity gradient, as it is always present in orbit.
- Simplification: consider an asymmetrical spacecraft in circular orbit with radius R around an spherical planets; elliptical orbits and/or deviations from speherical gravity (i.e. the J_2 perturbation) introduce higher-order terms that we do not analyze (they produce the so-called librations: oscillations around the stable orientation).
- Angular velocity is defined as usual in body axes with respect to inertial, but the selected Euler angles are w.r.t. the orbit frame, which is non-inertial. This subtlety has to be taken into account in the analysis.
- The situation is as in the figure of the next slide. N axes are inertial, A axes are from the orbit frame (to be defined) and B the body axes (principal axes of inertia).

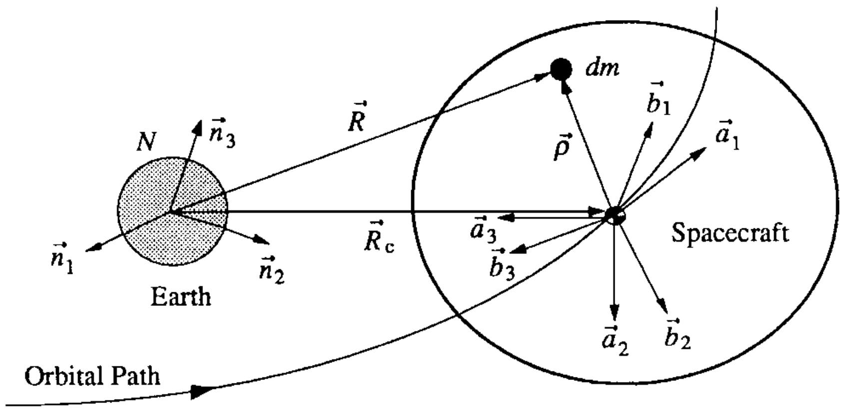
Gravity gradient.



- Orbit frame: centered in the spacecraft. The direction z ($\vec{a}_3$) points towards Earth's center (rotation:yaw). The direction x ($\vec{a}_1$) along the orbital velocity (rotation:roll). The direction y ($\vec{a}_2$) opposite to the orbital angular momentum $\vec{h}$ (orthogonal to the orbital plane, rotation:pitch).
- These axis spin with respect to the inertial frame N about the $-\vec{a}_2$ axis with angular speed $n = \sqrt{\frac{\mu_\oplus}{R^3}}$.
- Thus the relationship between frames is as follows

$$N \xrightarrow[y^n]{-nt} A \xrightarrow[z^A]{\theta_3} S \xrightarrow[y^S]{\theta_2} S' \xrightarrow[x^{S'}]{\theta_1} B$$

Gravity gradient.

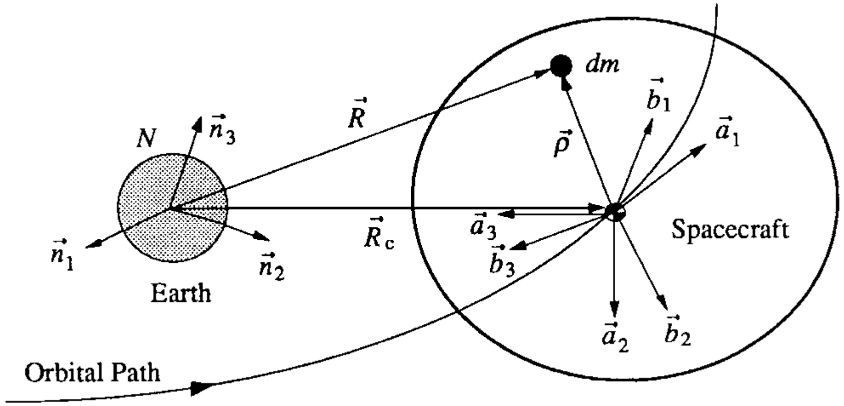


- The matrix C_A^B and its differential kinematic equation is:

$$C_A^B = \begin{bmatrix} c\theta_2 c\theta_3 & c\theta_2 s\theta_3 & -s\theta_2 \\ -c\theta_1 s\theta_3 + s\theta_1 s\theta_2 c\theta_3 & c\theta_1 c\theta_3 + s\theta_1 s\theta_2 s\theta_3 & s\theta_1 c\theta_2 \\ s\theta_1 s\theta_3 + c\theta_1 s\theta_2 c\theta_3 & -s\theta_1 c\theta_3 + c\theta_1 s\theta_2 s\theta_3 & c\theta_1 c\theta_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{c\theta_2} \begin{bmatrix} c\theta_2 & s\theta_2 s\theta_1 & s\theta_2 c\theta_1 \\ 0 & c\theta_1 c\theta_2 & -s\theta_1 c\theta_2 \\ 0 & s\theta_1 & c\theta_1 \end{bmatrix} \vec{\omega}_{B/A}^B$$

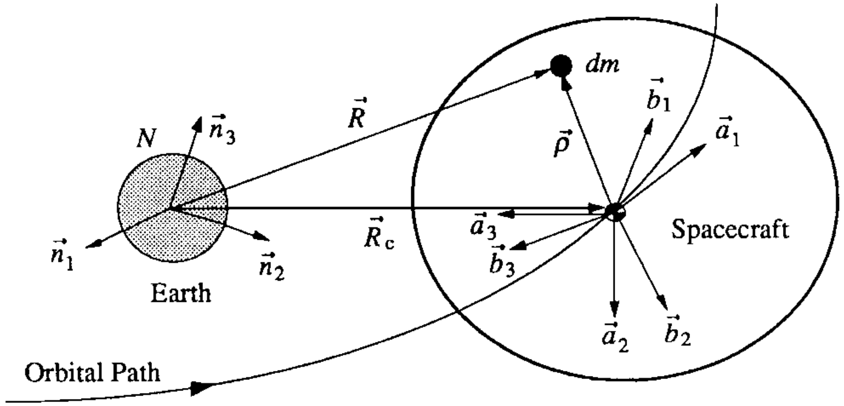
Gravity gradient.



- First let us derive the gravity gradient torque. For each dm of the spacecraft, there is an acting (gravity) force $d\vec{F} = -\frac{\mu\vec{R}}{R^3}dm = -\frac{\mu(\vec{R}_c+\vec{\rho})}{|\vec{R}_c+\vec{\rho}|^3}dm.$
- The moment of the forces is therefore:

$$\vec{M} = \int_V \rho \times d\vec{F} = -\mu \int_V \rho \times \frac{\vec{R}_c + \vec{\rho}}{|\vec{R}_c + \vec{\rho}|^3} dm = -\mu \int_V \frac{\rho \times \vec{R}_c}{|\vec{R}_c + \vec{\rho}|^3} dm$$

Gravity gradient.



- Since $|\vec{\rho}| \ll |\vec{R}_c|$, $|\vec{R}_c + \vec{\rho}|^{-3} \approx \frac{1}{R_c^3} - 3\frac{\vec{R}_c \cdot \vec{\rho}}{R_c^5}$. Then:

$$\begin{aligned}
\vec{M} &\approx -\frac{\mu}{R_c^3}\int_V \rho \times \vec{R}_c dm + 3\frac{\mu}{R_c^5}\int_V \rho \times \vec{R}_c(\vec{R}_c \cdot \vec{\rho})dm \\
&= 3\frac{\mu}{R_c^5}\int_V \rho \times \vec{R}_c(\vec{R}_c \cdot \vec{\rho})dm = -3\frac{\mu}{R_c^5}\vec{R}_c^{\times}\left(\int_V \vec{\rho}\vec{\rho}^T dm\right)\vec{R}_c \\
&= 3\frac{\mu}{R_c^5}\vec{R}_c^{\times}\mathcal{I}\vec{R}_c - 3\frac{\mu}{R_c^5}\vec{R}_c^{\times}\left(\int_V (|\vec{\rho}|^2)dm\right)\vec{R}_c = 3\frac{\mu}{R_c^5}\vec{R}_c^{\times}\mathcal{I}\vec{R}_c
\end{aligned}$$

Gravity gradient.

- Thus $\vec{M} = 3\frac{\mu}{R_c^5}\vec{R}_c^{\times}\mathcal{I}\vec{R}_c$. In the A axes, $\vec{R}_c^A = [0\ 0\ \ -R_c]^T$. Thus, in the B frame:

$$\vec{R}_c^B = C_A^B\vec{R}_c^A = -R_c\begin{bmatrix} -\mathrm{s}\theta_2 \\ \mathrm{s}\theta_1\mathrm{c}\theta_2 \\ \mathrm{c}\theta_1\mathrm{c}\theta_2 \end{bmatrix}$$

- Thus:

$$\vec{M}^B = 3\frac{\mu}{R_c^3}\begin{bmatrix} 0 & -\mathrm{c}\theta_1\mathrm{c}\theta_2 & \mathrm{s}\theta_1\mathrm{c}\theta_2 \\ \mathrm{c}\theta_1\mathrm{c}\theta_2 & 0 & \mathrm{s}\theta_2 \\ -\mathrm{s}\theta_1\mathrm{c}\theta_2 & -\mathrm{s}\theta_2 & 0 \end{bmatrix}\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{bmatrix} -\mathrm{s}\theta_2 \\ \mathrm{s}\theta_1\mathrm{c}\theta_2 \\ \mathrm{c}\theta_1\mathrm{c}\theta_2 \end{bmatrix}$$

- Operating:

$$\begin{aligned} \vec{M}^B &= 3n^2\begin{bmatrix} 0 & -\mathrm{c}\theta_1\mathrm{c}^2\theta_2 & \mathrm{s}\theta_1\mathrm{c}\theta_2 \\ \mathrm{c}\theta_1\mathrm{c}\theta_2 & 0 & \mathrm{s}\theta_2 \\ -\mathrm{s}\theta_1\mathrm{c}\theta_2 & -\mathrm{s}\theta_2 & 0 \end{bmatrix}\begin{bmatrix} -\mathrm{s}\theta_2 I_1 \\ \mathrm{s}\theta_1\mathrm{c}\theta_2 I_2 \\ \mathrm{c}\theta_1\mathrm{c}\theta_2 I_3 \end{bmatrix} \\ &= 3n^2\begin{bmatrix} -\mathrm{c}\theta_1\mathrm{c}^2\theta_2\mathrm{s}\theta_1(I_2 - I_3) \\ \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_2(I_3 - I_1) \\ \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_2(I_1 - I_2) \end{bmatrix} \end{aligned}$$

Gravity gradient.

- Replacing the gravity gradient torque in Euler's equations, we get ODEs for the angular velocity:

$$
\begin{aligned}
I_1\dot\omega_1 &= \left[\omega_2\omega_3 - 3n^2 \mathrm{c}\theta_1 \mathrm{c}^2\theta_2 \mathrm{s}\theta_1\right](I_2 - I_3)\\
I_2\dot\omega_2 &= \left[\omega_1\omega_3 + 3n^2 \mathrm{c}\theta_1 \mathrm{c}\theta_2 \mathrm{s}\theta_2\right](I_3 - I_1)\\
I_3\dot\omega_3 &= \left[\omega_2\omega_1 + 3n^2 \mathrm{s}\theta_1 \mathrm{c}\theta_2 \mathrm{s}\theta_2\right](I_1 - I_2)
\end{aligned}
$$

- On the other hand since $\vec\omega^B_{B/N} = \vec\omega^B_{B/A} + \vec\omega^B_{A/N} = \vec\omega^B_{B/A} + C^B_A \vec\omega^A_{A/N}$, there follows:

$$
\vec\omega^B_{B/A} = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} - C^B_A \begin{bmatrix}0\\ -n\\ 0\end{bmatrix} = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} + n\begin{bmatrix}\mathrm{c}\theta_2\mathrm{s}\theta_3\\ \mathrm{c}\theta_1\mathrm{c}\theta_3 + \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{s}\theta_3\\ -\mathrm{s}\theta_1\mathrm{c}\theta_3 + \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{s}\theta_3\end{bmatrix}
$$

- Then the kinematic ODEs are

$$
\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix} = \frac{1}{\mathrm{c}\theta_2}\begin{bmatrix}\mathrm{c}\theta_2 & \mathrm{s}\theta_2\mathrm{s}\theta_1 & \mathrm{s}\theta_2\mathrm{c}\theta_1\\ 0 & \mathrm{c}\theta_1\mathrm{c}\theta_2 & -\mathrm{s}\theta_1\mathrm{c}\theta_2\\ 0 & \mathrm{s}\theta_1 & \mathrm{c}\theta_1\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} + \frac{n}{\mathrm{c}\theta_2}\begin{bmatrix}\mathrm{s}\theta_3\\ \mathrm{c}\theta_2\mathrm{c}\theta_3\\ \mathrm{s}\theta_2\mathrm{s}\theta_3\end{bmatrix}
$$

Stable orientation

- System of 6 nonlinear ODEs. Making zero the derivatives we can find the equilibria:

$$
\begin{aligned}
0 &= \left[\omega_2\omega_3 - 3n^2 \mathrm{c}\theta_1 \mathrm{c}^2\theta_2 \mathrm{s}\theta_1\right](I_2 - I_3) \\
0 &= \left[\omega_1\omega_3 + 3n^2 \mathrm{c}\theta_1 \mathrm{c}\theta_2 \mathrm{s}\theta_2\right](I_3 - I_1) \\
0 &= \left[\omega_2\omega_1 + 3n^2 \mathrm{s}\theta_1 \mathrm{c}\theta_2 \mathrm{s}\theta_2\right](I_1 - I_2) \\
\vec{0} &= \frac{1}{\mathrm{c}\theta_2}\begin{bmatrix} \mathrm{c}\theta_2 & \mathrm{s}\theta_2\mathrm{s}\theta_1 & \mathrm{s}\theta_2\mathrm{c}\theta_1 \\ 0 & \mathrm{c}\theta_1\mathrm{c}\theta_2 & -\mathrm{s}\theta_1\mathrm{c}\theta_2 \\ 0 & \mathrm{s}\theta_1 & \mathrm{c}\theta_1 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \frac{n}{\mathrm{c}\theta_2}\begin{bmatrix} \mathrm{s}\theta_3 \\ \mathrm{c}\theta_2\mathrm{c}\theta_3 \\ \mathrm{s}\theta_2\mathrm{s}\theta_3 \end{bmatrix}
\end{aligned}
$$

- One equilibrium is $\omega_1 = \omega_3 = 0$, $\omega_2 = -n$, $\theta_1 = \theta_2 = \theta_3 = 0$,. Warning:there are other possible equilibria (i.e. $\theta_1 = \pi$).
- If we are close to the equilibrium and to analyze its stability, we can consider small angles and linealize the equations, finding

$$
\begin{aligned}
\dot{\omega}_1 &= -\left[n\omega_3 + 3n^2\theta_1\right](I_2 - I_3) \\
\dot{\omega}_2 &= 3n^2\theta_2(I_3 - I_1) \\
\dot{\omega}_3 &= -n\omega_1(I_1 - I_2) \\
\dot{\theta}_1 &= \omega_1 + n\theta_3 \\
\dot{\theta}_2 &= \omega_2 \\
\dot{\theta}_3 &= \omega_3 - n\theta_1
\end{aligned}
$$

Stable orientation

- Taking a derivative in the angle equations

$$\begin{aligned}\ddot{\theta}_1 &= \dot{\omega}_1 + n\dot{\theta}_3 \\ \ddot{\theta}_2 &= \dot{\omega}_2 \\ \ddot{\theta}_3 &= \dot{\omega}_3 - n\dot{\theta}_1\end{aligned}$$

- Using these equations to eliminate the ω_i's we find

$$\begin{aligned}I_1\ddot{\theta}_1 &= -\left[n\dot{\theta}_3 + 4n^2\theta_1\right](I_2 - I_3) + nI_1\dot{\theta}_3 \\ I_2\ddot{\theta}_2 &= 3n^2\theta_2(I_3 - I_1) \\ I_3\ddot{\theta}_3 &= -n(\dot{\theta}_1 - n\theta_3)(I_1 - I_2) - nI_3\dot{\theta}_1\end{aligned}$$

- The second equation is stable if $I_3 < I_1$. The first and third are more challenging. Writing the system matrix:

$$\frac{d}{dt}\begin{bmatrix}\theta_1 \\ \theta_3 \\ \dot{\theta}_1 \\ \dot{\theta}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4n^2\frac{I_3-I_2}{I_1} & 0 & 0 & n\frac{I_3-I_2+I_1}{I_1} \\ 0 & n^2\frac{I_1-I_2}{I_3} & n\frac{I_2-I_1-I_3}{I_3} & 0\end{bmatrix}\begin{bmatrix}\theta_1 \\ \theta_3 \\ \dot{\theta}_1 \\ \dot{\theta}_3\end{bmatrix}$$

- Define $k_1 = \frac{I_2-I_3}{I_1}$ y $k_3 = \frac{I_2-I_1}{I_3}$. Since $I_1 + I_2 > I_3$, $I_2 + I_3 > I_1$, $I_1 + I_3 > I_2$, one gets $k_1, k_3 \in [-1, 1]$.

Stable orientation

- The matrix writes

$$\frac{d}{dt}\begin{bmatrix}\theta_1\\ \theta_3\\ \dot\theta_1\\ \dot\theta_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -4n^2k_1 & 0 & 0 & n(1-k_1)\\ 0 & -n^2k_3 & n(k_3-1) & 0\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_3\\ \dot\theta_1\\ \dot\theta_3\end{bmatrix}$$

- Studying the eigenvalues of the matrix, we find the characteristic polynomial:
$\lambda^4 + \lambda^2 n^2(1 + k_1(3 + k_3)) + 4n^4 k_1 k_3 = 0$, cuya solución es:

$$\lambda = \pm n\sqrt{\frac{-(1 + k_1(3 + k_3)) \pm \sqrt{(1 + k_1(3 + k_3))^2 - 16k_1k_3}}{2}}$$

- Eigenvalues are stable (non-positive real part) if and only if the two options insde the square root are real and negative, this is: $-(1 + k_1(3 + k_3)) \pm \sqrt{(1 + k_1(3 + k_3))^2 - 16k_1k_3} < 0$. This only happens if:
  - $-(1 + k_1(3 + k_3)) < 0$, this is, $1 + k_1(3 + k_3) > 0$.
  - $\sqrt{(1 + k_1(3 + k_3))^2 - 16k_1k_3}$ is real, this is,$(1 + k_1(3 + k_3))^2 - 16k_1k_3 > 0$.
  - $16k_1k_3 > 0$ (if not there would be a positive number inside the root)

Stable orientation

- Plotting the conditions in a chart:

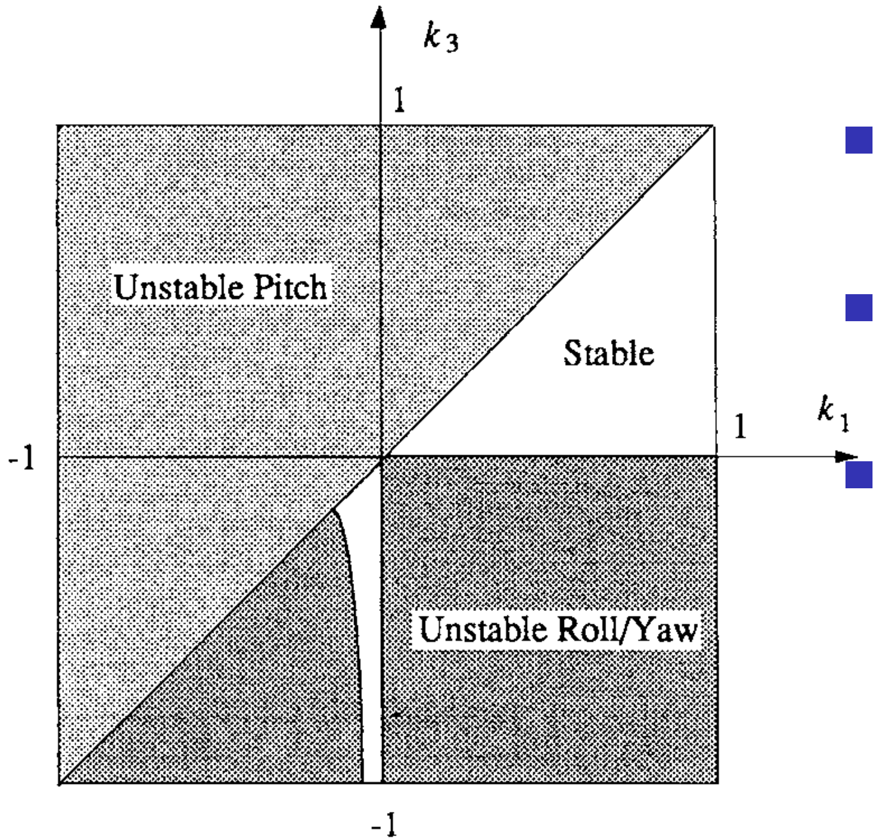


Fig. 6.9 Gravity-gradient stability plot.

- From $16k_1k_3 > 0$, we obtain k_1 and k_3 with the same sign.
- Since $I_3 < I_1$, one gets that $k_1 - k_3 > 0$.
- if $k_1 > k_3 > 0$ we obtain "Lagrange's region" (right-upper triangle).
- There is another region (known as "De Bra-Delp") obtained from $(1 + k_1(3 + k_3))^2 - 16k_1k_3 > 0$. However it is sensitive to energy dissipation, which makes it unstable.

Stable orientation

- In summary:

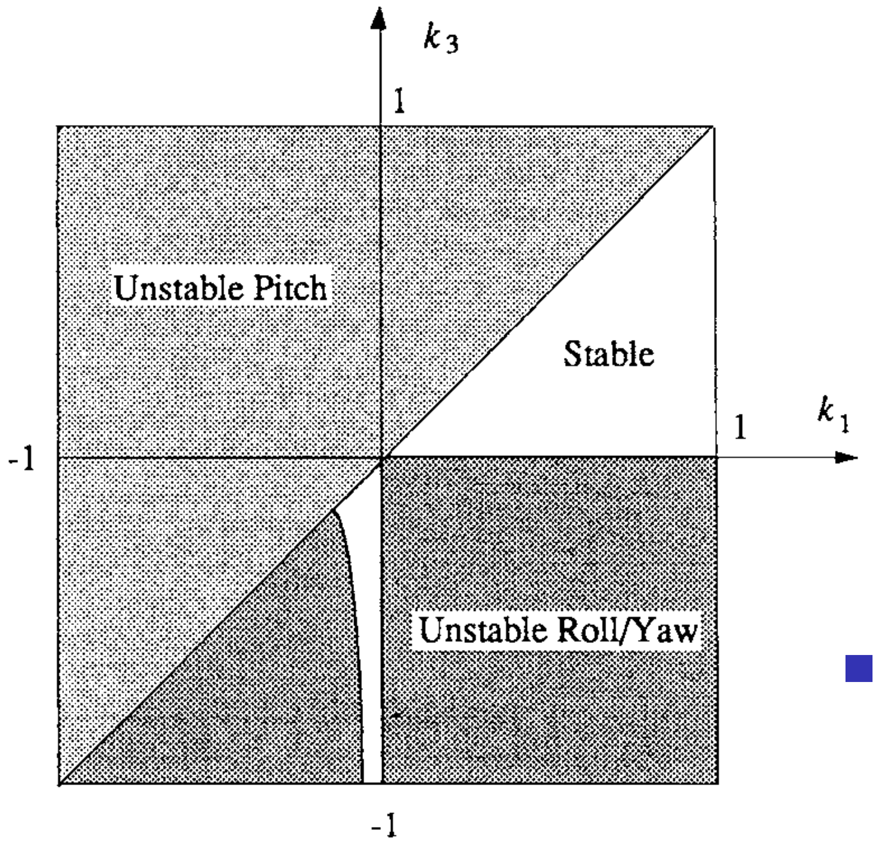


Fig. 6.9 Gravity-gradient stability plot.

- The practical stable zone corresponds to $k_1 > k_3 > 0$, which in turn implies that $I_2 > I_1$ and $I_2 > I_3$. Before we already obtained $I_3 < I_1$. Thus axis 2 (orthogonal to the orbital plane) must be the major axis, axis 3 (pointing to the planet) the minor axis of inertia, and axis 1 (in the direction of orbital velocity) the intermediate.
- Careful: the angles at the equilibrium are 0° but they can also be 180° (the “opposite” attitude is also stable!).
- How many stable equilibria? How many unstable equilibria?