Spacecraft Dynamics

Lesson 2: Attitude Representation

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October 18, 2021

Spacecraft Attitude

- The attitude of a Spacecraft is its orientation with respect to a given reference frame (typically, inertial or orbit axes).
- Under the hypothesis of the spacecraft being a rigid body, it is enough to know the orientation of the body axes (i.e., a reference frame fixed to the spacecraft). Thus one needs to study the orientation of a reference frame w.r.t. another.
- The set of orientations between two frames is denoted as SO(3): the special orthogonal group of dimension 3.
- Aircraft classically use Euler angles (yaw, pitch, roll). For spacecraft there are several alternatives (also applicable to aircraft), with their corresponding advantages and disadvantages:
  - Director Cosine Matrix (DCM)
  - Euler Angles (12 possible sets)
  - Euler's Angle and Axis (a.k.a. Eigenaxis)
  - Rotation vector
  - Quaternions
  - Rodrigues parameters (a.k.a. Gibbs' vector)
  - Modified Rodrigues parameters

SO(3) Representations: Main features

- Each representation has advantages and disadvantages, as will be seen.
- Each representation is defined by n parameters.
  - If $n = 3$ the representation is *minimal* (since there are 3 degrees of freedom). However, minimal representations always have singularities.
  - If $n > 3$ then there will be $n - 3$ constraints for the parameters.
- For a given representation, it might happen that two different values of the parameters represent the same physical attitude. Then, it is said that the representation has ambiguities. The set of parameters that needs to be eliminated to avoid ambiguities is called the "shadow set".
- In this lesson we study:
  - How to switch between different representations
  - How to compose attitudes for each representation when there are more than 2 reference frames

SO(3) Representations: Main features

- Another interesting feature is the capacity to generate smooth "paths" of attitude, this is, a continuous set of rotations to get from an initial attitude to a final attitude.
- One can talk about passive and active interpretations between reference frames.
- In the passive representation (a.k.a. "alias") one transform the reference frames (i.e. their basis vectors). Then, vectors also transform since the reference frame change. However, they do so in the opposite way. For instance, if the x-y axes rotate 45º (along the z axis), a vector would rotate 45º in the opposite direction (along the -z axis). This is the preferred interpretation. Plot it!
- The active interpretation (a.k.a. "alibi") looks at the transformation of vectors (therefore reference frames transform in the opposite way).

Director Cosine Matrix (DCM) I

- Let S and S' be reference frames, respectively, with unitary basis vectors $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ and $(\vec{e}_{x'}, \vec{e}_{y'}, \vec{e}_{z'})$. The orientation (attitude) of S' w.r.t. S is totally determined by the change of basis matrix $C_S^{S'}$. This matrix allows, given any generic vector $\vec{v}$ expressed in the basis of S as $\vec{v}^S$, to change its basis as follows: $\vec{v}^{S'} = C_S^{S'} \vec{v}^S$. Denote:

$$C_S^{S'} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

- Note: $\vec{e}_x^{S'} = C_S^{S'} e_x^S = C_S^{S'} [1\ 0\ 0]^T = [c_{11}\ c_{21}\ c_{31}]^T$.
- Therefore:
  $\vec{e}_{x'} \cdot \vec{e}_x = (\vec{e}_{x'}^{S'})^T \vec{e}_x^{S'} = [1\ 0\ 0][c_{11}\ c_{21}\ c_{31}]^T = c_{11}$.
- In addition:

$$\begin{aligned} c_{21} &= \vec{e}_{y'} \cdot \vec{e}_x, \quad c_{31} = \vec{e}_{z'} \cdot \vec{e}_x \\ c_{12} &= \vec{e}_{x'} \cdot \vec{e}_y, \quad c_{22} = \vec{e}_{y'} \cdot \vec{e}_y, \quad c_{32} = \vec{e}_{z'} \cdot \vec{e}_y \\ c_{13} &= \vec{e}_{x'} \cdot \vec{e}_z, \quad c_{23} = \vec{e}_{y'} \cdot \vec{e}_z, \quad c_{33} = \vec{e}_{z'} \cdot \vec{e}_z \end{aligned}$$

Director Cosine Matrix (DCM) II

- Thus:

$$C_S^{S'} = \begin{bmatrix} \vec{e}_{x'} \cdot \vec{e}_x & \vec{e}_{x'} \cdot \vec{e}_y & \vec{e}_{x'} \cdot \vec{e}_z \\ \vec{e}_{y'} \cdot \vec{e}_x & \vec{e}_{y'} \cdot \vec{e}_y & \vec{e}_{y'} \cdot \vec{e}_z \\ \vec{e}_{z'} \cdot \vec{e}_x & \vec{e}_{z'} \cdot \vec{e}_y & \vec{e}_{z'} \cdot \vec{e}_z \end{bmatrix}$$

- By a similar reasoning:

$$C_{S'}^{S} = \begin{bmatrix} \vec{e}_{x'} \cdot \vec{e}_x & \vec{e}_{y'} \cdot \vec{e}_x & \vec{e}_{z'} \cdot \vec{e}_x \\ \vec{e}_{x'} \cdot \vec{e}_y & \vec{e}_{y'} \cdot \vec{e}_y & \vec{e}_{z'} \cdot \vec{e}_y \\ \vec{e}_{x'} \cdot \vec{e}_z & \vec{e}_{y'} \cdot \vec{e}_z & \vec{e}_{z'} \cdot \vec{e}_z \end{bmatrix} = (C_S^{S'})^T$$

- And since $C_{S'}^{S} = (C_S^{S'})^{-1}$, we get that $C_{S'}^{S}$ is *orthogonal*, this is: $(C_S^{S'})^{-1} = (C_S^{S'})^T$. The name "Director Cosine Matrix" is also justified since the dot product of unitary vectors is the cosine of the angle they form.
- Another property is that $\det(C_{S'}^{S}) = 1$. This is due to the fact that $1 = \det(\mathrm{Id}) = \det((C_{S'}^{S})(C_{S'}^{S})^{-1}) = \det((C_{S'}^{S})(C_{S'}^{S})^{T}) = \left(\det(C_{S'}^{S})\right)^2$. Therefore $\det(C_{S'}^{S}) = \pm 1$. The sign $+$ corresponds to both S and S' being right-handed reference frames, which are the ones used in practice.

Director Cosine Matrix (DCM) III

- This attitude representation has 9 parameters. These are dependent from each other, this is, the coefficients of the C matrix cannot be arbitrary (the matrix has to be orthogonal and with determinant 1). In particular, one must have 6 independent constraints which determine that the matrix is orthogonal.
- Composition: assume that the attitude of S_2 w.r.t S_1 is given by $C_{S_1}^{S_2}$ and the attitude of S_3 w.r.t S_2 is given by $C_{S_2}^{S_3}$. Then it it easy to see that the attitude of S_3 w.r.t. S_1 can be found by applying the succesive transformations, this is, $C_{S_1}^{S_3} = C_{S_2}^{S_3} C_{S_1}^{S_2}$. Therefore attitude "composition" is given by a simple matrix product (note that the order matters: non-commutativity of rotations).

Euler angles I

- In general attitude can be mathematically described by three rotations in the main axes, where any axis can be selected for the first, second and third rotation with the only rule that one cannot repeat a consecutive axis (i.e. 1st and 2nd, and 2nd and 3rd must be different).
- As an example, the classical aircraft rotation sequence is:

$$n \xrightarrow[z^n]{\psi} S \xrightarrow[y^S]{\theta} S' \xrightarrow[x^{S'}]{\varphi} BFS$$

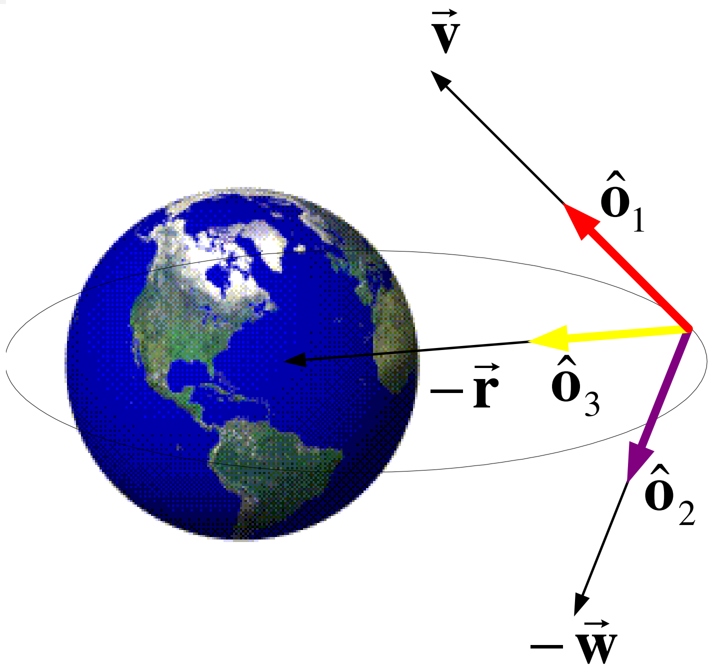
- There exists other options, more suited to spacecraft:

$$n \xrightarrow[x^n]{\theta_1} S \xrightarrow[y^S]{\theta_2} S' \xrightarrow[z^{S'}]{\theta_3} BFS \qquad n \xrightarrow[z^n]{\Omega} S \xrightarrow[x^S]{i} S' \xrightarrow[z^{S'}]{\omega} BFS$$

- There are 12 possible sequences of Euler angles to represent the attititude. This is a minimal representation (3 angles).
- One can obtain the DCM from Euler angles by multiplying elementary rotation matrices. For instance $C_n^b(\psi,\theta,\varphi) = C_{S'}^b(\varphi) C_S^{S'}(\theta) C_n^S(\psi)$.

Euler angles II

- In the figure, the typical aircraft Euler angles are used w.r.t. orbit axes.
  - First a rotation around the axis labelled as 3 (yellow): yaw.
  - Next, a rotation about the resulting axis 2: pitch
  - Finally, a rotation about the resulting axis 3: roll
- Notice that a rotation affects the position of the axes for the next rotations.
- This sequence is denoted as (3,2,1). The other sequences of Euler angles contained in the previous slide are, respectively, (1,2,3) and (3,1,3).
- One can choose a sequence depending on the angles which are of interest for a given application or study (see Lesson 5).



- Other possible sequences: (1,2,1), (1,3,1), (1,3,2), (2,1,2), (2,1,3), (2,3,1), (2,3,2), (3,1,2), (3,2,3).

Euler angles III

- For the sequence (3,2,1) with angles denoted as (ψ, θ, φ), one has:

$$C_n^b = \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ -c\varphi s\psi + s\varphi s\theta c\psi & c\varphi c\psi + s\varphi s\theta s\psi & s\varphi c\theta \\ s\varphi s\psi + c\varphi s\theta c\psi & -s\varphi c\psi + c\varphi s\theta s\psi & c\varphi c\theta \end{bmatrix}$$

- Notice that $(180^{\circ} + \psi, 180^{\circ} - \theta, 180^{\circ} + \varphi)$ defines the same attitude that (ψ, θ, φ). Therefore typically one limits $\theta \in [-90^{\circ}, 90^{\circ}]$ (the angles that are excluded from these values constitute the shadow set).
- Given the DCM, to obtain the Euler angles, one can derive the following formulas:
  1. $\theta = -\arcsin c_{13}$.
  2. From $\cos\psi = c_{11}/\cos\theta$, $\sin\psi = c_{12}/\cos\theta$, obtain ψ.
  3. From $\sin\varphi = c_{23}/\cos\theta$, $\cos\varphi = c_{33}/\cos\theta$, obtain φ.
- For other sequences, one can get similar relations from the explicit expression of the DCM.

Euler angles IV

- Main advantage: physically meaningful.
- One has, however, to be careful when composing attitude.
- Suppose the attitude of S_2 w.r.t. S_1 is given by $(\psi_1, \theta_1, \varphi_1)$ and the attitude of S_3 w.r.t. S_2 is given by $(\psi_2, \theta_2, \varphi_2)$. Denote as $(\psi_3, \theta_3, \varphi_3)$ the attitude of S_3 w.r.t. S_1. In general $\psi_3 \neq \psi_1 + \psi_2$, $\theta_3 \neq \theta_1 + \theta_2$, $\varphi_3 \neq \varphi_1 + \varphi_2$!!
- The best way to obtain $(\psi_3, \theta_3, \varphi_3)$ is to compute them from $C_{S_1}^{S_3} = C_{S_2}^{S_3}(\psi_2, \theta_2, \varphi_2) C_{S_1}^{S_2}(\psi_1, \theta_1, \varphi_1)$. This is, going to a DCM representation, composing, and going back to Euler angles.
- This shows that it might be complex to work with Euler angles.
- Main disadvantage: singularities (as will be seen in Lesson 4).

Euler's angle and axis I

- Euler's Rotation Theorem: "the most general movement of a solid with a fixed point is a single rotation around a unique axis."
- Note: We are considering a rotation at a given time (a "snapshot"), not a rotation that is changing as time evolves (that is the subject of Lesson 4).
- Let us call a unit vector in the direction of that axis (Euler's Axis) as $\vec{e}_{S/S'}$, and the magnitude of the rotation (Euler's Angle) as θ.
- Thus, $\|\vec{e}_{S/S'}\| = 1$ and if we write $\vec{e}^{S'}_{S/S'} = [e_x \; e_y \; e_z]^T$ it follows that $e_x^2 + e_y^2 + e_z^2 = 1$.
- A useful formalism is the following. Given a vector $\vec{v} = [v_x \; v_y \; v_z]^T$ define the operator $\times$ acting on $\vec{v}$ (denoted $\vec{v}^{\times}$) as follows:

$$\vec{v}^{\times} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$$

Euler's angle and axis II

- The operation $\vec{v}^{\times}$ helps to quickly compute the cross product $\vec{v} \times \vec{w}$, for any vector $\vec{w}$, in a reference frame S: $(\vec{v} \times \vec{w})^S = \left(\vec{v}^S\right)^{\times} \vec{w}^S$.
- Thus, if the attitude using Euler's angle and axis is given by $(\vec{e}^{S'}_{S/S'}, \theta)$, how to go from there to the DCM and the other way around? The $\times$ operator helps.
- One has
  $C_S^{S'} = \cos\theta \mathrm{Id} + (1-\cos\theta)\vec{e}^{S'}_{S/S'}(\vec{e}^{S'}_{S/S'})^T - \sin\theta \left(\vec{e}^{S'}_{S/S'}\right)^{\times}$.
  This is known as the Euler-Rodrigues formula and it is mathematically proven later.
- On the other hand, $C_S^{S'}$, and computing $\mathrm{Tr}(C_S^{S'})$ and $(C_S^{S'})^T - C_S^{S'}$, one gets:

$$\begin{aligned} \cos\theta &= \frac{\mathrm{Tr}(C_S^{S'}) - 1}{2} \\ \left(\vec{e}^{S'}_{S/S'}\right)^{\times} &= \frac{1}{2\sin\theta}\left((C_S^{S'})^T - C_S^{S'}\right) \end{aligned}$$

Euler's angle and axis III

- Another relationship between Euler's angle and axis and the Director Cosine Matrix is given by the algebraic properties of the DCM.
- Since the DCM is orthogonal, it can be shown that 1 is always an eigenvalue of it. If C is the DCM, then the eigenvector associated to the 1 is the Euler's axis $\vec{e}$ since $C\vec{e} = \vec{e}$.
- On the other hand, the other two eigenvalues of the DCM are precisely $\mathrm{e}^{i\theta}$, $\mathrm{e}^{-i\theta}$.
- This is another way of computing Euler's angle and axis, by evaluating the eigenvalues and eigenvectors of the DCM.

Euler's angle and axis IV

- Therefore, in this representation, one describes the attitude with four parameters: three componentes of an unit vector and an angle. These have a clear physical meaning.
- Notice that the attitude given by $(\vec{e}^{S'}_{S/S'}, \theta)$ and by $(-\vec{e}^{S'}_{S/S'}, 360^{\text{o}} - \theta)$ is exactly the same. To avoid this ambiguity, one can constraint θ to $[0, 180^{\text{o}})$.
- The "opposite" attitude (the one from S w.r.t. S') is given by $(-\vec{e}^{S}_{S'/S}, \theta)$. Notice also that $e^{S}_{S'/S} = e^{S'}_{S'/S}$.
- Composition: if the attitude of S_2 w.r.t. S_1 is given by $(\vec{e}^{S_2}_{S_1/S_2}, \theta_1)$ and the attitude of S_3 w.r.t. S_2 is given by $(\vec{e}^{S_3}_{S_2/S_3}, \theta_2)$, then, denoting as $(\vec{e}^{S_3}_{S_1/S_3}, \theta_3)$ the attitude of S_3 w.r.t. S_1, one obtains:

$$\begin{aligned}
\cos\theta_3 &= -\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2(\vec{e}_{S_1/S_2}\cdot\vec{e}_{S_2/S_3}) \\
e^{S_3}_{S_1/S_3} &= \frac{1}{\sin\theta_3}\left(\sin\theta_1\cos\theta_2\vec{e}_{S_1/S_2} + \cos\theta_1\sin\theta_2\vec{e}_{S_2/S_3} + \sin\theta_1\sin\theta_2(\vec{e}_{S_1/S_2}\times\vec{e}_{S_2/S_3})\right)
\end{aligned}$$

Rotation vector

- A minimal attitude representation can be obtained by combining Euler's axis and angle in a single vector as follows: $\vec{\theta} = \theta \vec{e}$.
- This representation can be useful as it physically represents the angular speed one would need to maintain constant from a second for one reference frame respect to another, that start being the same, to obtain the attitude given by $(\vec{e}, \theta)$.
- On the other hand for large rotations it is not an adequate rotation. Note that a rotation of 0° and 360° are physically the same but the first is $\vec{\theta} = \vec{0}$ and the second is not univocally defined.
- Thus, the representation is reserved for theoretical analysis of for small angles (or to determine the angular velocity necessary to perform a fixed rotation).

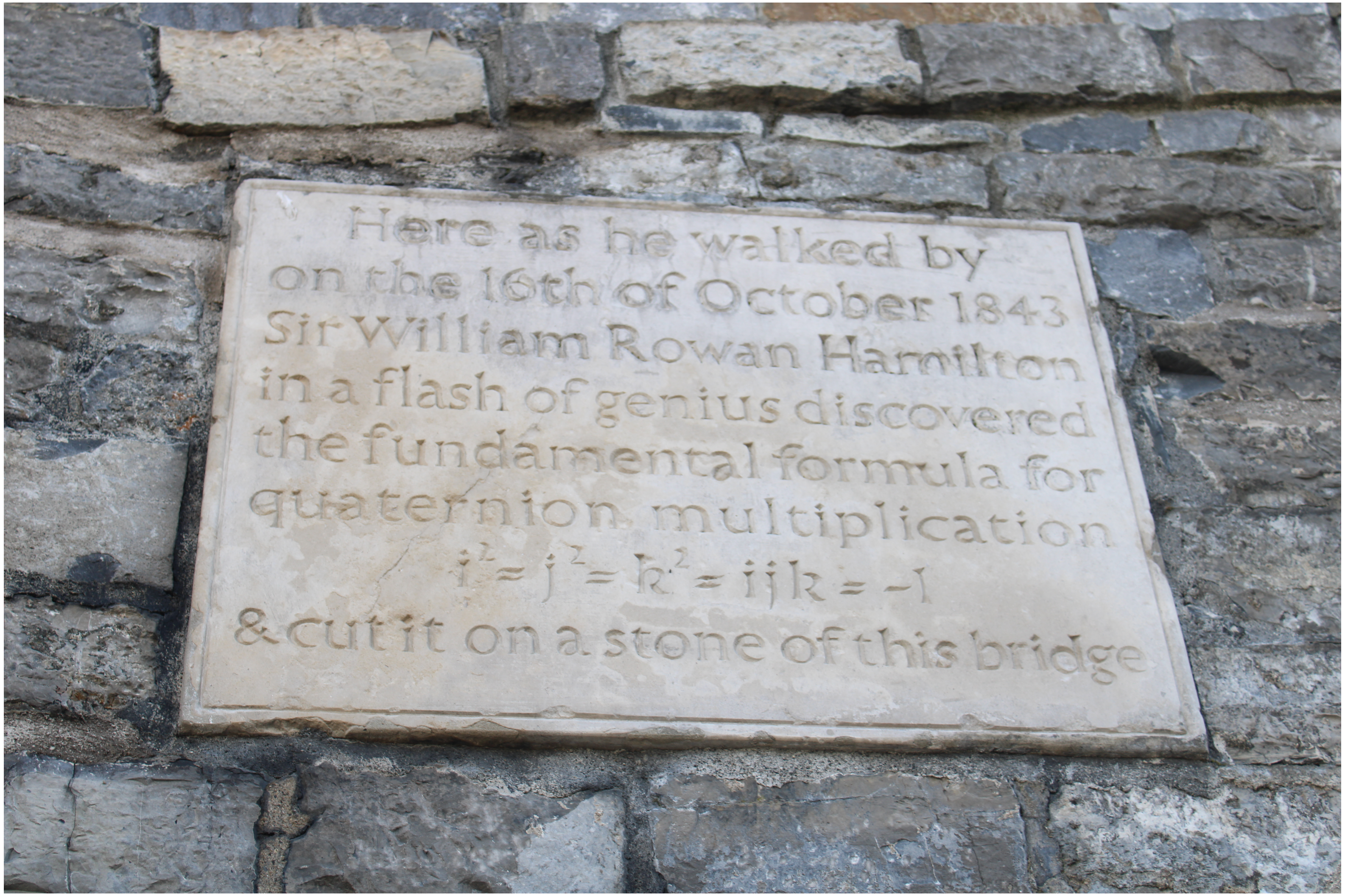
Quaternions

- Quaternions were first described by Hamilton (19th century), who considered them his greatest creation; he thought they were going to be used as Physics "universal language". However, they were soon substituted by vectors (Gibbs) and matrices (Cayley).
- Remember a complex number z can be thought of as a " 2-D vector", which can be written in terms of its components as $z = x + iy$. Complex number of unity modulus can be used to represent a 2-D rotation, since if $|z| = 1$, one can write $z = \mathrm{e}^{i\theta}$, and it is well-known multiplying by this number rotates the phase by an angle θ.
- Quaternions extend complex number to "4 dimensions". A quaternion q can be written as: $q = q_0 + iq_1 + jq_2 + kq_3$.
- q_0 is the scalar part and $\vec{q} = [q_1\ q_2\ q_3]^T$ the "vector part" of q.

Quaternion Algebra I

- To better understand Quaternions it's important to know their algebraic properties, this is, how to operate with Quaternions.
- Sum: Component-wise, i.e., given $q = q_0 + iq_1 + jq_2 + kq_3$ and $q' = q'_0 + iq'_1 + jq'_2 + kq'_3$, one has that $q'' = q + q' = q''_0 + iq''_1 + jq''_2 + kq''_3$ is given by the obvious formulae:
  $q''_0 = q_0 + q'_0$, $q''_1 = q_1 + q'_1$, $q''_2 = q_2 + q'_2$, $q''_3 = q_3 + q'_3$.
- Product: denote by $\star$, again, component-wise, knowing the following rules of multiplication:
  $i \star i = -1$, $i \star j = k$, $i \star k = -j$, $j \star i = -k$, $j \star j = -1$,
  $j \star k = i$, $k \star i = j$, $k \star j = -i$, $k \star k = -1$.
- Hamilton's formula follows: $i \star j \star k = -1$.
- Notice that $q \star q' \neq q' \star q$: Quaternion multiplication is NOT commutative!

Quaternions: Plaque on Broom Bridge (Dublin)



Quaternion Algebra II

- Matrix form of the product: It is possible to write the product $q'' = q' \star q$ in matrix form as follows:

$$\begin{bmatrix} q''_0 \\ q''_1 \\ q''_2 \\ q''_3 \end{bmatrix} = \begin{bmatrix} q'_0 & -q'_1 & -q'_2 & -q'_3 \\ q'_1 & q'_0 & -q'_3 & q'_2 \\ q'_2 & q'_3 & q'_0 & -q'_1 \\ q'_3 & -q'_2 & q'_1 & q'_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

- "vector" form of the product: $q''_0 = q'_0 q_0 - \vec{q}'^T \vec{q}$, $\vec{q}'' = q_0 \vec{q}' + q'_0 \vec{q} + \vec{q}' \times \vec{q}$.
- Conjugate: As for complex numbers, given $q = q_0 + iq_1 + jq_2 + kq_3$ one defines the conjugate of q as $q^* = q_0 - iq_1 - jq_2 - kq_3$.
- Modulus: The definition of the modulus of $q = q_0 + iq_1 + jq_2 + kq_3$ is $|q|^2 = q \star q^* = q_0^2 + q_1^2 + q_2^2 + q_3^2$. Property: $|q \star q'| = |q||q'|$.
- Division: One defines division using the conjugate: $q'/q = q'/q \star q^*/q^* = (q' \star q^*)/|q|^2$.

Attitude representation using Quaternions I

- Given the attitude represented by Euler's axis and angle, $\vec{e}$ and θ, one can "codify" that attitude in terms of Quaternions as follows: $q_0 = \cos\theta/2$, $\vec{q} = \sin\theta/2\vec{e}$.
- Notice therefore that if q represents an attitude, it follows that $|q| = 1$ (and vice-versa!).
- Remember the $\times$ operator and apply it to the quaternion $\vec{q}^{\times}$:
  $\vec{q}^{\times} = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$
- To go from DCM C to Quaternions, use:$q_0 = \frac{\sqrt{1+\mathrm{Tr}(C)}}{2}$ y $\vec{q}^{\times} = \frac{1}{4q_0}\left(C^T - C\right)$.
- To go from Quaternions to DCM use Euler-Rodrigues formula for Quaternions:
  $C = \left(q_0^2 - \vec{q}^T\vec{q}\right)\mathrm{Id} + 2\vec{q}\vec{q}^T - 2q_0\vec{q}^{\times}$.
- One can transform a vector $\vec{v}$ without need of the DCM using the formula: $\begin{bmatrix} 0 \\ \vec{v}^B \end{bmatrix} = q^*_{B/A} \star \begin{bmatrix} 0 \\ \vec{v}^A \end{bmatrix} \star q_{B/A}$

Attitude representation using Quaternions II

- Euler-Rodrigues formula in matrix form:

$$C(q) = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 + q_0 q_3) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 + q_0 q_1) \\ 2(q_1 q_3 + q_0 q_2) & 2(q_2 q_3 - q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

- Quaternions are an attitude representation that requires 4 parameters, with the additional constraint $|q| = 1$.
- Ambiguities: q and $-q$ represent the same attitude, since if q corresponds to $(\vec{e}, \theta)$, then $-q$ corresponds to $(-\vec{e}, 360 - \theta)$. Prove it!
- Disadvantage: no physical sense unless you have some experience using them.
- Notice: To convert from DCM to Quaternions and back no trig formulas are required, increasing the precission.
- If $q_{S'S}$ represents the attitude of S' w.r.t. S y $q_{S''S'}$ represents the attitude of S" w.r.t. S', then $q_{S''S}$, the attitude of S" w.r.t. S, can be computed $q_{S''S} = q_{S'S} \star q_{S''S'}$ (notice that the product is in the other direction, comparing with the DCM).

Computing Quaternions from Euler angles

- For the classical (3,2,1) sequence, notice that
  - The quaternion corresponding to the Euler angles $(\psi, 0, 0)$ is $q_\psi = \cos\psi/2 + k\sin\psi/2$.
  - The quaternion corresponding to the Euler angles $(0, \theta, 0)$ is $q_\theta = \cos\theta/2 + j\sin\theta/2$.
  - The quaternion corresponding to the Euler angles $(0, 0, \varphi)$ is $q_\varphi = \cos\varphi/2 + i\sin\varphi/2$.
- Thus, given the Euler angles (ψ, θ, φ) one obtains a corresponding quaternion using the composition rule as $q = q_\psi \star q_\theta \star q_\varphi$.
- Explicitly doing the product one gets

$$\begin{aligned}
q &= (\cos\psi/2\cos\theta/2\cos\varphi/2 + \sin\psi/2\sin\theta/2\sin\varphi/2) \\
&\quad + i\,(\cos\psi/2\cos\theta/2\sin\varphi/2 - \sin\psi/2\sin\theta/2\cos\varphi/2) \\
&\quad + j\,(\cos\psi/2\sin\theta/2\cos\varphi/2 + \sin\psi/2\cos\theta/2\sin\varphi/2) \\
&\quad + k\,(\sin\psi/2\cos\theta/2\cos\varphi/2 - \cos\psi/2\sin\theta/2\sin\varphi/2)\,.
\end{aligned}$$

Quaternions: a word of caution

- Careful: some authors (STK as well) write q_4 instead of q_0 so the scalar part is the last component of the quaternion.
- Some authors define the quaternion product in an opposite way, so $i \star j = -k$, etc. The consequence of this is that many formulas change:
  - The quaternion composition rule now is as for the matrices (from right to left).
  - The formula for vector transformation becomes $\begin{bmatrix} 0 \\ \vec{v}^B \end{bmatrix} = q_{B/A} \star \begin{bmatrix} 0 \\ \vec{v}^A \end{bmatrix} \star q^*_{B/A}$
- Also, if one wants to use our definition of quaternions but to rotate a vector (instead of changing its reference frame, this is, to use the active interpretation) then: $\begin{bmatrix} 0 \\ \vec{v}' \end{bmatrix} = q \star \begin{bmatrix} 0 \\ \vec{v} \end{bmatrix} \star q^*$ where $\vec{v}'$ is the vector $\vec{v}$ rotated by an axis and angle defined by q, which is the formula one may find over the internet.

Quaternions: shortest path and interpolation

- Given two quaternions q_0 and q_1 representing two different attitudes, can one construct a ‘ìnterpolation path,” continuous, $q(s)$ such that $q(0) = q_0$ and $q(1) = q_1$?
- The way to do it is to first find q_2 representing the attitude between q_0 and q_1 (the rotation quaternion): $q_2 = \frac{1}{q_0} \star q_1 = q_0^* q_1$. From this quaternion extract Euler's angle and axis (θ and $\vec{e}$):$q_2 = \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2\vec{e} \end{bmatrix}$.
- Now the solution of the problem is $q(s)$ which is the product of q_0 and another quaternion coming from Euler's axis $\vec{e}$ and angle $s\theta$, so that when $s = 0$ it is the unity quaternion (and the product is q_0) and when $s = 1$ it is q_2 (and the product is q_1):

$$q(s) = q_0 \star \begin{bmatrix} \cos(s\theta/2) \\ \sin(s\theta/2)\vec{e} \end{bmatrix}$$

Rodrigues Parameters I

- Atittude representation using Rodrigues Parameters (RP, also called Gibbs vector) can be easily obtained from the quaternion as $\vec{g} = \frac{\vec{q}}{q_0}$, obviously this is only valid if $q_0 > 0$ $(i.e.\theta < 180^\circ)$ because otherwise one gets a singularity. To recover the quaternion from $\vec{g}$:

$$\|\vec{g}\|^2 = \frac{\|\vec{q}\|^2}{q_0^2} = \frac{1 - q_0^2}{q_0^2}$$

Thus $q_0 = \frac{\pm 1}{\sqrt{1+\|\vec{g}\|^2}}$. And therefore:

$$q = \frac{\pm 1}{\sqrt{1 + \|\vec{g}\|^2}} \begin{bmatrix} 1 \\ \vec{g} \end{bmatrix}$$

- In terms of Euler's axis and angle, $\vec{g} = \vec{e} \tan \frac{\theta}{2}$.

Rodrigues Parameters II

- The relationship with the DCM is as follows:

$$C = \mathrm{Id} + 2\frac{\vec{g}^{\times}\vec{g}^{\times} - \vec{g}^{\times}}{1 + \|\vec{g}\|^2} = (\mathrm{Id} - \vec{g}^{\times})(\mathrm{Id} + \vec{g}^{\times})^{-1} = (\mathrm{Id} + \vec{g}^{\times})^{-1}(\mathrm{Id} - \vec{g}^{\times})$$

- On the other hand, since $q_0 = \frac{\sqrt{1+\mathrm{Tr}(C)}}{2}$ and $\vec{q}^{\times} = \frac{1}{4q_0}\left(C^T - C\right)$, one gets:

$$\vec{g}^{\times} = \frac{q^{\times}}{q_0} = \frac{1}{4q_0^2}\left(C^T - C\right) = \frac{C^T - C}{1 + \mathrm{Tr}(C)}$$

- Composition follows a simple rule.If $\vec{g}_{S'S}$ represents the attitude of S' w.r.t. S and $\vec{g}_{S''S'}$ represents the attitude of S" w.r.t. S', then $\vec{g}_{S''S}$, the attitude of S" w.r.t. S, is computed as:

$$\vec{g}_{S''S} = \frac{\vec{g}_{S''S'} + \vec{g}_{S'S} - \vec{g}_{S''S'} \times \vec{g}_{S'S}}{1 - \vec{g}_{S'S} \cdot \vec{g}_{S''S'}}$$

Modified Rodrigues Parameters

- The representation using Modified Rodrigues Parameters (MRP) is quite recent (1962) but popular in control applications. Similarly to RP, one can get it from the quaternion, by defining $\vec{p} = \frac{\vec{q}}{1+q_0}$. To recover the quaternion from the MRP:

$$\|\vec{p}\|^2 = \frac{\|\vec{q}\|^2}{(1+q_0)^2} = \frac{1-q_0^2}{(1+q_0)^2} = \frac{1-q_0}{1+q_0}$$

Then $q_0 = \frac{1-\|\vec{p}\|^2}{1+\|\vec{p}\|^2}$. Therefore:

$$q = \frac{1}{1+\|\vec{p}\|^2} \begin{bmatrix} 1-\|\vec{p}\|^2 \\ 2\vec{p} \end{bmatrix}$$

- In terms of Euler's axis and angle, $\vec{p} = \vec{e} \tan \frac{\theta}{4}$.

Modified Rodrigues Parameters II

- The relationship of MRP with the DCM is

$$C = \mathrm{Id} + \frac{8\vec{p}^{\times}\vec{p}^{\times} - 4(1-\|\vec{p}\|^2)\vec{p}^{\times}}{(1+\|\vec{p}\|^2)^2} = \left[(\mathrm{Id}-\vec{p}^{\times})(\mathrm{Id}+\vec{p}^{\times})^{-1}\right]^2$$

- Since q and $-q$ represent the same attitude, then $\vec{p} = \frac{\vec{q}}{1+q_0}$ and $\vec{p}' = \frac{-\vec{q}}{1-q_0}$ also represent the same attitude. How we can relate both?

$$\|\vec{p}\|^2 = \frac{1-q_0}{1+q_0} = \frac{1}{\|\vec{p}'\|^2}$$

- Thus $\vec{p}$ and $\frac{-\vec{p}}{\|\vec{p}\|^2}$ represent the same attitude. Limiting $\|\vec{p}\| \leq 1$ we avoid the ambiguity (notice however that there are some other ambiguities if $\|\vec{p}\| = 1$).
- Composition is complex compared to RPs. If $\vec{p}_{S'S}$ represents the attitude of S' w.r.t. S and $\vec{p}_{S''S'}$ represents the attitude of S'' w.r.t. S', then $\vec{p}_{S''S}$, the attitude of S'' w.r.t. S, is:

$$\vec{p}_{S''S} = \frac{(1-\|\vec{p}_{S'S}\|^2)\vec{p}_{S''S'} + (1-\|\vec{p}_{S''S'}\|^2)\vec{p}_{S'S} - 2\vec{p}_{S''S'} \times \vec{p}_{S'S}}{1+\|\vec{p}_{S'S}\|^2\|\vec{p}_{S''S'}\|^2 - 2\vec{p}_{S'S}\cdot\vec{p}_{S''S'}}$$

The error quaternion

- To consider errors or to linealize any (nonlinear) equation containing Quaternions around a (reference) value $\bar{q}$, the classical "aditive" formulation $q = \bar{q} + \delta q$ does not work well, because even if $\bar{q}$ and δq have unit modulus, the sum of them may not be unitary.
- It is more convenient to use a "multiplicative" formulation where $q = \bar{q} \star \delta q$, and$\delta q$ is known as the error quaternion which should be close to the unity quaternion $q = [1\ 0\ 0\ 0]^T$.
- δq has 4 components but, obviously, only 3 d.o.f.; these can be codified in a vector $\vec{a}$ "small" (in fact equivalent to $2\vec{g}$):

$$\delta q(\vec{a}) = \frac{1}{\sqrt{4 + \|\vec{a}\|^2}} \begin{bmatrix} 2 \\ \vec{a} \end{bmatrix}$$

- Notice that $\delta q(\vec{a})$ has unity modulus, as expected. If one finally needs to linealize, one gets:

$$\delta q(\vec{a}) \approx \begin{bmatrix} 1 \\ \vec{a}/2 \end{bmatrix}$$