Spacecraft Dynamics

Lesson 1: Introduction to Attitude Dynamics and Control and to ADCS (Attitude Determination and Control System)

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October 18, 2021

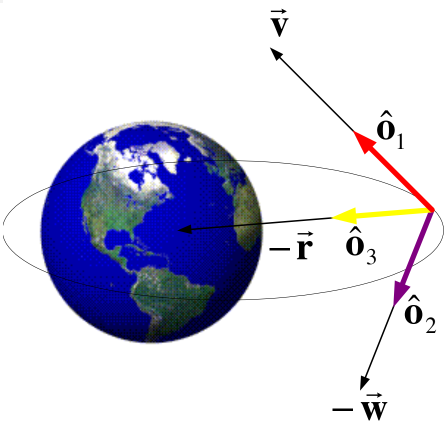
Introduction I

- The great majority of spacecraft have instruments or antennas which must point to one direction. For example:
  - Space telescopes (Hubble).
  - Communications satellites must point their antennas.
  - Solar panels must maximize their solar exposition.
  - Photography cameras must point to one location.
  - Radiators must be pointed to deep space.
  - The thrusters of a spacecraft must be correctly aligned.
  - Other scientific instruments and sensors.
- In addition there are other kinds of requirements:
  - Space telescopes (Hubble).
  - Target tracking.
  - Forbidden directions (e.g. the direction to the Sun for sensitive optics).
- A spacecraft's orientation (with respect to another frame of reference of interest, e.g. inertial or the orbit axes) is called attitude.

Introduction II

- The subsystem responsible for estimating and controlling the attitude is the ADCS (Attitude Determination and Control System) whose basic functions are:
  - Determine the current or instantaneous attitude, from the measurements of the sensors and the knowledge of the previous attitude (estimation problem).
  - Use the available actuators in order to stabilize the attitude and correct possible deviations with respect to a desired attitude (control problem).
- Other possible functions:
  - Generate attitude maneuvers (slew maneuvers), for example, in order to go from an initial attitude to a desired final one (attitude transfer problem)
  - Track a target (tracking problem).

Attitude Representation and Kinematics

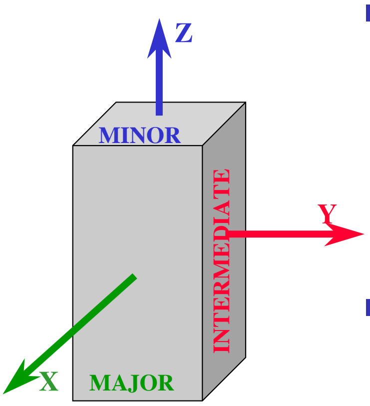


- Under the assumption of a rigid body, attitude is established by specifying the orientation of the body axes with respect to other axes of interest.
- For example, the orbit axes as shown in the figure, whose definition depends on the specific orbit.
- The relationship between two frames of reference can be represented in several ways: using matrices, Euler angles or other mathematical objects.
- Attitude kinematics is a combination of relationships (in the form of differential equations) between the spacecraft's angular velocity, $\vec{\omega}$, and its attitude, represented by any of the mathematical objects previously mentioned.

Attitude Dynamics

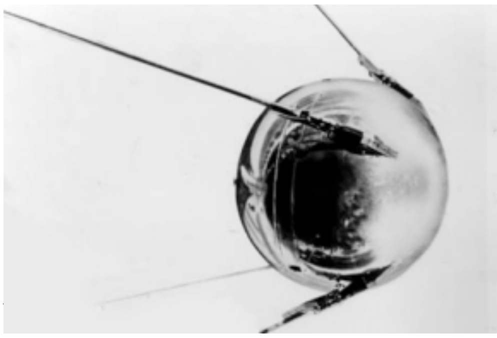
- Attitude dynamics relates the spacecraft's angular velocity with the moment of forces acting on it, and is based on the Angular Momentum Theorem; the resulting differential equations are known as Euler's Equations.
- The movement of a body in torque-free precession (moments equal to zero) is the most simple solution of these equations, and even explicit in the axisymmetric case; it is a precession of the rotation axes around another fixed axis.
- A body in rotation that is subject to a constant moment does not react "intuitively" but rather suffers perturbations in its initial rotation, causing precession and nutation movements.
- This resistance to perturbing moments is named gyroscopic effect. It is the basis of the spinning top's behavior.

Rotational Stability

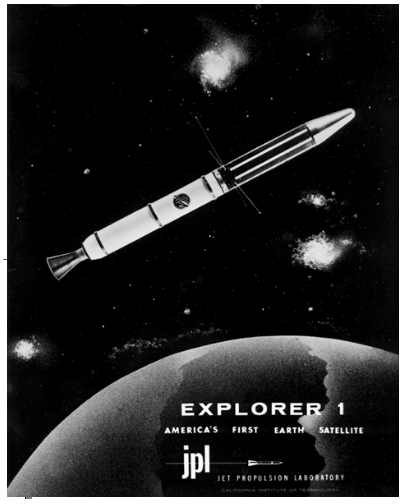


- For the body in the figure, $I_1 = I_x$, $I_2 = I_y$, $I_3 = I_x$ are the principal moments of inertia (given the shape of the body). In addition $I_1 > I_2 > I_3$ because of the apparent dimensions in the figure, so the x axis is the major axis of inertia, the y axis is the intermediate one, and the z axis is the minor axis of inertia.
- It can be shown that if a rigid body rotates around the major or the minor axes, these rotations are stable (they are actually neutrally stable: when the rotation is disturbed, the perturbation does not increase).
- However if the rotation is around the intermediate axis, this rotation is unstable (an initial perturbation would increase and the instantaneous axis of rotation would get away from the intermediate axis).
- These results change in the presence of dissipation of energy (which always exists in real life): The minor axis is unstable if there is dissipation of energy (Major Axis Rule).

Sputnik vs. Explorer I

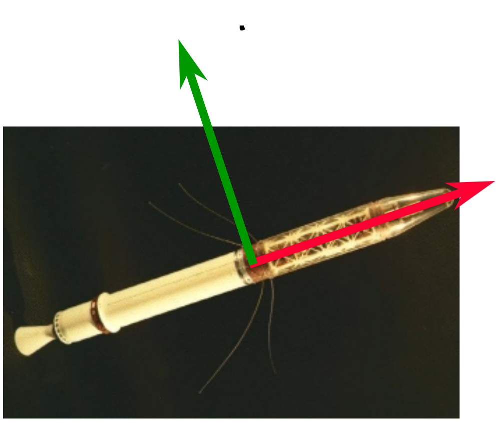


- Sputnik was launched in 1957
- The satellite was stabilized by rotation around its major axis.
- NASA engineers were not conscious of this fact, neither of the major axis rule (which cannot be deduced from a rigid body model).

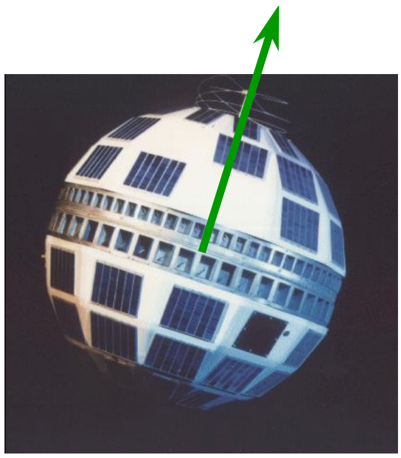


- Explorer I was launched in 1958, "stabilized" by rotation around its minor axis.

Sputnik vs. Explorer II

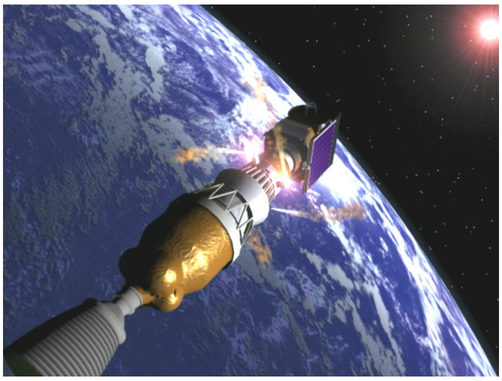


- Stabilization around the minor axis (red) did not work.
- In a few hours Explorer 1 started to spin around its major axis (green) with a quite chaotic movement, making communication with Earth difficult.



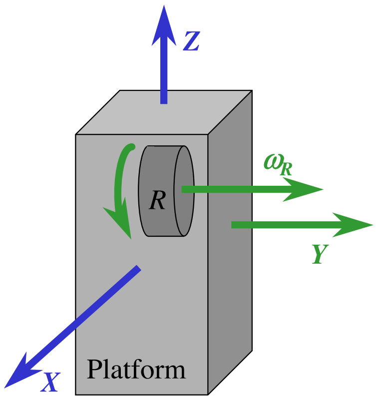
- Telstar I (the first communications satellite) was launched in 1962.
- It was stabilized by rotation around its major axis, spinning at 200 RPM.

Major Axis Rule: Exceptions



- The minor axis is unstable, but the characteristic time of the instability is slow (hours).
- Stabilization is typically achieved by rotation around the minor axis in the launch vehicles' later stages, before firing these stages.
- The gyroscopic effect induced by the rotations considerably reduces errors due to misalignment between the real and ideal axis of thrust.
- After ejecting the final stage, this rotation is typically stopped, for example with a yo-yo mechanism, or waiting long enough so that the dynamics transform the rotation to a major axis spin.
- Example: Mars Odissey.

Effect of a wheel in rotational dynamics



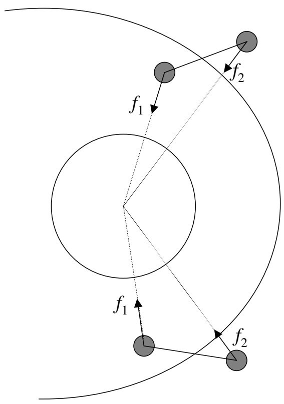
- A wheel, flywheel or rotor placed inside or outside the vehicle, and which is in rotation, produces a stabilization effect due to the gyroscopic effect it provides to the ensemble.
- In addition, the intermediate axis, or even the minor axis in presence of dissipation of energy, can be stabilized with a wheel.
- Moreover, rotations (maneuvers) can be performed as follows: if the wheel is accelerated in one direction, in the absence of (significant) external moments, the vehicle would rotate in the opposite direction due to the fact that the total angular momentum cannot change.
- The most extreme example of this principle is a CMG (control moment gyroscope); it consists of a wheel with high inertia and large fixed velocity but with moving axes.

Examples of Spacecraft with flywheels

- Navstar satellite (GPS).
- 4 flywheels spinning at several thousands of RPM.
- Auxiliary system: RCS (hydrazine).

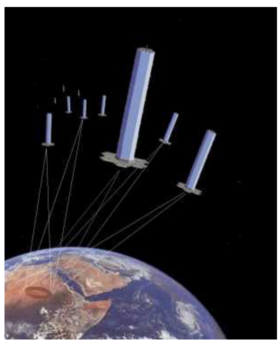
- DSP (Defense Support Program) satellites are part of the USA early warning system. They have infrared sensors.
- Stabilized by rotation with a flywheel.

Gravity gradient (G^2)



- The non spherical shape and mass distribution of a spacecraft produces the so-called gravitational torque, while it travels in its orbit, since $F = \mu m/r^2$.
- It can be seen as a "restorative force" which makes the spacecraft rotate as a pendulum, around its equilibrium position.

- "G^2" can be used for stabilization; however, it barely provides stability in yaw.
- The Moon is "stabilized" by G^2.
- The Polar BEAR satellite, stabilized by gravity, inverted its equilibrium position.



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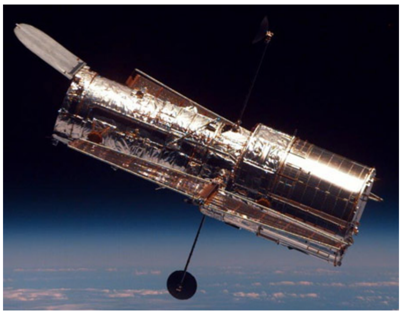
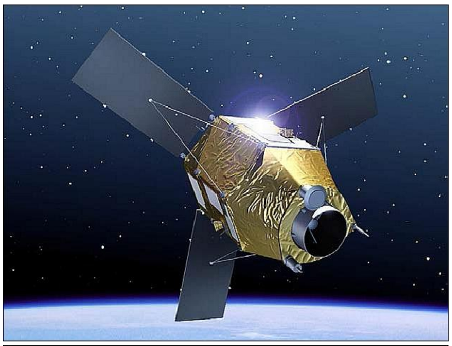
Three-axis stabilized systems

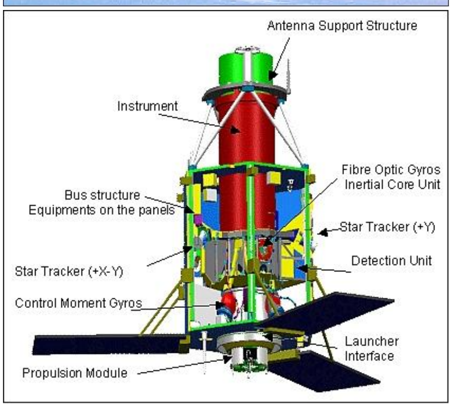
- Stellites with an ADCS system that totally controls their attitude are known as three-axis stabilized satellites.



- For example, the Hubble telescope's attitude control system is one of the most accurate systems ever built by man.
- The principal telescope has to be able of maintaining its position respect to a target with an accuracy of 0.007 arc seconds (a human hair width seen from a distance of 1.5 km).
- A golfer with that accuracy (and the required strength) would be able to achieve a "hole in one" in a golf course in Malaga executing the exit from Moscow, 19 out of 20 times!
- The Hubble performs its three-axis attitude control using flywheels.

Agile satellites





- Earth observation satellites have considerable attitude control requirements.
- The so-called "agile satellites" are prepared to obtain multiple images or even 3D images (taken twice from different angles).
- For example, the Pleiades constellation (2 CNES—French space agency—satellites) has the capacity to obtain images with a resolution < 1 m. from any point of the Earth!
- In order to take advantage of the optical capabilities, a large accuracy in the attitude control/determination is required, but also speed in the maneuvers; this is achieved with CMG (control moment gyros), star trackers and FOG (fiber optic gyros) of high resolution.

Spacecraft Dynamics

Lesson 2: Attitude Representation

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October 18, 2021

Spacecraft Attitude

- The attitude of a Spacecraft is its orientation with respect to a given reference frame (typically, inertial or orbit axes).
- Under the hypothesis of the spacecraft being a rigid body, it is enough to know the orientation of the body axes (i.e., a reference frame fixed to the spacecraft). Thus one needs to study the orientation of a reference frame w.r.t. another.
- The set of orientations between two frames is denoted as SO(3): the special orthogonal group of dimension 3.
- Aircraft classically use Euler angles (yaw, pitch, roll). For spacecraft there are several alternatives (also applicable to aircraft), with their corresponding advantages and disadvantages:
  - Director Cosine Matrix (DCM)
  - Euler Angles (12 possible sets)
  - Euler's Angle and Axis (a.k.a. Eigenaxis)
  - Rotation vector
  - Quaternions
  - Rodrigues parameters (a.k.a. Gibbs' vector)
  - Modified Rodrigues parameters

SO(3) Representations: Main features

- Each representation has advantages and disadvantages, as will be seen.
- Each representation is defined by n parameters.
  - If $n = 3$ the representation is *minimal* (since there are 3 degrees of freedom). However, minimal representations always have singularities.
  - If $n > 3$ then there will be $n - 3$ constraints for the parameters.
- For a given representation, it might happen that two different values of the parameters represent the same physical attitude. Then, it is said that the representation has ambiguities. The set of parameters that needs to be eliminated to avoid ambiguities is called the "shadow set".
- In this lesson we study:
  - How to switch between different representations
  - How to compose attitudes for each representation when there are more than 2 reference frames

SO(3) Representations: Main features

- Another interesting feature is the capacity to generate smooth "paths" of attitude, this is, a continuous set of rotations to get from an initial attitude to a final attitude.
- One can talk about passive and active interpretations between reference frames.
- In the passive representation (a.k.a. "alias") one transform the reference frames (i.e. their basis vectors). Then, vectors also transform since the reference frame change. However, they do so in the opposite way. For instance, if the x-y axes rotate 45º (along the z axis), a vector would rotate 45º in the opposite direction (along the -z axis). This is the preferred interpretation. Plot it!
- The active interpretation (a.k.a. "alibi") looks at the transformation of vectors (therefore reference frames transform in the opposite way).

Director Cosine Matrix (DCM) I

- Let S and S' be reference frames, respectively, with unitary basis vectors $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ and $(\vec{e}_{x'}, \vec{e}_{y'}, \vec{e}_{z'})$. The orientation (attitude) of S' w.r.t. S is totally determined by the change of basis matrix $C_S^{S'}$. This matrix allows, given any generic vector $\vec{v}$ expressed in the basis of S as $\vec{v}^S$, to change its basis as follows: $\vec{v}^{S'} = C_S^{S'} \vec{v}^S$. Denote:

$$C_S^{S'} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

- Note: $\vec{e}_x^{S'} = C_S^{S'} e_x^S = C_S^{S'} [1\ 0\ 0]^T = [c_{11}\ c_{21}\ c_{31}]^T$.
- Therefore:
  $\vec{e}_{x'} \cdot \vec{e}_x = (\vec{e}_{x'}^{S'})^T \vec{e}_x^{S'} = [1\ 0\ 0][c_{11}\ c_{21}\ c_{31}]^T = c_{11}$.
- In addition:

$$\begin{aligned} c_{21} &= \vec{e}_{y'} \cdot \vec{e}_x, \quad c_{31} = \vec{e}_{z'} \cdot \vec{e}_x \\ c_{12} &= \vec{e}_{x'} \cdot \vec{e}_y, \quad c_{22} = \vec{e}_{y'} \cdot \vec{e}_y, \quad c_{32} = \vec{e}_{z'} \cdot \vec{e}_y \\ c_{13} &= \vec{e}_{x'} \cdot \vec{e}_z, \quad c_{23} = \vec{e}_{y'} \cdot \vec{e}_z, \quad c_{33} = \vec{e}_{z'} \cdot \vec{e}_z \end{aligned}$$

Director Cosine Matrix (DCM) II

- Thus:

$$C_S^{S'} = \begin{bmatrix} \vec{e}_{x'} \cdot \vec{e}_x & \vec{e}_{x'} \cdot \vec{e}_y & \vec{e}_{x'} \cdot \vec{e}_z \\ \vec{e}_{y'} \cdot \vec{e}_x & \vec{e}_{y'} \cdot \vec{e}_y & \vec{e}_{y'} \cdot \vec{e}_z \\ \vec{e}_{z'} \cdot \vec{e}_x & \vec{e}_{z'} \cdot \vec{e}_y & \vec{e}_{z'} \cdot \vec{e}_z \end{bmatrix}$$

- By a similar reasoning:

$$C_{S'}^{S} = \begin{bmatrix} \vec{e}_{x'} \cdot \vec{e}_x & \vec{e}_{y'} \cdot \vec{e}_x & \vec{e}_{z'} \cdot \vec{e}_x \\ \vec{e}_{x'} \cdot \vec{e}_y & \vec{e}_{y'} \cdot \vec{e}_y & \vec{e}_{z'} \cdot \vec{e}_y \\ \vec{e}_{x'} \cdot \vec{e}_z & \vec{e}_{y'} \cdot \vec{e}_z & \vec{e}_{z'} \cdot \vec{e}_z \end{bmatrix} = (C_S^{S'})^T$$

- And since $C_{S'}^S = (C_S^{S'})^{-1}$, we get that $C_{S'}^S$ is *orthogonal*, this is: $(C_S^{S'})^{-1} = (C_S^{S'})^T$. The name "Director Cosine Matrix" is also justified since the dot product of unitary vectors is the cosine of the angle they form.
- Another property is that $\det(C_{S'}^S) = 1$. This is due to the fact that $1 = \det(\mathrm{Id}) = \det((C_{S'}^S)(C_{S'}^S)^{-1}) = \det((C_{S'}^S)(C_{S'}^S)^T) = \left(\det(C_{S'}^S)\right)^2$. Therefore $\det(C_{S'}^S) = \pm 1$. The sign + corresponds to both S and S' being right-handed reference frames, which are the ones used in practice.

Director Cosine Matrix (DCM) III

- This attitude representation has 9 parameters. These are dependent from each other, this is, the coefficients of the C matrix cannot be arbitrary (the matrix has to be orthogonal and with determinant 1). In particular, one must have 6 independent constraints which determine that the matrix is orthogonal.
- Composition: assume that the attitude of S_2 w.r.t S_1 is given by $C_{S_1}^{S_2}$ and the attitude of S_3 w.r.t S_2 is given by $C_{S_2}^{S_3}$. Then it it easy to see that the attitude of S_3 w.r.t. S_1 can be found by applying the succesive transformations, this is, $C_{S_1}^{S_3} = C_{S_2}^{S_3} C_{S_1}^{S_2}$. Therefore attitude "composition" is given by a simple matrix product (note that the order matters: non-commutativity of rotations).

Euler angles I

- In general attitude can be mathematically described by three rotations in the main axes, where any axis can be selected for the first, second and third rotation with the only rule that one cannot repeat a consecutive axis (i.e. 1st and 2nd, and 2nd and 3rd must be different).
- As an example, the classical aircraft rotation sequence is:

$$n \xrightarrow[z^n]{\psi} S \xrightarrow[y^S]{\theta} S' \xrightarrow[x^{S'}]{\varphi} BFS$$

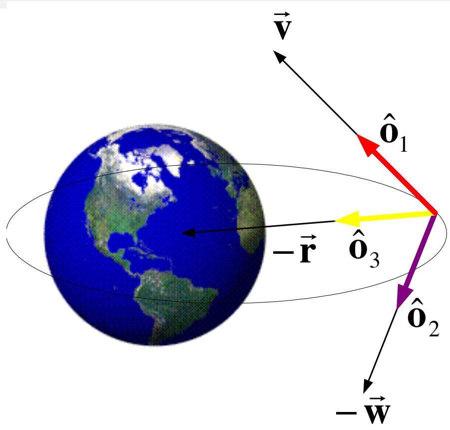
- There exists other options, more suited to spacecraft:

$$n \xrightarrow[x^n]{\theta_1} S \xrightarrow[y^S]{\theta_2} S' \xrightarrow[z^{S'}]{\theta_3} BFS \qquad n \xrightarrow[z^n]{\Omega} S \xrightarrow[x^S]{i} S' \xrightarrow[z^{S'}]{\omega} BFS$$

- There are 12 possible sequences of Euler angles to represent the attititude. This is a minimal representation (3 angles).
- One can obtain the DCM from Euler angles by multiplying elementary rotation matrices. For instance $C_n^b(\psi,\theta,\varphi) = C_{S'}^b(\varphi) C_S^{S'}(\theta) C_n^S(\psi)$.

Euler angles II

- In the figure, the typical aircraft Euler angles are used w.r.t. orbit axes.
  - First a rotation around the axis labelled as 3 (yellow): yaw.
  - Next, a rotation about the resulting axis 2: pitch
  - Finally, a rotation about the resulting axis 3: roll
- Notice that a rotation affects the position of the axes for the next rotations.
- This sequence is denoted as (3,2,1). The other sequences of Euler angles contained in the previous slide are, respectively, (1,2,3) and (3,1,3).
- One can choose a sequence depending on the angles which are of interest for a given application or study (see Lesson 5).



- Other possible sequences: (1,2,1), (1,3,1), (1,3,2), (2,1,2), (2,1,3), (2,3,1), (2,3,2), (3,1,2), (3,2,3).

Euler angles III

- For the sequence (3,2,1) with angles denoted as (ψ, θ, φ), one has:

$$C_n^b = \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ -c\varphi s\psi + s\varphi s\theta c\psi & c\varphi c\psi + s\varphi s\theta s\psi & s\varphi c\theta \\ s\varphi s\psi + c\varphi s\theta c\psi & -s\varphi c\psi + c\varphi s\theta s\psi & c\varphi c\theta \end{bmatrix}$$

- Notice that $(180^o + \psi, 180^o - \theta, 180^o + \varphi)$ defines the same attitude that (ψ, θ, φ). Therefore typically one limits $\theta \in [-90^o, 90^o]$ (the angles that are excluded from these values constitute the shadow set).
- Given the DCM, to obtain the Euler angles, one can derive the following formulas:
  1. $\theta = -\arcsin c_{13}$.
  2. From $\cos\psi = c_{11}/\cos\theta$, $\sin\psi = c_{12}/\cos\theta$, obtain ψ.
  3. From $\sin\varphi = c_{23}/\cos\theta$, $\cos\varphi = c_{33}/\cos\theta$, obtain φ.
- For other sequences, one can get similar relations from the explicit expression of the DCM.

Euler angles IV

- Main advantage: physically meaningful.
- One has, however, to be careful when composing attitude.
- Suppose the attitude of S_2 w.r.t. S_1 is given by $(\psi_1, \theta_1, \varphi_1)$ and the attitude of S_3 w.r.t. S_2 is given by $(\psi_2, \theta_2, \varphi_2)$. Denote as $(\psi_3, \theta_3, \varphi_3)$ the attitude of S_3 w.r.t. S_1. In general $\psi_3 \neq \psi_1 + \psi_2$, $\theta_3 \neq \theta_1 + \theta_2$, $\varphi_3 \neq \varphi_1 + \varphi_2$!!
- The best way to obtain $(\psi_3, \theta_3, \varphi_3)$ is to compute them from $C_{S_1}^{S_3} = C_{S_2}^{S_3}(\psi_2, \theta_2, \varphi_2) C_{S_1}^{S_2}(\psi_1, \theta_1, \varphi_1)$. This is, going to a DCM representation, composing, and going back to Euler angles.
- This shows that it might be complex to work with Euler angles.
- Main disadvantage: singularities (as will be seen in Lesson 4).

Euler's angle and axis I

- Euler's Rotation Theorem: "the most general movement of a solid with a fixed point is a single rotation around a unique axis."
- Note: We are considering a rotation at a given time (a "snapshot"), not a rotation that is changing as time evolves (that is the subject of Lesson 4).
- Let us call a unit vector in the direction of that axis (Euler's Axis) as $\vec{e}_{S/S'}$, and the magnitude of the rotation (Euler's Angle) as θ.
- Thus, $\|\vec{e}_{S/S'}\| = 1$ and if we write $\vec{e}_{S/S'}^{S'} = [e_x\ e_y\ e_z]^T$ it follows that $e_x^2 + e_y^2 + e_z^2 = 1$.
- A useful formalism is the following. Given a vector $\vec{v} = [v_x\ v_y\ v_z]^T$ define the operator $\times$ acting on $\vec{v}$ (denoted $\vec{v}^{\times}$) as follows:

$$\vec{v}^{\times} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$$

Euler's angle and axis II

- The operation $\vec{v}^{\times}$ helps to quickly compute the cross product $\vec{v} \times \vec{w}$, for any vector $\vec{w}$, in a reference frame S: $(\vec{v} \times \vec{w})^{S} = \left(\vec{v}^{S}\right)^{\times} \vec{w}^{S}$.
- Thus, if the attitude using Euler's angle and axis is given by $(\vec{e}^{S'}_{S/S'}, \theta)$, how to go from there to the DCM and the other way around? The $\times$ operator helps.
- One has $C_S^{S'} = \cos\theta \mathrm{Id} + (1 - \cos\theta)\vec{e}^{S'}_{S/S'}(\vec{e}^{S'}_{S/S'})^T - \sin\theta \left(\vec{e}^{S'}_{S/S'}\right)^{\times}$. This is known as the Euler-Rodrigues formula and it is mathematically proven later.
- On the other hand, $C_S^{S'}$, and computing $\mathrm{Tr}(C_S^{S'})$ and $(C_S^{S'})^T - C_S^{S'}$, one gets:

$$\begin{aligned} \cos\theta &= \frac{\mathrm{Tr}(C_S^{S'}) - 1}{2} \\ \left(\vec{e}^{S'}_{S/S'}\right)^{\times} &= \frac{1}{2\sin\theta}\left((C_S^{S'})^T - C_S^{S'}\right) \end{aligned}$$

Spacecraft Attitude. Representation methods.
Director Cosine Matrix
Euler Angles
Euler's angle and axis. Quaternions. Other representations.

Euler's angle and axis III

- Another relationship between Euler's angle and axis and the Director Cosine Matrix is given by the algebraic properties of the DCM.
- Since the DCM is orthogonal, it can be shown that 1 is always an eigenvalue of it. If C is the DCM, then the eigenvector associated to the 1 is the Euler's axis $\vec{e}$ since $C\vec{e} = \vec{e}$.
- On the other hand, the other two eigenvalues of the DCM are precisely $\mathrm{e}^{i\theta}$, $\mathrm{e}^{-i\theta}$.
- This is another way of computing Euler's angle and axis, by evaluating the eigenvalues and eigenvectors of the DCM.

Euler's angle and axis IV

- Therefore, in this representation, one describes the attitude with four parameters: three componentes of an unit vector and an angle. These have a clear physical meaning.
- Notice that the attitude given by $(\vec{e}^{S'}_{S/S'}, \theta)$ and by $(-\vec{e}^{S'}_{S/S'}, 360^{\mathrm{o}} - \theta)$ is exactly the same. To avoid this ambiguity, one can constraint θ to $[0, 180^{\mathrm{o}})$.
- The "opposite" attitude (the one from S w.r.t. S') is given by $(-\vec{e}^{S}_{S'/S}, \theta)$. Notice also that $e^{S}_{S'/S} = e^{S'}_{S'/S}$.
- Composition: if the attitude of S_2 w.r.t. S_1 is given by $(\vec{e}^{S_2}_{S_1/S_2}, \theta_1)$ and the attitude of S_3 w.r.t. S_2 is given by $(\vec{e}^{S_3}_{S_2/S_3}, \theta_2)$, then, denoting as $(\vec{e}^{S_3}_{S_1/S_3}, \theta_3)$ the attitude of S_3 w.r.t. S_1, one obtains:

$$\cos\theta_3 = -\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2(\vec{e}_{S_1/S_2}\cdot\vec{e}_{S_2/S_3})$$

$$e^{S_3}_{S_1/S_3} = \frac{1}{\sin\theta_3}\left(\sin\theta_1\cos\theta_2\vec{e}_{S_1/S_2} + \cos\theta_1\sin\theta_2\vec{e}_{S_2/S_3} + \sin\theta_1\sin\theta_2(\vec{e}_{S_1/S_2}\times\vec{e}_{S_2/S_3})\right)$$

Rotation vector

- A minimal attitude representation can be obtained by combining Euler's axis and angle in a single vector as follows: $\vec{\theta} = \theta\vec{e}$.
- This representation can be useful as it physically represents the angular speed one would need to maintain constant from a second for one reference frame respect to another, that start being the same, to obtain the attitude given by $(\vec{e}, \theta)$.
- On the other hand for large rotations it is not an adequate rotation. Note that a rotation of 0º and 360º are physically the same but the first is $\vec{\theta} = \vec{0}$ and the second is not univocally defined.
- Thus, the representation is reserved for theoretical analysis of for small angles (or to determine the angular velocity necessary to perform a fixed rotation).

Quaternions

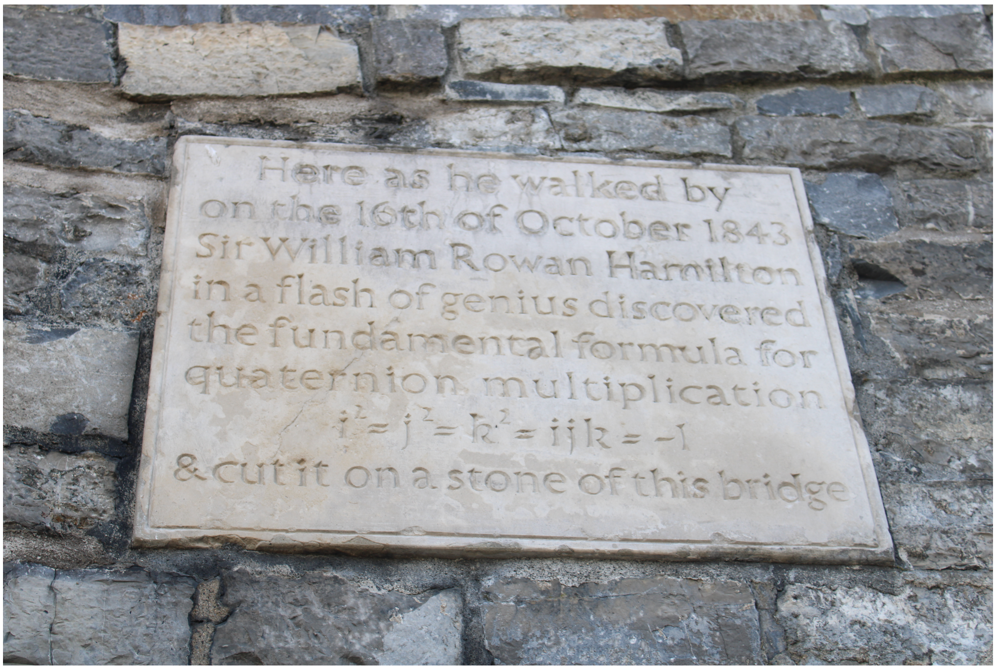
- Quaternions were first described by Hamilton (19th century), who considered them his greatest creation; he thought they were going to be used as Physics "universal language". However, they were soon substituted by vectors (Gibbs) and matrices (Cayley).
- Remember a complex number z can be thought of as a " 2-D vector", which can be written in terms of its components as $z = x + iy$. Complex number of unity modulus can be used to represent a 2-D rotation, since if $|z| = 1$, one can write $z = \mathrm{e}^{i\theta}$, and it is well-known multiplying by this number rotates the phase by an angle θ.
- Quaternions extend complex number to "4 dimensions". A quaternion q can be written as: $q = q_0 + iq_1 + jq_2 + kq_3$.
- q_0 is the scalar part and $\vec{q} = [q_1\ q_2\ q_3]^T$ the "vector part" of q.

Spacecraft Attitude. Representation methods.
Director Cosine Matrix
Euler Angles
Euler's angle and axis. Quaternions. Other representations.

Quaternion Algebra I

- To better understand Quaternions it's important to know their algebraic properties, this is, how to operate with Quaternions.
- Sum: Component-wise, i.e., given $q = q_0 + iq_1 + jq_2 + kq_3$ and $q' = q'_0 + iq'_1 + jq'_2 + kq'_3$, one has that $q'' = q + q' = q''_0 + iq''_1 + jq''_2 + kq''_3$ is given by the obvious formulae:
  $q''_0 = q_0 + q'_0$, $q''_1 = q_1 + q'_1$, $q''_2 = q_2 + q'_2$, $q''_3 = q_3 + q'_3$.
- Product: denote by $\star$, again, component-wise, knowing the following rules of multiplication:
  $i \star i = -1$, $i \star j = k$, $i \star k = -j$, $j \star i = -k$, $j \star j = -1$,
  $j \star k = i$, $k \star i = j$, $k \star j = -i$, $k \star k = -1$.
- Hamilton's formula follows: $i \star j \star k = -1$.
- Notice that $q \star q' \neq q' \star q$: Quaternion multiplication is NOT commutative!

Quaternions: Plaque on Broom Bridge (Dublin)



Quaternion Algebra II

- Matrix form of the product: It is possible to write the product $q'' = q' \star q$ in matrix form as follows:

$$\begin{bmatrix} q_0'' \\ q_1'' \\ q_2'' \\ q_3'' \end{bmatrix} = \begin{bmatrix} q_0' & -q_1' & -q_2' & -q_3' \\ q_1' & q_0' & -q_3' & q_2' \\ q_2' & q_3' & q_0' & -q_1' \\ q_3' & -q_2' & q_1' & q_0' \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

- "vector" form of the product: $q_0'' = q_0' q_0 - \vec{q}'^T \vec{q}$, $\vec{q}'' = q_0 \vec{q}' + q_0' \vec{q} + \vec{q}' \times \vec{q}$.
- Conjugate: As for complex numbers, given $q = q_0 + iq_1 + jq_2 + kq_3$ one defines the conjugate of q as $q^* = q_0 - iq_1 - jq_2 - kq_3$.
- Modulus: The definition of the modulus of $q = q_0 + iq_1 + jq_2 + kq_3$ is $|q|^2 = q \star q^* = q_0^2 + q_1^2 + q_2^2 + q_3^2$. Property: $|q \star q'| = |q||q'|$.
- Division: One defines division using the conjugate: $q'/q = q'/q \star q^*/q^* = (q' \star q^*)/|q|^2$.

Attitude representation using Quaternions I

- Given the attitude represented by Euler's axis and angle, $\vec{e}$ and θ, one can "codify" that attitude in terms of Quaternions as follows: $q_0 = \cos\theta/2$, $\vec{q} = \sin\theta/2\vec{e}$.
- Notice therefore that if q represents an attitude, it follows that $|q| = 1$ (and vice-versa!).
- Remember the $\times$ operator and apply it to the quaternion $\vec{q}^{\times}$:
$\vec{q}^{\times} = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$
- To go from DCM C to Quaternions, use:$q_0 = \frac{\sqrt{1+\mathrm{Tr}(C)}}{2}$ y $\vec{q}^{\times} = \frac{1}{4q_0}\left(C^T - C\right)$.
- To go from Quaternions to DCM use Euler-Rodrigues formula for Quaternions:
$C = \left(q_0^2 - \vec{q}^T\vec{q}\right)\mathrm{Id} + 2\vec{q}\vec{q}^T - 2q_0\vec{q}^{\times}$.
- One can transform a vector $\vec{v}$ without need of the DCM using the formula: $\begin{bmatrix} 0 \\ \vec{v}^B \end{bmatrix} = q^*_{B/A} \star \begin{bmatrix} 0 \\ \vec{v}^A \end{bmatrix} \star q_{B/A}$

Attitude representation using Quaternions II

- Euler-Rodrigues formula in matrix form:

$$C(q) = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_0q_2) & 2(q_2q_3-q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

- Quaternions are an attitude representation that requires 4 parameters, with the additional constraint $|q| = 1$.
- Ambiguities: q and $-q$ represent the same attitude, since if q corresponds to $(\vec{e}, \theta)$, then $-q$ corresponds to $(-\vec{e}, 360 - \theta)$. Prove it!
- Disadvantage: no physical sense unless you have some experience using them.
- Notice: To convert from DCM to Quaternions and back no trig formulas are required, increasing the precission.
- If $q_{S'S}$ represents the attitude of S' w.r.t. S y $q_{S''S'}$ represents the attitude of S'' w.r.t. S', then $q_{S''S}$, the attitude of S'' w.r.t. S, can be computed $q_{S''S} = q_{S'S} \star q_{S''S'}$ (notice that the product is in the other direction, comparing with the DCM).

Computing Quaternions from Euler angles

- For the classical (3,2,1) sequence, notice that
  - The quaternion corresponding to the Euler angles $(\psi, 0, 0)$ is $q_\psi = \cos\psi/2 + k\sin\psi/2$.
  - The quaternion corresponding to the Euler angles $(0, \theta, 0)$ is $q_\theta = \cos\theta/2 + j\sin\theta/2$.
  - The quaternion corresponding to the Euler angles $(0, 0, \varphi)$ is $q_\varphi = \cos\varphi/2 + i\sin\varphi/2$.
- Thus, given the Euler angles (ψ, θ, φ) one obtains a corresponding quaternion using the composition rule as $q = q_\psi \star q_\theta \star q_\varphi$.
- Explicitly doing the product one gets

$$\begin{aligned} q &= (\cos\psi/2\cos\theta/2\cos\varphi/2 + \sin\psi/2\sin\theta/2\sin\varphi/2) \\ &\quad + i\,(\cos\psi/2\cos\theta/2\sin\varphi/2 - \sin\psi/2\sin\theta/2\cos\varphi/2) \\ &\quad + j\,(\cos\psi/2\sin\theta/2\cos\varphi/2 + \sin\psi/2\cos\theta/2\sin\varphi/2) \\ &\quad + k\,(\sin\psi/2\cos\theta/2\cos\varphi/2 - \cos\psi/2\sin\theta/2\sin\varphi/2). \end{aligned}$$

Quaternions: a word of caution

- Careful: some authors (STK as well) write q_4 instead of q_0 so the scalar part is the last component of the quaternion.
- Some authors define the quaternion product in an opposite way, so $i \star j = -k$, etc. The consequence of this is that many formulas change:
  - The quaternion composition rule now is as for the matrices (from right to left).
  - The formula for vector transformation becomes $\begin{bmatrix} 0 \\ \vec{v}^B \end{bmatrix} = q_{B/A} \star \begin{bmatrix} 0 \\ \vec{v}^A \end{bmatrix} \star q^*_{B/A}$
- Also, if one wants to use our definition of quaternions but to rotate a vector (instead of changing its reference frame, this is, to use the active interpretation) then: $\begin{bmatrix} 0 \\ \vec{v}' \end{bmatrix} = q \star \begin{bmatrix} 0 \\ \vec{v} \end{bmatrix} \star q^*$ where $\vec{v}'$ is the vector $\vec{v}$ rotated by an axis and angle defined by q, which is the formula one may find over the internet.

Quaternions: shortest path and interpolation

- Given two quaternions q_0 and q_1 representing two different attitudes, can one construct a ‘interpolation path,” continuous, $q(s)$ such that $q(0) = q_0$ and $q(1) = q_1$?
- The way to do it is to first find q_2 representing the attitude between q_0 and q_1 (the rotation quaternion): $q_2 = \frac{1}{q_0} \star q_1 = q_0^* q_1$. From this quaternion extract Euler's angle and axis (θ and $\vec{e}$):$q_2 = \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2\vec{e} \end{bmatrix}$.
- Now the solution of the problem is $q(s)$ which is the product of q_0 and another quaternion coming from Euler's axis $\vec{e}$ and angle $s\theta$, so that when $s = 0$ it is the unity quaternion (and the product is q_0) and when $s = 1$ it is q_2 (and the product is q_1):

$$q(s) = q_0 \star \begin{bmatrix} \cos(s\theta/2) \\ \sin(s\theta/2)\vec{e} \end{bmatrix}$$

Rodrigues Parameters I

- Atittude representation using Rodrigues Parameters (RP, also called Gibbs vector) can be easily obtained from the quaternion as $\vec{g} = \frac{\vec{q}}{q_0}$, obviously this is only valid if $q_0 > 0$ $(i.e.\theta < 180^\circ)$ because otherwise one gets a singularity. To recover the quaternion from $\vec{g}$:

$$\|\vec{g}\|^2 = \frac{\|\vec{q}\|^2}{q_0^2} = \frac{1 - q_0^2}{q_0^2}$$

Thus $q_0 = \frac{\pm 1}{\sqrt{1+\|\vec{g}\|^2}}$. And therefore:

$$q = \frac{\pm 1}{\sqrt{1 + \|\vec{g}\|^2}} \begin{bmatrix} 1 \\ \vec{g} \end{bmatrix}$$

- In terms of Euler's axis and angle, $\vec{g} = \vec{e} \tan\frac{\theta}{2}$.

Rodrigues Parameters II

- The relationship with the DCM is as follows:

$$C = \mathrm{Id} + 2\frac{\vec{g}^\times \vec{g}^\times - \vec{g}^\times}{1 + \|\vec{g}\|^2} = (\mathrm{Id} - \vec{g}^\times)(\mathrm{Id} + \vec{g}^\times)^{-1} = (\mathrm{Id} + \vec{g}^\times)^{-1}(\mathrm{Id} - \vec{g}^\times)$$

- On the other hand, since $q_0 = \frac{\sqrt{1+\mathrm{Tr}(C)}}{2}$ and $\vec{q}^\times = \frac{1}{4q_0}\left(C^T - C\right)$, one gets:

$$\vec{g}^\times = \frac{q^\times}{q_0} = \frac{1}{4q_0^2}\left(C^T - C\right) = \frac{C^T - C}{1 + \mathrm{Tr}(C)}$$

- Composition follows a simple rule.If $\vec{g}_{S'S}$ represents the attitude of S' w.r.t. S and $\vec{g}_{S''S'}$ represents the attitude of S" w.r.t. S', then $\vec{g}_{S''S}$, the attitude of S" w.r.t. S, is computed as:

$$\vec{g}_{S''S} = \frac{\vec{g}_{S''S'} + \vec{g}_{S'S} - \vec{g}_{S''S'} \times \vec{g}_{S'S}}{1 - \vec{g}_{S'S} \cdot \vec{g}_{S''S'}}$$

Modified Rodrigues Parameters

- The representation using Modified Rodrigues Parameters (MRP) is quite recent (1962) but popular in control applications. Similarly to RP, one can get it from the quaternion, by defining $\vec{p} = \frac{\vec{q}}{1+q_0}$. To recover the quaternion from the MRP:

$$\|\vec{p}\|^2 = \frac{\|\vec{q}\|^2}{(1+q_0)^2} = \frac{1-q_0^2}{(1+q_0)^2} = \frac{1-q_0}{1+q_0}$$

Then $q_0 = \frac{1-\|\vec{p}\|^2}{1+\|\vec{p}\|^2}$. Therefore:

$$q = \frac{1}{1+\|\vec{p}\|^2}\begin{bmatrix} 1 - \|\vec{p}\|^2 \\ 2\vec{p} \end{bmatrix}$$

- In terms of Euler's axis and angle, $\vec{p} = \vec{e}\tan\frac{\theta}{4}$.

Modified Rodrigues Parameters II

- The relationship of MRP with the DCM is

$$C = \mathrm{Id} + \frac{8\vec{p}^{\times}\vec{p}^{\times} - 4(1-\|\vec{p}\|^2)\vec{p}^{\times}}{(1+\|\vec{p}\|^2)^2} = \left[(\mathrm{Id}-\vec{p}^{\times})(\mathrm{Id}+\vec{p}^{\times})^{-1}\right]^2$$

- Since q and $-q$ represent the same attitude, then $\vec{p} = \frac{\vec{q}}{1+q_0}$ and $\vec{p}' = \frac{-\vec{q}}{1-q_0}$ also represent the same attitude. How we can relate both?

$$\|\vec{p}\|^2 = \frac{1-q_0}{1+q_0} = \frac{1}{\|\vec{p}'\|^2}$$

- Thus $\vec{p}$ and $\frac{-\vec{p}}{\|\vec{p}\|^2}$ represent the same attitude. Limiting $\|\vec{p}\| \leq 1$ we avoid the ambiguity (notice however that there are some other ambiguities if $\|\vec{p}\| = 1$).
- Composition is complex compared to RPs. If $\vec{p}_{S'S}$ represents the attitude of S' w.r.t. S and $\vec{p}_{S''S'}$ represents the attitude of S" w.r.t. S', then $\vec{p}_{S''S}$, the attitude of S" w.r.t. S, is:

$$\vec{p}_{S''S} = \frac{(1-\|\vec{p}_{S'S}\|^2)\vec{p}_{S''S'} + (1-\|\vec{p}_{S''S'}\|^2)\vec{p}_{S'S} - 2\vec{p}_{S''S'}\times\vec{p}_{S'S}}{1+\|\vec{p}_{S'S}\|^2\|\vec{p}_{S''S'}\|^2 - 2\vec{p}_{S'S}\cdot\vec{p}_{S''S'}}$$

The error quaternion

- To consider errors or to linealize any (nonlinear) equation containing Quaternions around a (reference) value $\bar{q}$, the classical "aditive" formulation $q = \bar{q} + \delta q$ does not work well, because even if $\bar{q}$ and δq have unit modulus, the sum of them may not be unitary.
- It is more convenient to use a "multiplicative" formulation where $q = \bar{q} \star \delta q$, and$\delta q$ is known as the error quaternion which should be close to the unity quaternion $q = [1\ 0\ 0\ 0]^T$.
- δq has 4 components but, obviously, only 3 d.o.f.; these can be codified in a vector $\vec{a}$ "small" (in fact equivalent to $2\vec{g}$):

$$\delta q(\vec{a}) = \frac{1}{\sqrt{4+\|\vec{a}\|^2}}\begin{bmatrix} 2 \\ \vec{a} \end{bmatrix}$$

- Notice that $\delta q(\vec{a})$ has unity modulus, as expected. If one finally needs to linealize, one gets:

$$\delta q(\vec{a}) \approx \begin{bmatrix} 1 \\ \vec{a}/2 \end{bmatrix}$$

Spacecraft Dynamics

Lesson 3: Attitude Determination. Errors.

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June 30, 2022

Attitude determination

- Attitude determination is a process that estimates the present attitude by using sensors and applicable algorithms. It can be thought of as a "static" process that gives the picture of what the present attitude is.
- Attitude determination sensors, in general, determine a vector $\vec{v}$ in the body axes, this is, $\vec{v}^B$ (in fact they use "sensor axes" but the transformation to body axes should be known and it is implicitly applied). It is assumed that said vector is known in some reference axes (inertial axes or orbit axes), denoted as $\vec{v}^N$. As will be seen it is necessary to have two or more measurements of this kind to be able to solve the problem.
- In Lesson 6 we see sensors that from measurements of angular velocity $\vec{\omega}^B$ continuously determine the attitude (a more dynamic process that is typically referred to as attitude estimation).

Estimation from observations

- In general, consider we have n (2 or more) sensors that determine a vector $\vec{v}_i$, $i = 1, \ldots, n$, in body axes, this is, $\vec{v}_i^B$. The vector is assumed known in some reference axes (inertial axes or orbit axes, with respect to which we want to study the spacecraft attitude) and denoted in that frame as $\vec{v}_i^N$. Those are unit vectors since in principle only directions matter.
- Thus we have n equation written as $\vec{v}_i^B = C_N^B \vec{v}_i^N$ and we need to solve for C_N^B.
- To simplify write $\vec{W}_i = \vec{v}_i^B$, $\vec{V}_i = \vec{v}_i^N$, $A = C_N^B$. Thus, we have n equations $\vec{W}_i = A\vec{V}_i$ and need to solve for A.
- These vectors will contain some errors.
- If $n = 2$ there a simple method that can be applied known as TRIAD. We'll see other more general methods for $n \geq 2$.
- Question: what conditions would the measurements/references verify if they are exact??

TRIAD Method

- Start from two observations related to the references through the DCM: $\vec{W}_1 = A\vec{V}_1$ and $\vec{W}_2 = A\vec{V}_2$
- Define the following vectors: $\vec{r}_1 = \vec{V}_1$, $\vec{r}_2 = \frac{\vec{V}_1 \times \vec{V}_2}{|\vec{V}_1 \times \vec{V}_2|}$, and $\vec{r}_3 = \frac{\vec{V}_1 \times \vec{r}_2}{|\vec{V}_1 \times \vec{r}_2|}$. Similarly: $\vec{s}_1 = \vec{W}_1$, $\vec{s}_2 = \frac{\vec{W}_1 \times \vec{W}_2}{|\vec{W}_1 \times \vec{W}_2|}$, and $\vec{s}_3 = \frac{\vec{W}_1 \times \vec{s}_2}{|\vec{W}_1 \times \vec{s}_2|}$. It is rather obvious that one should have now: $\vec{s}_1 = A\vec{r}_1$, $\vec{s}_2 = A\vec{r}_2$, and $\vec{s}_3 = A\vec{r}_3$.
- Construct now the matrices $M_{ref} = [\vec{r}_1\ \vec{r}_2\ \vec{r}_3]$ and $M_{obs} = [\vec{s}_1\ \vec{s}_2\ \vec{s}_3]$. It holds that $M_{obs} = AM_{ref}$. In addition, the columns of M_{ref} are orthonormal between them. Thus, M_{ref} is invertible (and orthogonal!). Therefore we can solve for A as $A = M_{obs}M_{ref}^T$.
- Notice that the method is not symmetric, as the measurement labelled as 1 is given more importance. In practice, A will not be the exact DCM matrix due to errors in the sensors. Thus, one should use the "best" measurement as first.

Wahba's Problem

- Consider now n measures satisfying $\vec{W}_i = A\vec{V}_i$. We pose the problem as a least squares minimization problem.
- Define the function $L(A) = \frac{1}{2}\sum_{i=1}^n a_i|\vec{W}_i - A\vec{V}_i|^2$, where a_i are the weights given to each measurement (verifying $\sum_{i=1}^n a_i = 1$) and pose the mathematical objective of finding A (orthogonal) such $L(A)$ is minimized. In the literature this is known as **"Wahba's Problem"**.
- Since operating

$$|\vec{W}_i - A\vec{V}_i|^2 = (\vec{W}_i - A\vec{V}_i)^T(\vec{W}_i - A\vec{V}_i) = 2 - 2\vec{W}_i^T A\vec{V}_i,$$

one has

$$L(A) = 1 - \sum_{i=1}^n a_i \vec{W}_i^T A\vec{V}_i = 1 - g(A),$$

where $g(A) = \sum_{i=1}^n a_i \vec{W}_i^T A\vec{V}_i$. Minimizing $L(A)$ is thus equivalente to maximizing $g(A)$ (and notice $g(A) \leq 1!$).

Davenport's q method

- Writing A as a function of q by using Euler-Rodrigues ($A = (q_0^2 - \vec{q}^T\vec{q})\mathrm{I} + 2\vec{q}\vec{q}^T - 2q_0\vec{q}^\times$) we reach

$$g(A) = \sum_{i=1}^n a_i \vec{W}_i^T (q_0^2 - \vec{q}^T\vec{q})\vec{V}_i + 2\sum_{i=1}^n a_i \vec{W}_i^T \vec{q}\vec{q}^T \vec{V}_i - 2\sum_{i=1}^n a_i \vec{W}_i^T q_0 \vec{q}^\times \vec{V}_i$$

- Develop now each term trying to reach a bilineal form $g(q) = q^T K q$:
  - Starting with the second term

$$2\sum_{i=1}^n a_i \vec{W}_i^T \vec{q}\vec{q}^T \vec{V}_i = 2\sum_{i=1}^n a_i \vec{q}^T \vec{W}_i \vec{V}_i^T \vec{q} = 2\vec{q}^T B\vec{q} = \vec{q}^T (B + B^T)\,\vec{q}$$

  where $B = \sum_{i=1}^n a_i \vec{W}_i \vec{V}_i^T$.
  - The first term can be written as

$$\sum_{i=1}^n a_i \vec{W}_i^T (q_0^2 - \vec{q}^T\vec{q})\vec{V}_i = (q_0^2 - \vec{q}^T\vec{q})\sum_{i=1}^n a_i \vec{W}_i^T \vec{V}_i = q_0\sigma q_0 - \vec{q}^T(\sigma\mathrm{I})\vec{q}$$

  where $\sigma = \sum_{i=1}^n a_i \vec{W}_i^T \vec{V}_i = \mathrm{Tr}(B)$.

Davenport's q method

- Finally, the last term can be expressed as:

$$-2\sum_{i=1}^{n} a_i \vec{W}_i^T q_0 \vec{q}^\times \vec{V}_i = 2\sum_{i=1}^{n} a_i \vec{W}_i^T q_0 \vec{V}_i^\times \vec{q} = 2q_0 \vec{z}^T \vec{q} = q_0 \vec{z}^T \vec{q} + \vec{q}^T \vec{z} q_0$$

where $\vec{z}^T = \sum_{i=1}^{n} a_i \vec{W}_i^T \vec{V}_i^\times$, hence $\vec{z} = -\sum_{i=1}^{n} a_i \vec{V}_i^\times \vec{W}_i$.

- One has $(\vec{a}^\times \vec{b})^\times = \vec{b}\vec{a}^T - \vec{a}\vec{b}^T$, what can be shown from the identity $(\vec{a} \times \vec{b}) \times \vec{c}$. Observe then that

$$\vec{z}^\times = -\sum_{i=1}^{n} a_i (\vec{V}_i^\times \vec{W}_i)^\times = \sum_{i=1}^{n} a_i \vec{V}_i \vec{W}_i^T - \sum_{i=1}^{n} a_i \vec{W}_i \vec{V}_i^T = B^T - B$$

Davenport's q method

- Thus, the function g is expressed in terms of the quaternion as

$$g(q) = q^T K q$$

where the matrix K can be found from the coefficients of a newly defined matrix in terms of weights, measurements and references $B = \sum_{i=1}^{n} a_i \vec{W}_i \vec{V}_i^T$, as follows

$$\begin{aligned} \sigma &= \mathrm{Tr}(B), \\ S &= B + B^T, \\ \vec{z}^\times &= B^T - B \end{aligned}$$

being K a 4×4 matrix equal to

$$K = \begin{bmatrix} \sigma & \vec{z}^T \\ \vec{z} & S - \sigma \mathrm{Id} \end{bmatrix}$$

Davenport's q method

- Thus, the problem is now reduced to finding q (attitude quaternion, this is, a norm 1 vector of four components) such that $g(q) = q^T Kq$ is maximized.
- To solve a multivariable maximization problem with constraints ($q^T q = 1$) one can use Lagrange's multipliers:

$$H = q^T Kq - \lambda(q^T q - 1)$$

- Taking derivative w.r.t. q and setting it to zero: $\frac{\partial H}{\partial q} = 2q^T K - 2\lambda q^T = 0 \quad \longrightarrow \quad Kq = \lambda q.$
- Thus λ must be an eigenvalue of K and q the associated eigenvector of modulus 1 (there are two, but of opposing signs, thus representing the same attitude). To find which eigenvalue, replace the solution in $g(q)$: $g(q) = q^T Kq = q^T \lambda q = \lambda$
- Therefore, the maximum attained at the critical point is equal to the eigenvalue and the solution will be the eigenvector (of modulus 1) associated to the maximum eigenvalue.

The QUEST method

- Davenport's q method reduces the attitude determination problem to an eigenvalue/eigenvector problem, however this algebraic method might be problematic to solve on a satellite, depending on computational resources available onboard.
- In 1978 the QUEST (QUaternion ESTimator) method was developed to avoid the computational burden.
- The idea is to rewrite $Kq = \lambda q$ in terms of the K matrix:

$$\begin{bmatrix} \sigma & \vec{z}^T \\ \vec{z} & S - \sigma \mathrm{Id} \end{bmatrix} \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} = \lambda \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix}$$

- Therefore two equations can be extracted.

$$\sigma q_0 + \vec{z}^T \vec{q} = \lambda q_0, \quad q_0 \vec{z} + S\vec{q} - \sigma \vec{q} = \lambda \vec{q}$$

- Remembering Gibb's vector $\vec{g} = \frac{\vec{q}}{q_0}$, one can manipulate the second equation reaching

$$\vec{z} + [S - (\sigma + \lambda)\mathrm{I}]\,\vec{g} = 0$$

The QUEST Method

- Then $\vec{g} = [(\sigma + \lambda)\mathrm{I} - S]^{-1}\,\vec{z}$ (but we don't know λ, the maximum eigenvalue)
- A first approximation is to take $\lambda \approx 1$ (which would be the value if the measurements were without error). Then $\vec{g} = [(1 + \sigma)\mathrm{I} - S]^{-1}\,\vec{z}$
- A better approximation is to find an explicit expression for the maximum eigenvalue by finding the roots of the characteristic equation of K, which is:

$$\lambda^4 - (a + b)\lambda^2 - c\lambda + (ab + c\sigma - d) = 0$$

- Where the coefficients are

$$\begin{aligned} a &= \sigma - \mathrm{Tr}[\mathrm{adj}(S)], \\ b &= \sigma - \vec{z}^T\vec{z}, \\ c &= \det[S] + \vec{z}^T S\vec{z}, \\ d &= \vec{z}^T S^2\vec{z}. \end{aligned}$$

Errors in attitude determination

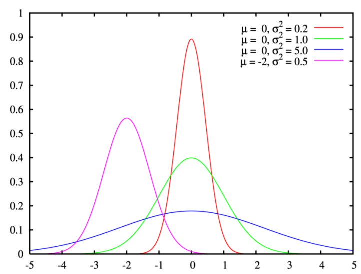
- Errors are, by definition, unknown. Since, if they were known, they would not be errors anymore!
- However, it is important to characterize errors in some way.
- The science that deals with unknowns is statistics (and its associated math field, probability).
- Engineers have to know about statistics, since it can be applied to many fields. Here, we give a refresher for some concepts necessary for estimating errors in attitude determination.
- We will always use normal distributions.
- We go from sensor errors (typically given by their technical specifications) to errors in attitude determination: propagation of uncertainty.

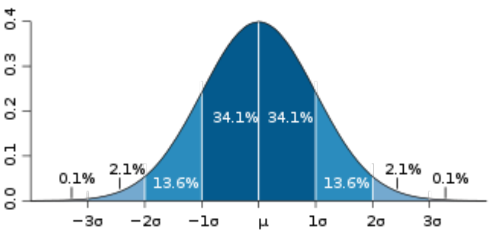
1-D Continuous RandomVariables

- Let $X \in \mathbb{R}$ be a random continuous variable.
- Remember that the cumulative distribution function (CDF) $F(x)$ is the probability that $X \leq x$, which is written as $F(x) = P(X \leq x)$.
- The CDF is computed from the probability density function (PDF) $f(x)$: $F(x) = \int_{-\infty}^{x} f(y)dy$.
- One defines the operator "mathematical expectation" acting over the function $g(x)$ as $E[g(X)] = \int_{-\infty}^{\infty} g(y)f(y)dy$. It is a linear operator: $E[\alpha_1 g_1(X) + \alpha_2 g_2(X)] = \alpha_1 E[g_1(X)] + \alpha_2 E[g_2(X)]$. Two importan examples are:
  - Mean: $m(X) = E[X] = \int_{-\infty}^{\infty} yf(y)dy$.
  - Variance: $V(X) = E[(X - m(X))^2] = E[X^2] - (E[X])^2$ (non-negative).
  - The typical deviation σ is the square root of the variance $\sigma = \sqrt{V(X)}$ to make it have the same units as the mean.
- Does it make sense for errors to have nonzero mean?

Normal (Gaussian) distribution I

- It is the most commonly used distribution in statistics. One writes $X \sim N(m, \sigma^2)$ and its PDF is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{Exp}\left(-\frac{(x-m)^2}{2\sigma^2}\right)$.
- Confidence intervals: if $X \sim N(m, \sigma^2)$ then:
  - 1-σ interval: $P(X \in [m - \sigma, m + \sigma]) = 68.3\%$.
  - 2-σ interval: $P(X \in [m - 2\sigma, m + 2\sigma]) = 95.45\%$.
  - 3-σ interval: $P(X \in [m - 3\sigma, m + 3\sigma]) = 99.74\%$.





Normal (Gaussian) distribution II

- The central limit theorem shows that the sum of independent random variables (with any kind of distribution), tends (in average) to a normal distribution. Since large-scale errors come from the sum and accumulation of many small-scale errors (think for example about temperature fluctuations), this justifies using normal distributions as a good model for errors.
- An important property of a normal distribution is that the sum of independent normals is again normal, this is, if $X \sim N(m_x, \sigma_x^2)$ and $Y \sim N(m_y, \sigma_y^2)$, and they are independent, then $Z = X + Y$ is distributed as $Z \sim N(m_x + m_y, \sigma_x^2 + \sigma_y^2)$.
- Therefore $\sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2}$, this is, the typical deviation of the sum of errors is the square root of the sum of squares of the typical deviation of errors.
- This rule is known as Root-Sum-of-Squares (RSS) and it is of high importance when dealing with accumulated errors.

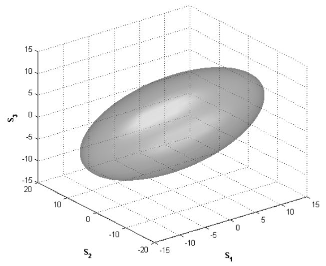
Multivariate Continuous Random Variables

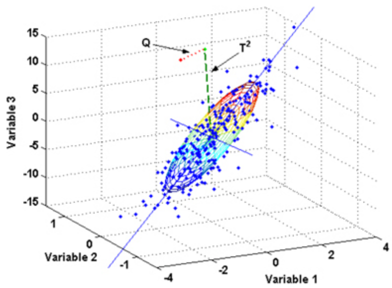
- Let $\vec{X} \in \mathbb{R}^n$ be a multivariate continuous random variables.
- Each component of $\vec{X}$ follows a 1-D distribution (i.e. is a 1-D random variable).
- Following the 1-D case, we now define a *joint* CDF that is computed from a joint PDF $f(\vec{x})$.
- Similarly $E[g(\vec{X})] = \int_{\mathbb{R}^n} g(\vec{y}) f(\vec{y}) dy$. Important cases:
  - Mean: $\vec{m}(\vec{X}) = E[\vec{X}] = \int_{\mathbb{R}^n} \vec{y} f(\vec{y}) dy$.
  - Covariance: $Cov(\vec{X}) = E[(\vec{X} - m(\vec{X}))(\vec{X} - m(\vec{X}))^T] = \Sigma$. A symmetric, non-negative definite matrix. The values of its diagonal represent the variance the corresponding component of $\vec{X}$, whereas off-diagonal coefficients represent the correlation between two components of $\vec{X}$. One has $\Sigma = E[(\vec{X}\vec{X}^T] - m(\vec{X})m(\vec{X})^T$.
- For instance for $n = 3$ and writing $\vec{X} = [X, Y, Z]$:

$$\Sigma = \begin{bmatrix} \sigma_x^2 & E[(X-m_x)(Y-m_y)] & E[(X-m_x)(Z-m_z)] \\ E[(X-m_x)(Y-m_y)] & \sigma_y^2 & E[(Y-m_y)(Z-m_z)] \\ E[(X-m_x)(Z-m_z)] & E[(Y-m_y)(Z-m_z)] & \sigma_z^2 \end{bmatrix}$$

Multivariate normal distribution I

- One writes $\vec{X} \sim N_n(\vec{m}, \Sigma)$ and its PDF is $f(\vec{x}) = \frac{1}{\mathrm{Det}(\Sigma)(2\pi)^{n/2}} \mathrm{Exp}\left(-\frac{1}{2}(\vec{x}-\vec{m})^T \Sigma^{-1} (\vec{x}-\vec{m})\right)$.
- Confidence intervals become regions in $\mathbb{R}^n$, defined by $P(\vec{X} \in \Omega) = P_\Omega$.
- The shape of these regions is a multidimensional ellipsoid described by $(\vec{x}-\vec{m})^T \Sigma^{-1} (\vec{x}-\vec{m}) = d^2$, where d depends on P_Ω. The size of the eigenvalues of Σ determines the size of the ellipsoid, whereas the direction of the ellipsoid axes is given by the eigenvectors of Σ.





Multivariate normal distribution II

- A classical example from aerial navigation or orbital mechanics, one can describe an aircraft/spacecraft position in some axes as $\delta\vec{r} = [\delta x \; \delta y \; \delta z]^T$, as a multivariate normal with $n = 3$, with mean zero (centered in the expected position of the vehicle) and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{bmatrix}$$

- Then one can visualize the movement of the vehicle with the movement of the whole ellipsoid, representing a region (tube) where the vehicle can be found with some degree of certainty.
- Property: If $\vec{X} \sim N_n(\vec{m}_x, \Sigma_x)$ and $\vec{Y} \sim N_n(\vec{m}_y, \Sigma_y)$ and they are independent, then if $\vec{Z} = \vec{X} + \vec{Y}$ it follows that $\vec{Z} \sim N_n(\vec{m}_x + \vec{m}_y, \Sigma_x + \Sigma_y)$.
- Similarly if $A\vec{X} + \vec{b}$ where A and b are non-random (known) it follows that $A\vec{X} + \vec{b} \sim N_n(A\vec{m}_x + \vec{b}, A\Sigma_x A^T)$.

Errors in attitude determination

- How can one characterize attitude errors?
- It will depend on the chosen attitude representation.
- For instance if one chooses quaternions, then one could use the quaternion error, parameterized $\delta q(\vec{a})$ and give a multivariate distribution for $\vec{a}$. Typically with zero mean and some covariance. Then the approximate attitude $\hat{q}$ is related to the real attitude q as in Lesson 2: $\hat{q} = q \star \delta q$ where

$$\delta q(\vec{a}) = \frac{1}{\sqrt{4 + \|\vec{a}\|^2}} \begin{bmatrix} 2 \\ \vec{a} \end{bmatrix}$$

- If one uses the DCM, it is required to find a way to represent some kind of "DCM error".
- It does not make sense to use a 9-dimensional distribution function to characterize the error of each component since attitude does have 3 degrees of freedom, as we know.
- Since errors are (or should be) small, we next characterize DMC errors with an approximation for "small" DMC.

DCM for small angles I

- Let A and B be two reference frames related as follows

$$A \xrightarrow[x^A]{d\theta_1} S_1 \xrightarrow[y^{S_1}]{d\theta_2} S_2 \xrightarrow[z^{S_2}]{d\theta_3} B$$

where we assume that $d\theta_i$ are small angles, so we can make the approximations $\cos d\theta_i \simeq 1$ and $\sin d\theta_i \simeq d\theta_i$.

- Writting the DCMs taking into account the approximations:

$$C_A^{S_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & d\theta_1 \\ 0 & -d\theta_1 & 1 \end{bmatrix}, C_{S_1}^{S_2} = \begin{bmatrix} 1 & 0 & -d\theta_2 \\ 0 & 1 & 0 \\ d\theta_2 & 0 & 1 \end{bmatrix}, C_{S_2}^{B} = \begin{bmatrix} 1 & d\theta_3 & 0 \\ -d\theta_3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- Then, since $C_A^B = C_{S_2}^B C_{S_1}^{S_2} C_A^{S_1}$, and neglecting all double products of angles (i.e. $d\theta_i d\theta_j \simeq 0$), one gets:

$$C_A^B = \begin{bmatrix} 1 & d\theta_3 & -d\theta_2 \\ -d\theta_3 & 1 & d\theta_1 \\ d\theta_2 & -d\theta_1 & 1 \end{bmatrix} = \mathrm{Id} - \begin{bmatrix} 0 & -d\theta_3 & d\theta_2 \\ d\theta_3 & 0 & -d\theta_1 \\ -d\theta_2 & d\theta_1 & 0 \end{bmatrix} = \mathrm{Id} - d\vec{\theta}^{\times},$$

DCM for small angles II

- In the previous slides the definition $d\vec{\theta} = [d\theta_1 \; d\theta_2 \; d\theta_3]^T$ was made, and the matrix

$$d\vec{\theta}^\times = \begin{bmatrix} 0 & -d\theta_3 & d\theta_2 \\ d\theta_3 & 0 & -d\theta_1 \\ -d\theta_2 & d\theta_1 & 0 \end{bmatrix},$$

is the result of the operator $\times$ as was defined in Lesson 2.
- Notice that under these hypothesis (small angles) it does not matter the order of rotations and the angles add up, however not all sets of Euler angles could be used since no axes can be repeated (meaning: 1-2-3 o 3-2-1 or any similar set works, but 1-2-1 would not).
- Exercise: work out the (very simple!) relationship between the small angles vector and the vector $\vec{a}$ used in quaternion errors by using Euler's axis and angle.

Error of a DCM

- To model errors for a DCM we use the "small angles vector" just defined, which will be randomly distributed.
- Denote $\hat{C}_N^B$ the matrix with errors (or actually $\hat{C}_N^B = C_n^{\hat{B}}$), where:

$$N \longrightarrow B \xrightarrow[x^b]{\delta\phi_x} S_1 \xrightarrow[y^{S_1}]{\delta\phi_y} S_2 \xrightarrow[z^{S_2}]{\delta\phi_z} \hat{B}$$

- Then $C_N^{\hat{B}} = C_B^{\hat{B}} C_N^B$ and thus $C_N^B = C_{\hat{B}}^B C_N^{\hat{B}}$, and we define $\delta C_N^B = C_N^B - \hat{C}_N^B = C_{\hat{B}}^B \hat{C}_N^B - \hat{C}_N^B = (C_{\hat{B}}^B - \mathrm{Id})\hat{C}_N^B$.
- Assuming $\delta\vec{\phi} = [\delta\phi_x \; \delta\phi_y \; \delta\phi_z]^T$ are small, one has $C_B^{\hat{B}} = \mathrm{Id} - \delta\vec{\phi}^\times$ (and $C_{\hat{B}}^B = \mathrm{Id} + \delta\vec{\phi}^\times$).
- Then the relationship between the "error matrix" δC_N^B and $\delta\vec{\phi}$ is $\delta C_N^B = (\mathrm{Id} + \delta\vec{\phi}^\times - \mathrm{Id})\hat{C}_N^B = \delta\vec{\phi}^\times \hat{C}_N^B$.And one has $C_N^B = (\mathrm{Id} + \delta\vec{\phi}^\times)\hat{C}_N^B$.

Covariance matrix for TRIAD

- For TRIAD, one can model the error as a small angles vector $\delta\vec{\phi}$ given by a multivariate normal with zero mean and covariance $P_{\phi\phi}$. One can prove:

$$P_{\phi\phi} = \sigma_1^2 \mathrm{Id} + \frac{1}{|\vec{W}_1 \times \vec{W}_2|^2}\left((\sigma_2^2 - \sigma_1^2)W_1 W_1^T + \sigma_1^2 (W_1^T W_2)(W_1 W_2^T + W_2 W_1^T)\right)$$

  where σ_1 represents the angular error (given as typical deviation) of the first measurement and σ_2 the error of the second measurement.
- Notice, as expected, that the first measurement has more influence on the final error.
- If the measurements are orthogonal, then:

$$P_{\phi\phi} = \sigma_1^2 \mathrm{Id} + (\sigma_2^2 - \sigma_1^2)W_1 W_1^T$$

- Imagine for instance if W_1 is the x axis, then this results in $P_{\phi\phi}$ diagonal, with the (1,1) entry as σ_2^2 and the other diagonal coefficients as σ_1^2: Can you interpret this?

Attitude determination: algorithms
Errors in attitude determination
A brief refresher of statistics
Errors in attitude determination methods

Covariance matrix for q

- Now $\vec{a}$ represents the attitude error (via $\delta q(\vec{a})$) and therefore we model $\vec{a}$ as a multivariate distributed vector with zero mean and covariance matrix P_a.
- In the q algorithm each measurement has an error represented by its variance σ_i^2. The global error of q depends on the chosen weights and one can prove the following relationship

$$P_a = \left[\mathrm{Id} - \sum_{i=1}^n a_i \vec{W}_i \vec{W}_i^T\right]^{-1} \left[\sum_{i=1}^n a_i^2 \sigma_i^2 \left[\mathrm{Id} - \vec{W}_i \vec{W}_i^T\right]\right] \left[\mathrm{Id} - \sum_{i=1}^n a_i \vec{W}_i \vec{W}_i^T\right]^{-1}$$

- A good rule of thumb for a_i is make it proportional to the inverse of the variances σ_i^2, however since the a_i's add up to 1, one chooses $a_i = \frac{\frac{1}{\sigma_i^2}}{\sum_{j=1}^n \frac{1}{\sigma_j^2}}$ so $P_a = \left[\sum_{j=1}^n \frac{1}{\sigma_j^2}\mathrm{Id} - \sum_{i=1}^n \frac{1}{\sigma_i^2}\vec{W}_i \vec{W}_i^T\right]^{-1}$.
- Note that $\left[\mathrm{Id} - \sum_{i=1}^n a_i \vec{W}_i \vec{W}_i^T\right]$ should be invertible.
- Exercise: consider the particular case analyzed for TRIAD with equal weights and compare.

Spacecraft Dynamics

Lesson 4: Attitude Kinematics

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June 30, 2022

Attitude Differential Kinematic Equations

- Remember that, when talking about displacements, the differential kinematic equations (for short:kinematics) relate the position and velocity vectors whereas the differential dynamic equations (dynamics) relate the velocity and force vectors.
- For attitude, the kinematics relate the chosen representation of attitude (DCM, Euler angles, quaternions,...) with the angular velocity $\vec{\omega}$ (normally, expressed in body axes). Typically these equations are non-linear.
- In attitude estimation (which is a part of inertial navigation), gyros measure $\vec{\omega}$ and one uses kinematics (integrating the equation) to compute attitude (Lesson 6).
- Thus, it is important to know the kinematics for the different representations, to see the possible computational advantages (hint: quaternions win).

DCM kinematics I

- Suppose we want to compute the attitude of a frame B w.r.t. to A, using the DCM $C_A^B(t)$, knowing B is *rotating* w.r.t. A at an angular velocity $\vec{\omega}_{B/A}^B$.
- By definition $\frac{d}{dt}C_A^B = \frac{C_A^B(t+dt)-C_A^B(t)}{dt}$ (if someone prefers limits the reasoning is analogous)
- Fixing A, we can imagine that B is moving, so in fact $B = B(t)$ and, formally, we can write $C_A^B(t) = C_A^{B(t)}$.
- Using this reasoning, $C_A^B(t+dt) = C_A^{B(t+dt)} = C_{B(t)}^{B(t+dt)} C_A^{B(t)}$. Then:

$$A \longrightarrow B(t) \longrightarrow B(t+dt)$$

- During a time dt, the reference frame B has rotated w.r.t to itself just a small angle; remembering Lesson 3: $C_{B(t)}^{B(t+dt)} = \mathrm{Id} - \left(d\vec{\theta}^B\right)^\times$, where $d\vec{\theta}^B$ is a small angles vector.

DCM kinematics II

- Then: $\frac{d}{dt}C_A^B = \frac{C_A^B(t+dt)-C_A^B(t)}{dt} = \frac{C_{B(t)}^{B(t+dt)}C_A^B(t)-C_A^B(t)}{dt} = \frac{(\mathrm{Id}-\left(d\vec{\theta}^B\right)^\times)C_A^B(t)-C_A^B(t)}{dt} = -\frac{\left(d\vec{\theta}^B\right)^\times}{dt}C_A^B(t)$
- The matrix $\frac{\left(d\vec{\theta}^B\right)^\times}{dt}$ is written

$$\frac{\left(d\vec{\theta}^B\right)^\times}{dt} = \begin{bmatrix} 0 & -\frac{d\theta_3}{dt} & \frac{d\theta_2}{dt} \\ \frac{d\theta_3}{dt} & 0 & -\frac{d\theta_1}{dt} \\ -\frac{d\theta_2}{dt} & \frac{d\theta_1}{dt} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix},$$

where $\vec{\omega}_{B/A}^B = [\omega_1\ \omega_2\ \omega_3]^T$ since $d\vec{\theta}^B$ is the angle the body rotates in a dt seen from its own frame, w.r.t. reference system A: by definition this is the angular velocity. Then

$$\left(\vec{\omega}_{B/A}^B\right)^\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix},$$

- Thus: $\frac{d}{dt}C_A^B = \dot{C}_A^B = -\left(\vec{\omega}_{B/A}^B\right)^\times C_A^B$.

DCM kinematics III

- A variation: transposing both sides of $\dot{C}_A^B = -\left(\vec{\omega}_{B/A}^B\right)^\times C_A^B$ we reach $\dot{C}_B^A. = C_B^A\left(\vec{\omega}_{B/A}^B\right)^\times$
- DCM kinematics: matrix differential equation, solved component-wise (system of 9 coupled scalar ODEs).
- Main difficulty in numerical resolution: conservation of orthogonality. Notice that, since $I = (C_A^B)(C_A^B)^T$, taking derivative:

$$\left[\frac{d}{dt}(C_A^B)\right](C_A^B)^T + C_A^B\frac{d}{dt}(C_A^B)^T$$
$$= -\left(\vec{\omega}_{B/A}^B\right)^\times C_A^B(C_A^B)^T + C_A^B C_B^A\left(\vec{\omega}_{B/A}^B\right)^\times$$
$$= -\left(\vec{\omega}_{B/A}^B\right)^\times + \left(\vec{\omega}_{B/A}^B\right)^\times = 0$$

- Thus kinematics preserve orthogonality. But numerical schemes will not.

DCM kinematics IV

- There exists algorithms to find, given a certain matrix, another orthogonal matrix "closest" to the starting one in some sense.
- For instance, given M, one can compute

$$Q = M(M^TM)^{-1/2}$$

which is orthogonal (and equal to M if it was orthogonal to start with).
- Problem: computing the square root of a matrix is not simple. An iterative method that avoids the computation is the following.
- Start: $Q_0 = M$; iterate $Q_{k+1} = 2M(Q_k^{-1}M + M^TQ_k)^{-1}$, and it's easy to see that this converges to Q when $k \to \infty$, with the condition that M is close to some orthogonal matrix (and therefore invertible).
- If M is very close to being orthogonal to start with, convergence is quite fast!

Euler angles kinematics I

- Example: aircraft set of Euler angles (yaw,pitch,roll). Start from the definition:

$$n \xrightarrow[z^n]{\psi} S \xrightarrow[y^S]{\theta} S' \xrightarrow[x^{S'}]{\varphi} b$$

- Angular velocity can be decomposed between frames as $\vec{\omega}_{b/n} = \vec{\omega}_{b/S'} + \vec{\omega}_{S'/S} + \vec{\omega}_{S/n}$.
- Writing the equation in b: $\vec{\omega}^b_{b/n} = \vec{\omega}^b_{b/S'} + \vec{\omega}^b_{S'/S} + \vec{\omega}^b_{S/n}$
- On the other hand: $\vec{\omega}^b_{b/S'} = [\dot{\varphi}\ 0\ 0]^T$, $\vec{\omega}^{S'}_{S'/S} = [0\ \dot{\theta}\ 0]^T$, $\vec{\omega}^S_{S/n} = [0\ 0\ \dot{\psi}]^T$.
- Then: $\vec{\omega}^b_{b/n} = \vec{\omega}^b_{b/S'} + C^b_{S'}\vec{\omega}^{S'}_{S'/S} + C^b_S\vec{\omega}^S_{S/n}$ and since $C^b_S = C^b_{S'}C^{S'}_S$, we reach: $\vec{\omega}^b_{b/n} = \vec{\omega}^b_{b/S'} + C^b_{S'}\vec{\omega}^{S'}_{S'/S} + C^b_{S'}C^{S'}_S\vec{\omega}^S_{S/n}$

Euler angles kinematics II

- Developing:

$$\begin{aligned}
\vec{\omega}^b_{b/n} &= \begin{bmatrix} \dot{\varphi} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\varphi & \mathrm{s}\varphi \\ 0 & -\mathrm{s}\varphi & \mathrm{c}\varphi \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\varphi & \mathrm{s}\varphi \\ 0 & -\mathrm{s}\varphi & \mathrm{c}\varphi \end{bmatrix} \begin{bmatrix} \mathrm{c}\theta & 0 & -\mathrm{s}\theta \\ 0 & 1 & 0 \\ \mathrm{s}\theta & 0 & \mathrm{c}\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \\
&= \begin{bmatrix} \dot{\varphi} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{c}\varphi\dot{\theta} \\ -\mathrm{s}\varphi\dot{\theta} \end{bmatrix} + \begin{bmatrix} -\mathrm{s}\theta\dot{\psi} \\ \mathrm{s}\varphi\mathrm{c}\theta\dot{\psi} \\ \mathrm{c}\varphi\mathrm{c}\theta\dot{\psi} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & -\mathrm{s}\theta \\ 0 & \mathrm{c}\varphi & \mathrm{s}\varphi\mathrm{c}\theta \\ 0 & -\mathrm{s}\varphi & \mathrm{c}\varphi\mathrm{c}\theta \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}
\end{aligned}$$

Euler angles kinematics III

- What we actually need is an expression for the time derivatives of angles as a function of $\vec{\omega}^b_{b/n} = [\omega_1\ \omega_2\ \omega_3]^T$, therefore, inverting the matrix we reach

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\mathrm{s}\theta \\ 0 & \mathrm{c}\varphi & \mathrm{s}\varphi\mathrm{c}\theta \\ 0 & -\mathrm{s}\varphi & \mathrm{c}\varphi\mathrm{c}\theta \end{bmatrix}^{-1} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \frac{1}{\mathrm{c}\theta} \begin{bmatrix} \mathrm{c}\theta & \mathrm{s}\theta\mathrm{s}\varphi & \mathrm{s}\theta\mathrm{c}\varphi \\ 0 & \mathrm{c}\varphi\mathrm{c}\theta & -\mathrm{s}\varphi\mathrm{c}\theta \\ 0 & \mathrm{s}\varphi & \mathrm{c}\varphi \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

- Notice these are 3 non-linear ODEs, with several trig functions.
- There is a singularity at $\theta = \pm 90^{\mathrm{o}}$. In fact Euler angles are not well defined for this attitude. This singularity is the reason why Euler angles are frequently avoided in inertial navigation (for aircraft or spacecraft).
- All other sets of Euler angles also exhibit singularities; there is no combination of angles free of them.

Euler's axis and angle kinematics

- Representation as Euler's axis and angle, namely $(\vec{e}^{\,b}_{b/n}, \theta)$, has the following kinematics:
- For Euler's angle: $\dot{\theta} = (\vec{e}^{\,b}_{b/n})^T \vec{\omega}^b_{b/n}$
- For Euler's axis:

$$\dot{\vec{e}}^{\,b}_{b/n} = \frac{1}{2}\left[\left(\vec{e}^{\,b}_{b/n}\right)^{\times} + \frac{1}{\tan\theta/2}\left(\mathrm{Id} - \vec{e}^{\,b}_{b/n}(\vec{e}^{\,b}_{b/n})^T\right)\right]\vec{\omega}^b_{b/n}$$

- These are 4 ODEs, non-linear.
- They exhibit a singularity at $\theta = 0$.
- If $\vec{\omega}$ has a constant direction equal to the initial axis $\vec{e}$, then kinematics simplify to $\dot{\vec{e}} = \vec{0}$ (this is, $\vec{e}(t) = \vec{e}(0)$) and $\dot{\theta} = \|\vec{\omega}\|$ (important case!).
- In practice these are seldom used; we just apply them as an intermediate step towards quaternion kinematics.

Quaternion kinematics I

- Remember the attitude quaternion defined from Euler's angle and axis:
  $q_0 = \cos\theta/2$, $\vec{q} = \sin\theta/2\vec{e}^{\,b}_{b/n}$.
- Taking derivative in the q_0 definition and substituting the kinematics for θ, one gets
  $\dot{q}_0 = -\frac{1}{2}\sin\theta/2\dot{\theta} = -\frac{1}{2}\sin\theta/2(\vec{e}^{\,b}_{b/n})^T\vec{\omega}^{\,b}_{b/n} = -\frac{1}{2}\vec{q}^{\,T}\vec{\omega}^{\,b}_{b/n}$
- Taking derivative now in the $\vec{q}$ definition:

$$\dot{\vec{q}} = \frac{1}{2}\cos\theta/2\vec{e}^{\,b}_{b/n}\dot{\theta} + \sin\theta/2\dot{\vec{e}}^{\,b}_{b/n}$$

- Substituting Euler's axis and angle kinematics:

$$\begin{aligned}\dot{\vec{q}} &= \frac{1}{2}\cos\theta/2\vec{e}^{\,b}_{b/n}(\vec{e}^{\,b}_{b/n})^T\vec{\omega}^{\,b}_{b/n} \\ &+\frac{1}{2}\sin\theta/2\left[\left(\vec{e}^{\,b}_{b/n}\right)^{\times} + \frac{1}{\tan\theta/2}\left(\mathrm{Id} - \vec{e}^{\,b}_{b/n}(\vec{e}^{\,b}_{b/n})^T\right)\right]\vec{\omega}^{\,b}_{b/n} \\ &= \frac{1}{2}\left[q^{\times} + q_0\mathrm{Id}\right]\vec{\omega}^{\,b}_{b/n}\end{aligned}$$

Quaternion kinematics II

- Quaternion kinematics in matrix form:

$$\frac{d}{dt}\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

  where $\vec{\omega}^{\,b}_{b/n} = [\omega_x \; \omega_y \; \omega_z]^T$.
- These are 4 bilinear ODEs, without singularities.
- Notice the absence of trig functions, which helps precision.
- These properties of quaternion kinematics are perhaps the most important reasons why its use is wide among the aerospace community to represent spacecraft (and aircraft!) attitude. All computations can be done (internally) with quaternions, and if necessary one can transform them to other representations for visualization or other purposes, depending on the application.

Quaternion kinematics III

- Remembering the definition of quaternion product as a matrix, one can notice some similarities with the differential kinematic equation. In fact, defining a "quaternion" q_ω with zero scalar part and whose vector part is equal to the components of the angular velocity, namely:

$$q_\omega = \left[\ 0\ \omega_x\ \omega_y\ \omega_z\ \right]^T$$

kinematics can be expressed very simply as

$$\dot{q} = \frac{1}{2} q \star q_\omega$$

- The only drawback of using quaternion kinematics is that numerical errors can creep in and make the quaternion modulus different from 1. However, unlike the DCM, making the quaternions verify its constraint is easy; just normalizing the quaternion (dividing by its modulus) we can make its modulus stay at one.

Attitude Differential Kinematic Equations
DCM
Euler angles
Quaternions

Other kinematics

- RP:

$$\dot{\vec{g}} = \frac{1}{2}\left[\mathrm{Id} + \vec{g}^\times + \vec{g}\vec{g}^T\right]\vec{\omega}$$

- MRP:

$$\dot{\vec{p}} = \frac{1+\|\vec{p}\|^2}{4}\left[\mathrm{Id} + 2\frac{\vec{p}^\times + \vec{p}^\times\vec{p}^\times}{1+\|\vec{p}\|^2}\right]\vec{\omega}$$

- Rotation vector:

$$\dot{\vec{\theta}} = \vec{\omega} + \frac{1}{2}\vec{\theta}\times\vec{\omega} + \frac{1}{\theta}\left(1 - \frac{\theta}{2\tan\theta/2}\right)\vec{\theta}\times(\vec{\theta}\times\vec{\omega})$$

Slew maneuvers

- Given two different attitudes expressed as quaternions, q_0 and q_1 and some time interval T, can we *construct* a continuous angular velocity $\vec{\omega}(t)$ such that $q(t=0)=q_0$ and $q(t=T)=q_1$?
- The key to do it is, as in interpolation, to find the so-called rotation quaternion q_2 representing the attitude between q_0 and q_1: $q_2 = \frac{1}{q_0} \star q_1 = q_0^* q_1$. From this quaternion extract Euler's angle θ_1 and axis $\vec{e}$ which verify $q_2 = \begin{bmatrix} \cos\theta_1/2 \\ \vec{e}\sin\theta_1/2 \end{bmatrix}$, this is, $\theta_1 = 2\arccos(q_{20})$ and $\vec{e} = \frac{\vec{q}_2}{\sin\theta_1/2}$
- The solution angular speed $\vec{\omega}(t)$ goes in the direction of $\vec{e}$ and represents the shortest rotation. Call its modulus $\omega(t)$. Then $\theta(t) = \int_0^t \omega(\tau)d\tau$ and the attitude evolves as
$$q(t) = q_0 \star \begin{bmatrix} \cos(\theta(t)/2) \\ \sin(\theta(t)/2)\vec{e} \end{bmatrix}$$
- Any $\omega(t)$ such that $\int_0^T \omega(\tau)d\tau = \theta_1$ is a solution.

Linearizing quaternion kinematics I

- Linearizing is crucial in many aerospace guidance and control applications. Asume we have a reference angular speed $\vec{\omega}_r$ that generates a reference quaternion $\bar{q}$ according to kinematics. If $\vec{\omega} = \vec{\omega}_r + \delta\vec{\omega}$, where $\delta\vec{\omega}$ is "small," what is the new resulting quaternion due to this small change?
- Use the error quaternion as $q = \bar{q} \star \delta q$, and let us determine δq. Taking derivative:
$$\dot{q} = \dot{\bar{q}} \star \delta q + \bar{q} \star \dot{\delta q} = \frac{1}{2} q \star q_\omega$$
- Using $\dot{\bar{q}} = \frac{1}{2}\bar{q} \star q_{\omega_r}$:
$$\frac{1}{2}\bar{q} \star q_{\omega_r} \star \delta q + \bar{q} \star \dot{\delta q} = \frac{1}{2}\bar{q} \star \delta q \star q_\omega$$
- Left-multiplying by $\bar{q}^*$ and solving for $\dot{\delta q}$, one gets:
$$\dot{\delta q} = \frac{1}{2}\delta q \star q_\omega - \frac{1}{2} q_{\omega_r} \star \delta q$$

Linearizing quaternion kinematics II

- Express now $\vec{\omega} = \vec{\omega}_r + \delta\vec{\omega}$ and remember the linearization of δq as a function of the parameter $\vec{a}$:

$$\frac{d}{dt}\left[\begin{array}{c} 1 \\ \vec{a}/2 \end{array}\right] \approx \frac{1}{2}\left[\begin{array}{c} 1 \\ \vec{a}/2 \end{array}\right] \star \left[\begin{array}{c} 0 \\ \vec{\omega}_r + \delta\vec{\omega} \end{array}\right] - \frac{1}{2}\left[\begin{array}{c} 0 \\ \vec{\omega}_r \end{array}\right] \star \left[\begin{array}{c} 1 \\ \vec{a}/2 \end{array}\right]$$

- Remebering: $\left[\begin{array}{c} q_0' \\ \vec{q}' \end{array}\right] \star \left[\begin{array}{c} q_0 \\ \vec{q} \end{array}\right] = \left[\begin{array}{c} q_0' q_0 - \vec{q}'^T \vec{q} \\ q_0 \vec{q}' + q_0' \vec{q} + \vec{q}' \times \vec{q} \end{array}\right]$, one has:

$$\frac{d}{dt}\left[\begin{array}{c} 1 \\ \vec{a}/2 \end{array}\right] \approx \frac{1}{2}\left[\begin{array}{c} -\vec{a}^T/2(\vec{\omega}_r + \delta\vec{\omega}) + \vec{\omega}_r^T \vec{a}/2 \\ \vec{\omega}_r + \delta\vec{\omega} + \vec{a}/2 \times (\vec{\omega}_r + \delta\vec{\omega}) - \vec{\omega}_r - \vec{\omega}_r \times \vec{a}/2 \end{array}\right]$$

- Since we are linearizing $\|\vec{a}\|\|\delta\vec{\omega}\| \approx 0$ because it is a double product of small terms. Operating:

$$\frac{d}{dt}\left[\begin{array}{c} 1 \\ \vec{a}/2 \end{array}\right] \approx \frac{1}{2}\left[\begin{array}{c} 0 \\ \delta\vec{\omega} + \vec{a} \times \vec{\omega}_r \end{array}\right]$$

- This is: $\dot{\vec{a}} \approx \delta\vec{\omega} + \vec{a} \times \vec{\omega}_r$. A quite simple expression. Thus the reference angular velocity also influences $\vec{a}$.

Spacecraft Dynamics

Lesson 5: Attitude Dynamics

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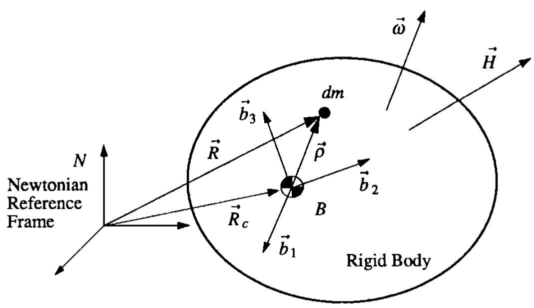
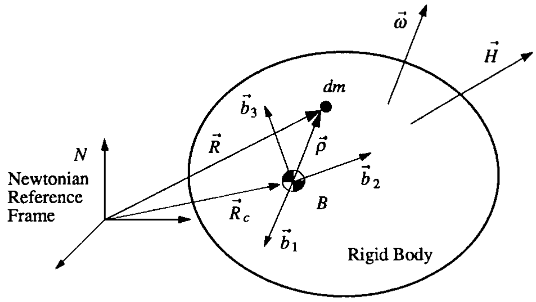
December 18, 2020

Spacecraft attitude dynamics

- Spacecraft attitude dynamics are given by the equations of rotational dynamics. These describe the relation between causes (torques exerted on the vehicle) and effects (angular velocity). Solved together with kinematics.
- Main hypothesis: The vehicle is a rigid body (rigid-body hypothesis). If there are flexible/mobile parts, the model needs to be extended to include them. Thus we can define the rotation of the body frame (fixed at the center of mass of the body) w.r.t. the inertial frame, as in previous lessons.

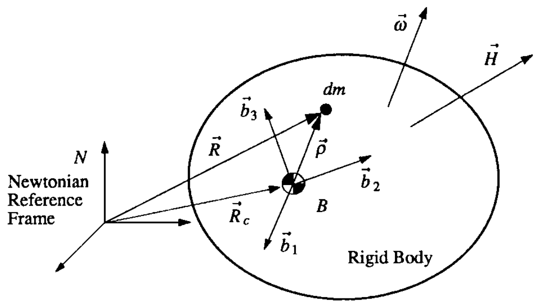


Angular momentum and Torque I



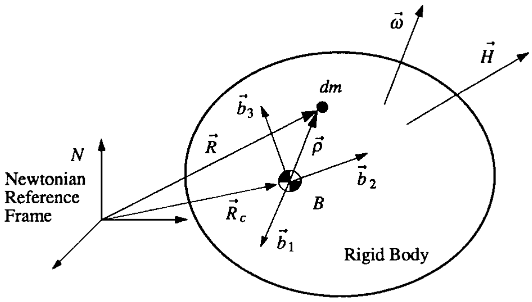
- For each point of the body with mass dm, one has $\ddot{\vec{R}}dm = d\vec{F}$. Taking moment with respect to the center of mass B, we get $\vec{\rho} \times \ddot{\vec{R}}dm = \vec{\rho} \times d\vec{F} = d\vec{M}_B$, and integrating over the volume V, we get a relation involving the total moment of the forces with respect to B (the total *torque*): $\int_V \vec{\rho} \times \ddot{\vec{R}}dm = \vec{M}_B$.
- Notice that these time-derivatives are considered w.r.t. the inertial frame.

Angular momentum and Torque II



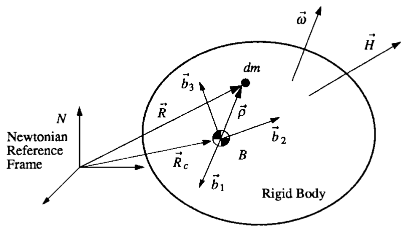
- The absolute angular momentum with respect to B, $\vec{\Gamma}_B$, is defined as: $\vec{\Gamma}_B = \int_V \vec{\rho} \times \dot{\vec{R}}dm$.
- Note $\dot{\vec{\Gamma}}_B = \int_V \dot{\vec{\rho}} \times \dot{\vec{R}}dm + \int_V \vec{\rho} \times \ddot{\vec{R}}dm$.
- Since $\vec{R} = \vec{R}_C + \vec{\rho}$, replacing it in the first term we get: $\dot{\vec{\Gamma}}_B = \int_V \dot{\vec{\rho}} \times \dot{\vec{\rho}}dm + \int_V \dot{\vec{\rho}} \times \dot{\vec{R}}_c dm + \vec{M}_B$
- The first term is zero. The second verifies $\int_V \dot{\vec{\rho}} \times \dot{\vec{R}}_c dm = \left(\frac{d}{dt} \int_V \vec{\rho}dm\right) \times \dot{\vec{R}}_c = \vec{0}$.
- Therefore $\dot{\vec{\Gamma}}_B = \vec{M}_B$

Angular momentum and Inertia I



- The angular momentum $\vec{\Gamma}_B$ verifies
  $\vec{\Gamma}_B = \int_V \vec{\rho} \times \dot{\vec{R}} dm = \int_V \vec{\rho} \times \dot{\vec{R}}_c dm + \int_V \vec{\rho} \times \dot{\vec{\rho}} dm = \int_V \vec{\rho} \times \dot{\vec{\rho}} dm.$
- Remember Coriolis' equation $\left(\frac{d}{dt}\vec{\rho}\right)_N = \left(\frac{d}{dt}\vec{\rho}\right)_B + \vec{\omega}_{B/N} \times \vec{\rho}$, where N is an inertial frame and B the body axes. Then, $\left(\frac{d}{dt}\vec{\rho}\right)_N = \vec{\omega}_{B/N} \times \vec{\rho}$.
- Therefore:
  $\vec{\Gamma}_B = \int_V \vec{\rho} \times (\vec{\omega}_{B/N} \times \vec{\rho}) dm = \left(- \int_V \vec{\rho}^\times \vec{\rho}^\times dm\right) \vec{\omega}_{B/N}$
- Define the inertia tensor
  $\mathcal{I} = - \int_V \vec{\rho}^\times \vec{\rho}^\times dm = \int_V \left[(\rho^T \vec{\rho})\mathrm{Id} - \rho \vec{\rho}^T\right] dm$

Angular momentum and Inertia II



- Thus $\vec{\Gamma}_B = \mathcal{I} \cdot \vec{\omega}_{B/N}$. The explicit expression of the inertia tensor is $\mathcal{I} = \begin{bmatrix} \int_V(\rho_2^2+\rho_3^2)dm & -\int_V \rho_1\rho_2 dm & -\int_V \rho_1\rho_3 dm \\ -\int_V \rho_1\rho_2 dm & \int_V(\rho_1^2+\rho_3^2)dm & -\int_V \rho_2\rho_3 dm \\ -\int_V \rho_1\rho_3 dm & -\int_V \rho_2\rho_3 dm & \int_V(\rho_1^2+\rho_2^2)dm \end{bmatrix}$
- Since the matrix is symmetric: it is diagonalizable. Thus one can find the *principal axes* where $\mathcal{I}$ is diagonal:
  $$\mathcal{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$
- The largest moment of inertia I_i is about an axis which is denoted as major axis; the smallest, about the minor axis. The remaining one is about the intermediate axis.

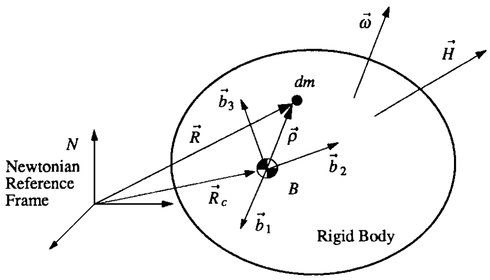
Angular momentum and Inertia III

- Assume we have a vehicle composed of n parts, each of them with known mass M_k, center of mass $\vec{r}_{ck}$ and inertia tensor $\mathcal{I}_k$. Then one can find the inertial tensor of the spacecraft as

$$\mathcal{I} = \sum_{k=1}^{n} \left[M_k \left(\|\vec{r}_{ck}\|^2 \mathrm{Id} - \vec{r}_{ck} \vec{r}_{ck}^{\,T} \right) + \mathcal{I}_k \right]$$

- Note that $\vec{r}_{ck}$ is the vector joining the center of mass of the k part with the whole spacecraft center of mass.
- Spacecraft are formed by a number of structural elements so this is a widely used formula. However, we will not need it in general for our lessons.

Kinetic energy



- Kinetic energy is defined as $T = \frac{1}{2}\int_V \dot{\vec{\rho}} \cdot \dot{\vec{\rho}} dm$.
- Using $\left(\frac{d}{dt}\vec{\rho}\right)_N = \vec{\omega}_{B/N} \times \vec{\rho}$, we get $T = \frac{1}{2}\int_V \dot{\vec{\rho}} \cdot (\vec{\omega}_{B/N} \times \vec{\rho}) dm = \frac{1}{2}\vec{\omega}_{B/N} \cdot \int_V (\vec{\rho} \times \dot{\vec{\rho}}) dm = \frac{1}{2}\vec{\omega}_{B/N} \cdot \vec{\Gamma}_B = \frac{1}{2}\vec{\omega}_{B/N} \cdot \mathcal{I} \cdot \vec{\omega}_{B/N}$.
- In principal axes, if $\vec{\omega}_{B/N} = [\omega_1 \ \omega_2 \ \omega_3]^T$, one gets:
$$\vec{\Gamma}_B = \begin{bmatrix} \omega_1 I_1 \\ \omega_2 I_2 \\ \omega_3 I_3 \end{bmatrix}$$
- Thus: $T = \dfrac{\omega_1^2 I_1 + \omega_2^2 I_2 + \omega_3^2 I_3}{2}$

Euler's Equations

- Start from $\dot{\vec{\Gamma}} = \vec{M}$. Since the time-derivative is in the inertial frame, taking it in body axes we get:
$\left(\frac{d}{dt}\vec{\Gamma}\right)_N = \left(\frac{d}{dt}\vec{\Gamma}\right)_B + \vec{\omega}_{B/N} \times \vec{\Gamma} = \vec{M}$.
- Replacing the expression of angular momentum in terms of the inertia tensor: $\left(\frac{d}{dt}\mathcal{I} \cdot \vec{\omega}_{B/N}\right)_B + \vec{\omega}_{B/N} \times \left(\mathcal{I} \cdot \vec{\omega}_{B/N}\right) = \vec{M}$
- Using the rigid-body hypothesis $\left(\frac{d}{dt}\mathcal{I}\right)_B = 0$, we get:
$\mathcal{I} \cdot \dot{\vec{\omega}}_{B/N} + \vec{\omega}^{\times}_{B/N}\mathcal{I} \cdot \vec{\omega}_{B/N} = \vec{M}$.
- Developing in principal axes and writing $\vec{M} = [M_1 \ M_2 \ M_3]^T$

$$\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= M_1 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= M_2 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= M_3
\end{aligned}$$

Torque-Free rotation

- Our first detailed study is of torque-free rotation, this is, when torque is zero: $\vec{M} = \vec{0}$. Under this assumption, the angular momentum of the system is preserved.
- This does not ever happen in reality since there are always some small perturbing torques (albeit they can be small).
- We will see some analytical solutions but the most interesting results are those concerning the stability of the rotation; in particular, we will find the major axis rule.
- We consider two cases: axisymmetric (two equal moments of inertia: the spinning top) and asymmetric (the three moments of inertia are different)
- The totally symmetric case ($I_1 = I_2 = I_3$) decouples Euler's equations and can be trivially solved (the resulting angular velocities are constant and independent from each other).

Axisymmetric case. Analytical solution.

- Consider $I_1 = I_2 = I$, $I_3 \neq I$.
- Euler's equations now read:

$$\begin{aligned} I\dot{\omega}_1 + (I_3 - I)\omega_2\omega_3 &= 0 \\ I\dot{\omega}_2 + (I - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 &= 0 \end{aligned}$$

- First, we obtain $\omega_3 = \mathrm{Cst} = n$ (spin rate of the spacecraft about it symmetry axes). Define $\lambda = \frac{I-I_3}{I}n$, denoted as the "relative spin rate". The first two equations result in

$$\begin{aligned} \dot{\omega}_1 - \lambda\omega_2 &= 0 \\ \dot{\omega}_2 + \lambda\omega_1 &= 0 \end{aligned}$$

This is the ODE of a harmonic oscillator, whose solution is:

$$\begin{aligned} \omega_1 &= \omega_1(0)\cos\lambda t + \omega_2(0)\sin\lambda t \\ \omega_2 &= \omega_2(0)\cos\lambda t - \omega_1(0)\sin\lambda t \end{aligned}$$

Axisymmetric case. Analytical solution.

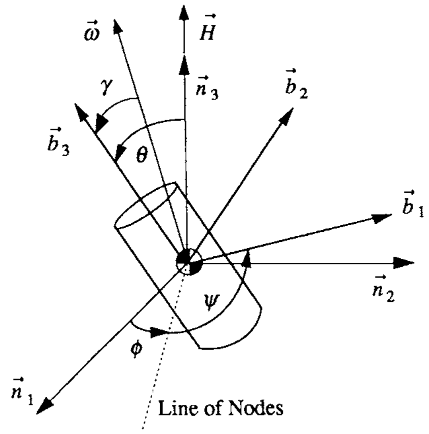
- It is easy to see that $\omega_1^2 + \omega_2^2 = \mathrm{Cst} = \omega_{12}^2$, the so-called transverse angular velocity. Thus, $\|\omega\| = \sqrt{\omega_{12}^2 + n^2} = \mathrm{Cst}$ and its third component is also constant. Therfore, $\vec{\omega}$ seen in the body frame describes a cone about the body symmetry axes, of angle $\gamma = \arctan\left(\frac{\omega_{12}}{n}\right)$.
- On the other hand $\vec{\Gamma} = \vec{\mathrm{Cst}}$ in the inertial frame by conservation of angular momentum. We choose the z axis of the inertial frame as pointing in the direction of $\vec{\Gamma}$ ($\vec{H}$ in the figure). In addition $\Gamma = \|\vec{\Gamma}\|$ must be constant as well.
- In body axes, $\vec{\Gamma} = [I\omega_1 \ I\omega_2 \ I_3 n]^T$, so that $\vec{\Gamma} \cdot \vec{e}_z^{\,b} = I_3 n = \cos\theta\Gamma$, this is, the angle between $\vec{\Gamma}$ and the body z axis is constant; this angle, θ, is the nutation angle. In addition:

$$\tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{\Gamma^2 - I_3^2 n^2}}{I_3 n} = \frac{I\omega_{12}}{I_3 n} = \frac{I}{I_3}\tan\gamma$$

- Exercise: prove that the angle between $\vec{\Gamma}$ y $\vec{\omega}$ is $\theta - \gamma = cst$.

Axisymmetric case. Analytical solution.

- Thus the situation is as in the figure (where $\vec{H} = \vec{\Gamma}$).



- This justifies introducing Euler angles to describe the movement, in the sequence (3,1,3), where one already knows that $\theta = \mathrm{Cst}$.

$$n \xrightarrow[z^n]{\phi} S \xrightarrow[x^S]{\theta} S' \xrightarrow[z^{S'}]{\psi} BFS$$

Rotational dynamics
Torque-Free rotation
Non-zero torque spins
Analytical/geometrical resolution.
Stability. Major axis rule
Effect of a wheel on rotational dynamics

Axisymmetric case. Analytical solution.

- For the sequence

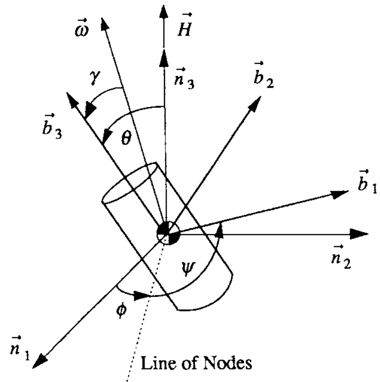
$$n \xrightarrow[z^n]{\phi} S \xrightarrow[x^S]{\theta} S' \xrightarrow[z^{S'}]{\psi} BFS$$

the kinematics are, replacing $\theta = \mathrm{Cst}$:

$$\begin{aligned}\omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi = \dot{\phi}\sin\theta\sin\psi \\ \omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi = \dot{\phi}\sin\theta\cos\psi \\ \omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta\end{aligned}$$

- Applying $\omega_1^2 + \omega_2^2 = \omega_{12}^2$ we obtain: $\omega_{12} = \dot{\phi}\sin\theta$. Thus $\dot{\phi} = \frac{\omega_{12}}{\sin\theta} = \mathrm{Cst}$, the precession rate. Finally $\dot{\psi} = n - \dot{\phi}\cos\theta = n - \frac{\omega_{12}}{\tan\theta} = n - \frac{I_3 n}{I} = n\frac{I-I_3}{I} = \lambda = \mathrm{Cst}$.
- Similarly $\dot{\phi} = \frac{\omega_{12}}{\sin\theta} = \frac{I_3 n}{I\cos\theta} = \frac{I_3(\dot{\psi}+\dot{\phi}\cos\theta)}{I\cos\theta}$, from where $\dot{\phi} = \frac{I_3\dot{\psi}}{(I-I_3)\cos\theta}$.

Axisymmetric case. Geometrical interpretation.

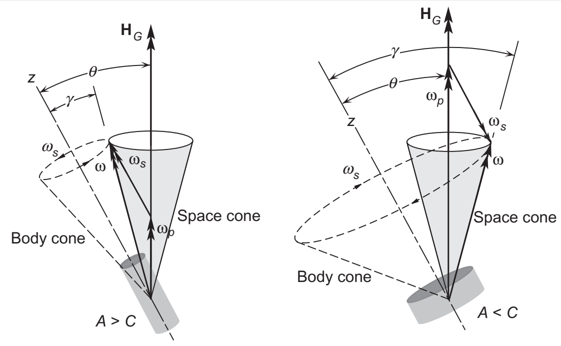


$$n \xrightarrow[z^n]{\phi} S \xrightarrow[x^S]{\theta} S' \xrightarrow[z^{S'}]{\psi} BFS$$

- Considering the sequence and taking into account the fact that the nutation angle is constant and the other two angles change uniformly, one can imagine the movement as the rolling of one cone over another without slipping (with constant angular speeds $\dot{\phi}$ and $\dot{\psi}$); the point of contact is where the angular velocity $\vec{\omega}$ lies.

Rotational dynamics
Torque-Free rotation
Non-zero torque spins
Analytical/geometrical resolution.
Stability. Major axis rule
Effect of a wheel on rotational dynamics

Axisymmetric case. Geometrical interpretation.



(a) Prograde precession (b) Retrograde precession

- Remember $\tan\gamma = \tan\theta\frac{I_3}{I}$ y $\dot{\phi} = \frac{I_3\dot{\psi}}{(I-I_3)\cos\theta}$. Two cases arise:
  - Prolate body (thin symmetry axis, $I_3 < I$): this is case (a). Since $\gamma < \theta$ the cones roll one outside the other and since the signs of $\dot{\phi}$ and $\dot{\psi}$ are equal the rotation is in the same direction (prograde precession).
  - Oblate body (thick symmetry axis, $I_3 > I$): this is case (b). Since $\gamma > \theta$ the cones roll one inside the other and since the signs of $\dot{\phi}$ y $\dot{\psi}$ are opposite the rotation is in the opposite direction (retrograde precession).

Torque-free rotation of an asymmetrical body

- In the asymmetrical case, there exists a major, minor and intermediate axis. The equations cannot be solved in terms of conventional functions.

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= 0 \end{aligned}$$

- Some authors solve these equations by using Jacobi's "elliptical functions". However, it is not easy to understand/interpret these functions, so we take a more "geometric" path.
- Notice that, due to conservation of angular momentum, $\vec{\Gamma}$ is constant (in inertial axes). Therefore $\|\vec{\Gamma}\| = \Gamma$ is constant no matter what axes are used to write $\vec{\Gamma}$. In particular, in the body frame, $\vec{\Gamma} = [I_1\omega_1 \ I_2\omega_2 \ I_3\omega_3]^T$, therefore $\Gamma^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \mathrm{Cst.}$

Rotational dynamics | Analytical/geometrical resolution.
Torque-Free rotation | Stability. Major axis rule
Non-zero torque spins | Effect of a wheel on rotational dynamics

Torque-free rotation of an asymmetrical body

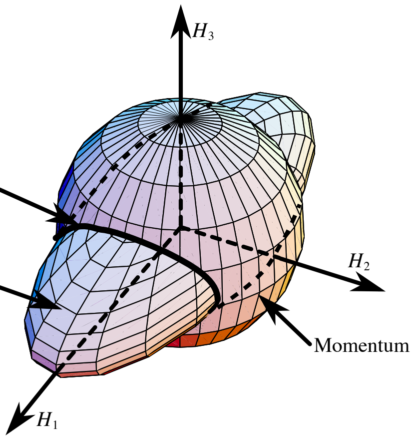
- Similarly, in torque-free rotations the kinetic energy T is also preserved. Which impliese $2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = \mathrm{Cst}'$
- Therefore the components of the angular velocity, $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$, no matter their values, must satisfy

$$\begin{aligned} \frac{\omega_1^2}{\frac{\Gamma^2}{I_1^2}} + \frac{\omega_2^2}{\frac{\Gamma^2}{I_2^2}} + \frac{\omega_3^2}{\frac{\Gamma^2}{I_3^2}} &= 1 \\ \frac{\omega_1^2}{\frac{2T}{I_1}} + \frac{\omega_2^2}{\frac{2T}{I_2}} + \frac{\omega_3^2}{\frac{2T}{I_3}} &= 1 \end{aligned}$$

- These are the equations of two ellipsoids: the angular momentum ellipsoid and the kinetic energy ellipsoid. Thus the angular velocity vector must always lie in the intersection of these two ellipsoids; these intersections are known as "polhode curves".

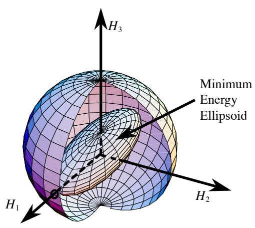
Polhode curves

- In general the curves, for given ellipsoids, are two disjoint, closed curves.

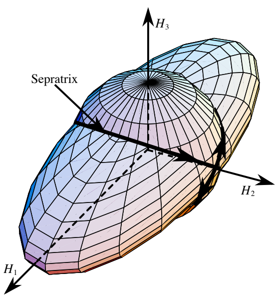


- In two cases the intersection reduces to two points: when the ellipsoids are tangent. These cases correspond to maxima or minima of the energy. In addition, there is a saddle point when the intermediate axes coincide, and the resulting curve is called the separatrix.

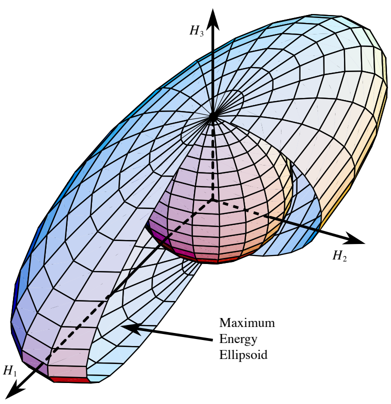
Polhode curves: special cases



(i) Minimum Energy Case



(ii) Intermediate Energy Case



(iii) Maximum Energy Case

Torque-free rotation of an asymmetrical body

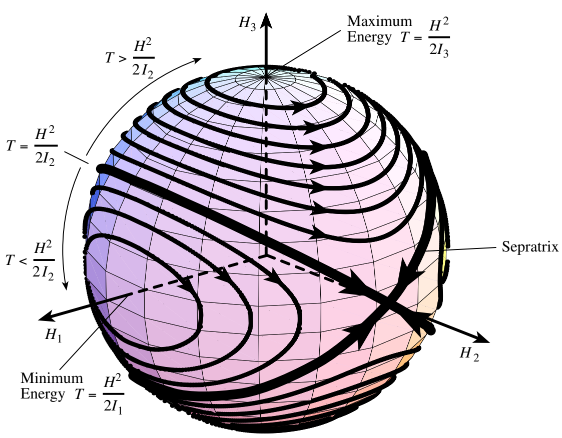
- Assume that $I_3 < I_2 < I_1$ (if not re-index the axes). Define $I^* = \frac{\Gamma^2}{2T}$. Subtracting the ellipsoid equations and multiplying by Γ^2, one gets:

$$I_1\omega_1^2\left(I_1 - I^*\right) + I_2\omega_2^2\left(I_2 - I^*\right) + I_3\omega_3^2\left(I_3 - I^*\right) = 0$$

- Note that if $I^* < I_3$ all terms are positive (for non-zero angular speed) so they cannot add to zero. Similarly if $I^* > I_1$ all terms are negative. Thus, $I^* \in [I_3, I_1]$. For fixed Γ, this implies that kinetic energy has to lie inside an interval. The extrema are $I^* = I_1$ (minimal energy, implies $\omega_2 = \omega_3 = 0$ and thus a rotation about the 1 axis, the major one) and $I^* = I_3$ (maximal energy, implies $\omega_1 = \omega_2 = 0$ and thus a rotation about the 3rd axis, the minor oner)
- The case $I^* = I_2$ has additional solutions besides pure rotations about the 2 axis ($\omega_1 = \omega_3 = 0$); these are called separatrices.

Polhode curves for fixed Γ

- If Γ (H in the figure) is fixed and we vary the energy, we obtain all possible polhode curves over the surface of the momentum ellipsoid, including the separatrices.



Stability of spinning spacecraft about a principal axis

- The simplest solutions of torque-free motion are pure rotations (spins) about a principal axis. Next, we start from the solution of equilibrium $\bar{\omega}_3 = n = \mathrm{Cst}$ and $\bar{\omega}_1 = \bar{\omega}_2 = 0$. We study the stability of this equilibrium as a function of whether the 3rd axis is major, minor or intermediate.
- Let us perturb the equilibrium, defining $\omega_1 = \delta\omega_1$, $\omega_2 = \delta\omega_2$ and $\omega_3 = n + \delta\omega_3$. Substituting in Euler's equations:

$$\begin{aligned} I_1\delta\dot{\omega}_1 + (I_3 - I_2)\delta\omega_2(n + \delta\omega_3) &= 0 \\ I_2\delta\dot{\omega}_2 + (I_1 - I_3)\delta\omega_1(n + \delta\omega_3) &= 0 \\ I_3\delta\dot{\omega}_3 + (I_2 - I_1)\delta\omega_2\delta\omega_1 &= 0 \end{aligned}$$

- Neglecting second-order terms:

$$\begin{aligned} I_1\delta\dot{\omega}_1 + n(I_3 - I_2)\delta\omega_2 &= 0 \\ I_2\delta\dot{\omega}_2 + n(I_1 - I_3)\delta\omega_1 &= 0 \\ I_3\delta\dot{\omega}_3 &= 0 \end{aligned}$$

Rotational dynamics
Torque-Free rotation
Non-zero torque spins
Analytical/geometrical resolution.
Stability. Major axis rule
Effect of a wheel on rotational dynamics

Stability of spinning spacecraft about a principal axis

- The equation of $\delta\omega_3$ defines a marginally stable equilibrium: the perturbed solutions don't grow, but they don't dissipate either.
- The equations for $\delta\omega_1$ and $\delta\omega_2$ can be combined as

$$\delta\ddot{\omega}_1 + \frac{n^2(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\delta\omega_1 = 0$$

- The stability of the solution to this equation depends on the sign of $(I_3 - I_2)(I_3 - I_1)$. For a positive sign, solutions are oscillatory (again, they don't grow or dissipate: marginally stable). If the sign is negative, the solutions are exponential and one of the solutions grows in time (unstable)
- If 3 is the major axis: $(I_3 - I_2)(I_3 - I_1) = + \times + > 0$: stable.
- If 3 is the minor axis:$(I_3 - I_2)(I_3 - I_1) = - \times - > 0$: stable.
- If 3 is the intermediate axis:, $(I_3 - I_2)(I_3 - I_1) = + \times - < 0$: unstable.

Stability of spinning spacecraft with energy dissipation

- While the previous calculation is correct under a rigid-body assumption (Euler's Equations), real-life solids are not perfectly rigid.
- There is always some deviation from the rigid body that can cause some energy dissipation (flexibility effects, friction between mobile parts, fuel sloshing). This modifies the previous calculation as the system tends to go to an energy minima.
- Assume again $I_1 > I_2 > I_3$. One idea (energy sink model) is to, starting from physical principles (conservation of angular momentum), find a minima of energy given the angular momentum. This is, solve the mathematical minimization problem

$$\min I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$$
$$\text{subject to } \; I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \Gamma^2$$

Stability of spinning spacecraft with energy dissipation

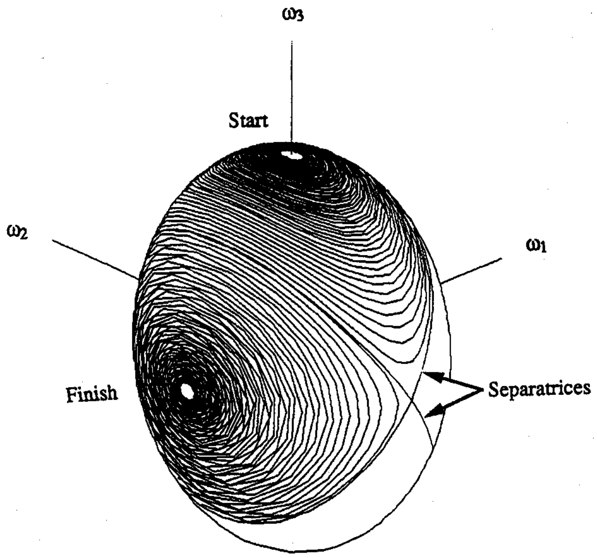
- Using Lagrange multipliers:

$$L(\omega_1,\omega_2,\omega_3,\lambda) = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + \lambda(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 - \Gamma^2)$$

- One has $0 = \frac{\partial L}{\partial \omega_i} = 2I_i\omega_i(1+\lambda I_i), \quad i = 1,2,3$
- Therefore there are three solutions:
  - $\omega_2{=}\omega_3{=}0$, $\lambda = -\frac{1}{I_1}$, $\omega_1 = \frac{\Gamma}{I_1}$. $T = \frac{\Gamma^2}{2I_1}$.
  - $\omega_1{=}\omega_3{=}0$, $\lambda = -\frac{1}{I_2}$, $\omega_2 = \frac{\Gamma}{I_2}$. $T = \frac{\Gamma^2}{2I_2}$.
  - $\omega_1{=}\omega_2{=}0$, $\lambda = -\frac{1}{I_3}$, $\omega_3 = \frac{\Gamma}{I_3}$. $T = \frac{\Gamma^2}{2I_3}$.
- Comparing the values of the objective function (the energy), clearly the minimum is given by the first solution (the second is a saddle point and third one is the maximum). Thus the only spin which is mathematically stable and at the same time a minimum for the energy are rotations about the major axis.
- Based on this argument, we can now state the major axis rule: "For spacecraft with dissipation of energy, the only stable spins are those about the major axis".

Stability of spinning spacecraft with energy dissipation

- The geometrical effect of the major axis rule is that polhodes become a single closed spiral curve that goes from the maximum of energy to the minimum of energy:



Example: fuel sloshing

- Consider a satellite with a spherical tank filled with viscous fuel, so that the fuel (with inertia J and friction coefficient Δ) can be modelled as a "solid bubble" with its own angular speed $\vec{\sigma} = [\sigma_1 \ \sigma_2 \ \sigma_3]^T$ relative to the satellite.
- ExtraCstd from C.D. Rahn, P.M. Barba, "Reorientation Maneuver for Spinning Spacecraft", AIAA Journal of Guidance, Dynamics and Control, Vol. 14, 1991.

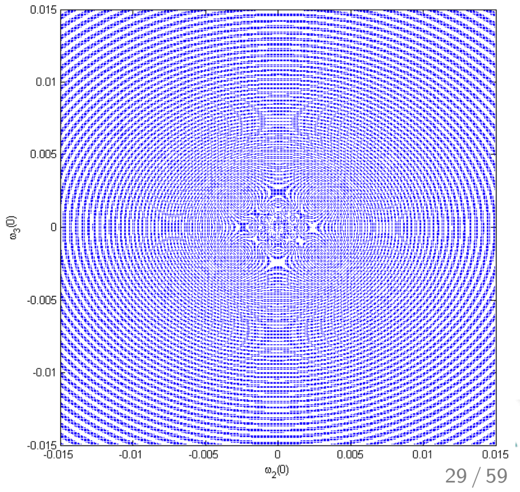
$$\begin{aligned}
(I_1 - J)\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= \Delta\sigma_1 \\
(I_2 - J)\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= \Delta\sigma_2 \\
(I_3 - J)\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= \Delta\sigma_3 \\
\dot{\sigma}_1 + \dot{\omega}_1 + \omega_2\sigma_3 - \omega_3\sigma_2 &= -\frac{\Delta\sigma_1}{J} \\
\dot{\sigma}_2 + \dot{\omega}_2 + \omega_3\sigma_1 - \omega_1\sigma_3 &= -\frac{\Delta\sigma_2}{J} \\
\dot{\sigma}_3 + \dot{\omega}_3 + \omega_1\sigma_2 - \omega_2\sigma_1 &= -\frac{\Delta\sigma_3}{J}
\end{aligned}$$

- By dissipation, any starting spin ends up a major axis spin; however, it is not possible to know a priori the orientation of the rotation, since the equations display strange (chaotics) dynamics.

Example: fuel sloshing

- The fact that the equations have chaotic dynamics means that the sense of rotation totally depends on the initial condition, to the point that any change on the initial condition, no matter how small, can produce a variation in the sense of rotation.

- Thus, to all practical effect, it is not possible to predict the final sense of the rotation.
- A plot in which one marks with the same color the initial conditions producing the same sense of rotation becomes enormously complex, due to this chaotic property of the equation. These kind of plots are known as fractals.



Major axis rule. Additional comments.

- The instability of minor axis spinners is, from the point of view of time-scales, much slower than the instability of intermediate axis spinners, depending on the rate of energy dissipation.
- If one desires a major axis spin one can amplify energy dissipation by adding dampers, such as nutation dampers (pendula with added friction).
- However, if for some reason one needs a minor axis spin this is no issue if it is only required for a short period of time and dissipation is not too large. Later the body will return to a major axis spin naturally.
- Important: the presence of mobile part such as inertia wheels may change these theoretical results.

Rotational dynamics with a wheel

- Let us start with how Euler's equations are modified by the presence of k wheels.
- For each wheel i, assumed axisymmetric, define I_{Ri} as its momentum of inertia in the rotation direction $\vec{e}_i$ and its relative (to the spacecraft) angular speed as ω_{Ri}.
- Since a wheel is symmetric, it does not change the distribution of mass: total spacecraft inertia does not change at all.
- The angular moment of the spacecraft + wheels is: $\vec{\Gamma} = \mathcal{I}\vec{\omega}_{B/N} + \sum_{i=0}^{k} \vec{e}_i I_{Ri}\omega_{Ri}$
- Expressing the derivative $\dot{\vec{\Gamma}} = \vec{M}$ in the body frame one can obtain the differential equations of motion.

Rotational dynamics
Torque-Free rotation
Non-zero torque spins
Analytical/geometrical resolution.
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Three wheels in principal axes

- If there is a wheel about each principal axis, the spacecraft angular momentum is $\vec{\Gamma} = \mathcal{I}\vec{\omega}_{B/N} + \begin{bmatrix} \omega_{R1} I_{R1} \\ \omega_{R2} I_{R2} \\ \omega_{R3} I_{R3} \end{bmatrix}$
- Thus the dynamics is given by

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + I_{R1}\dot{\omega}_{R1} + I_{R3}\omega_{R3}\omega_2 - I_{R2}\omega_{R2}\omega_3 &= M_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 + I_{R2}\dot{\omega}_{R2} + I_{R1}\omega_{R1}\omega_3 - I_{R3}\omega_{R3}\omega_1 &= M_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 + I_{R3}\dot{\omega}_{R3} + I_{R2}\omega_{R2}\omega_1 - I_{R1}\omega_{R1}\omega_2 &= M_3 \end{aligned}$$

- One needs to add the equations describen the wheels' spin. For instance, if for each axis an electric motor with (internal) torque J_{Ri} drives the wheels, these equations would be

$$\begin{aligned} I_{R1}(\dot{\omega}_1 + \dot{\omega}_{R1}) &= J_1 \\ I_{R2}(\dot{\omega}_2 + \dot{\omega}_{R2}) &= J_3 \\ I_{R3}(\dot{\omega}_3 + \dot{\omega}_{R3}) &= J_3 \end{aligned}$$

One wheel about the 3rd axis

- Assume that a spacecraft has an inertial wheel about the 3rd axis, with inertia I_R, and spinning at a velocity ω_R relative to the spacecraft. It could even be a part of the spacecraft (see dual spin-stabilization in lesson 7).
- Angular momentum is $\Gamma = [I_1\omega_1 \; I_2\omega_2 \; I_3\omega_3 + I_R\omega_R]^T$.
- Rotational dynamics become

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + I_R\omega_R\omega_2 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 - I_R\omega_R\omega_1 &= 0 \\ I_3\dot{\omega}_3 + I_R\dot{\omega}_R + (I_2 - I_1)\omega_2\omega_1 &= 0 \end{aligned}$$

- One needs to add $I_R(\dot{\omega}_3 + \dot{\omega}_R) = J$, where J is the torque driving the wheel (if any).

Spin stability with a wheel.

- One can use the motor to produce a torque that mantains ω_R constant. Then:

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + I_R\omega_R\omega_2 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 - I_R\omega_R\omega_1 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= 0 \end{aligned}$$

- New terms appear that modify the previous stability analysis. Even the intermediate axis can be made stable! Repeating the steps for mathematical stability:

$$\delta\ddot{\omega}_1 + \frac{(n(I_3 - I_2) + I_R\omega_R)(n(I_3 - I_1) + I_R\omega_R)}{I_1 I_2}\delta\omega_1 = 0$$

- Now if 1 is the minor axis and 2 the major, the condition for stability is $n(I_3 - I_2) + I_R\omega_R > 0$, this is, $\omega_R > \frac{I_2 - I_3}{I_R} n$.
- Next, we repeat the analysis in the case of energy dissipation by using the energy sink method.

Spin stability with a wheel and energy dissipation.

- Let us minimize the energy fixing the angular momentum (since it is a torque-free motion).
- Then

$$\begin{aligned} 2T &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + I_R\omega_R^2, \\ \Gamma^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + (I_3\omega_3 + I_R\omega_R)^2 \end{aligned}$$

- The last term of the energy can be ignored since it is a constant and does not influence the minimization process. The problem is posed as

$$\begin{aligned} &\min I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \\ &\text{subject to } \; I_1^2\omega_1^2 + I_2^2\omega_2^2 + (I_3\omega_3 + I_R\omega_R)^2 = \Gamma^2 \end{aligned}$$

Spin stability with a wheel and energy dissipation.

- Using Lagrange multipliers

$$L(\omega_1,\omega_2,\omega_3,\lambda) = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + \lambda(I_1^2\omega_1^2 + I_2^2\omega_2^2 + (I_3\omega_3 + I_R\omega_R)^2 - \Gamma^2)$$

- One gets $0 = \frac{\partial L}{\partial \omega_i} = 2I_i\omega_i(1+\lambda I_i), \quad i = 1,2$ y $0 = \frac{\partial L}{\partial \omega_3} = 2I_3(\omega_3 + \lambda(I_3\omega_3 + I_R\omega_R))$
- Several solutions exist, we take

$$\omega_1 = \omega_2 = 0, \quad \omega_3 = n, \quad \lambda = -\frac{n}{I_3 n + I_R\omega_R}.$$

- To identify if it is a minimum or not, we use the following theorem: Let $L(x,y,z) = F(x,y,z) + \lambda G(x,y,z)$ be the Lagrangian of the system so that F is the function to minimize and $G(x,y,z) = 0$ the constraint. Then, construct the matrices:

$$H_3 = \begin{bmatrix} 0 & \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} \\ \frac{\partial G}{\partial y} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial y^2} \end{bmatrix}, \quad H_4 = \begin{bmatrix} 0 & \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial x \partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial y^2} & \frac{\partial^2 L}{\partial y \partial z} \\ \frac{\partial G}{\partial z} & \frac{\partial^2 L}{\partial x \partial z} & \frac{\partial^2 L}{\partial y \partial z} & \frac{\partial^2 L}{\partial z^2} \end{bmatrix},$$

Spin stability with a wheel and energy dissipation.

- If x^*, λ^* is the critical point under analysis (i.e., the point that makes the first derivatives of L zero), to determine if there is a minimum or not, it follows that if:
  1. $\frac{\partial G}{\partial x}(x^*, y^*, z^*) \neq 0$
  2. $\mathrm{Det}(H_3(x^*, y^*, z^*, \lambda^*)) < 0$
  3. $\mathrm{Det}(H_4(x^*, y^*, z^*, \lambda^*)) < 0$

  then there is a minimum at the critical point (sufficient condition, not necessary!).
- In our particular case, to verify the theorem, define $x = \omega_3$, $y = \omega_1$, $z = \omega_2$. Then:

$$H_3 = \begin{bmatrix} 0 & 2I_3(I_3 n + I_r\omega_R) & 0 \\ 2I_3(I_3 n + I_r\omega_R) & 2I_3(1+\lambda I_3) & 0 \\ 0 & 0 & 2I_1(1+\lambda I_1) \end{bmatrix},$$

$$H_4 = \begin{bmatrix} 0 & 2I_3(I_3 n + I_r\omega_R) & 0 & 0 \\ 2I_3(I_3 n + I_r\omega_R) & 2I_3(1+\lambda I_3) & 0 & 0 \\ 0 & 0 & 2I_1(1+\lambda I_1) & 0 \\ 0 & 0 & 0 & 2I_2(1+\lambda I_2) \end{bmatrix}.$$

Spin stability with a wheel and energy dissipation.

- The (sufficient) conditions for a minimum are:
  1. $\frac{\partial G}{\partial x}(x^*, y^*, z^*) = 2I_3(I_3 n + I_r\omega_R) \neq 0$ (since if the other two angular speeds are zero, one has $I_3 n + I_r\omega_R = \pm\Gamma \neq 0$).
  2. $\mathrm{Det}(H_3(x^*, y^*, z^*, \lambda^*)) = -8I_3^2(I_3 n + I_r\omega_R)^2 I_1(1+\lambda I_1) < 0$
  3. $\mathrm{Det}(H_4(x^*, y^*, z^*, \lambda^*)) = \mathrm{Det}(H_3)2I_2(1+\lambda I_2) < 0$
- Two conditions are then reached

$$\begin{aligned} 1 + \lambda I_1 &> 0, \\ 1 + \lambda I_2 &> 0. \end{aligned}$$

- Using the value of λ that we derived before:

$$\begin{aligned} 1 - \frac{I_1 n}{I_3 n + I_R\omega_R} &> 0, \\ 1 - \frac{I_2 n}{I_3 n + I_R\omega_R} &> 0. \end{aligned}$$

- One has to be careful with the sign of $I_3 n + I_R\omega_R$ since when solving for ω_R the sign of the inequality can change.

Spin stability with a wheel and energy dissipation.

- Instead of solving for ω_R we can simplify the fraction, reaching:

$$\frac{(I_3 - I_1)n + I_R\omega_R}{I_3 n + I_R\omega_R} > 0,$$
$$\frac{(I_3 - I_2)n + I_R\omega_R}{I_3 n + I_R\omega_R} > 0,$$

- Two cases:
  1. If $I_3 n + I_R\omega_R > 0$, this is, $\omega_R > -\frac{I_3 n}{I_R}$, the conditions reduce to $\omega_R > \frac{(I_1 - I_3)n}{I_R}$, $\omega_R > \frac{(I_2 - I_3)n}{I_R}$.
  2. If $I_3 n + I_R\omega_R < 0$, this is, $\omega_R < -\frac{I_3 n}{I_R}$, the conditions reduce to$\omega_R < \frac{(I_1 - I_3)n}{I_R}$, $\omega_R < \frac{(I_2 - I_3)n}{I_R}$.
- Notice that these conditions are similar (but more restrictive) than the ones obtained without energy dissipation!

Spin stability with a wheel: Example.

- Consider a satellite with a wheel in the 3rd axis with:

$$\mathcal{I} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 20 \end{bmatrix} \mathrm{kg}\cdot\mathrm{m}^2, \qquad n = 60\ \mathrm{r.p.m.}, \qquad I_R = 2\ \mathrm{kg}\cdot\mathrm{m}^2.$$

- Need to study the required spinning speed for the weel for the 3rd axis (intermediate) to be stable.
- With the rigid-body hypothesis (no dissipation): $(n(I_2 - I_3) - I_R\omega_R)(n(I_3 - I_1) + I_R\omega_R) < 0$. Two cases
  1. First parenthesis is negative, second positive. Conditions become: $\omega_R > \frac{n(I_2 - I_3)}{I_R} = 300$ r.p.m. and $\omega_R > \frac{n(I_3 - I_1)}{I_R} = -300$ r.p.m.. Since the first condition is more stringent: $\omega_R > 300$ r.p.m..
  2. Second parenthesis is negative, first positive. Conditions become: $\omega_R < \frac{n(I_2 - I_3)}{I_R} = 300$ r.p.m. and $\omega_R < \frac{n(I_3 - I_1)}{I_R} = -300$ r.p.m.. Now the second condition is more restrictive, therefore $\omega_R < -300$ r.p.m..
- Thus, the spin is stable if $\omega_R > 300$ r.p.m. or if $\omega_R < -300$ r.p.m., but unstable if $\omega_R \in [-300, 300]$ r.p.m.

Spin stability with a wheel and energy dissipation: Example.

- With energy dissipation, two cases show up again:
  1. If $I_3 n + I_R \omega_R > 0$, this is $\omega_R > -\frac{I_3 n}{I_R} = -600$ r.p.m., then $\omega_R > \frac{(I_1 - I_3) n}{I_R} = -300$ r.p.m., $\omega_R > \frac{(I_2 - I_3) n}{I_R} = 300$ r.p.m.. The third condition is more restrictive so $\omega_R > 300$ r.p.m..
  2. If $I_3 n + I_R \omega_R < 0$, this $\omega_R < -\frac{I_3 n}{I_R} = -600$ r.p.m., then $\omega_R < \frac{(I_1 - I_3) n}{I_R} = -300$ r.p.m., $\omega_R < \frac{(I_2 - I_3) n}{I_R} = 300$ r.p.m.. The first condition is the more stringent, thus $\omega_R < -600$ r.p.m..
- Thus, the spin is stable if $\omega_R > 300$ r.p.m. or if $\omega_R < -600$ r.p.m., but unstable if $\omega_R \in [-600, 300]$ r.p.m..
- Notice in $\omega_R \in [-600, -300]$ r.p.m. the two models differ; however, the model with dissipation is more realistic, so the conclusion is that the rigid-body model is failing in that interval of ω_R!

Non-zero torque spins

- In practice there are always some perturbation torques. While typically of small magnitude, they might be persistent (such as gravity gradient which acts in the full orbit at all times). They might be large as well, for instance in the case of imperfectly aligned thrusters during manoeuvres.
- We analyze two cases:
  - Perturbation torque acting on a spinning solid (gyroscopic effect).
  - Gravity gradient stability.

Spinning body subject to a constant external torque.

- Hypothesis:
  - Axisymmetrical spacecraft: $I_1 = I_2 = I$.
  - Spinning spacecraft with speed n about axis 3, this is, $\omega_3 = n$.
  - Perturbation torque M_1 constant about the axis 1. No torque about the other axes.
- Example: spin-stabilized spacecraft making a propulsive manoeuvre with slight unalignment of the thruster axis with the center of mass. If there is no spin, the resulting torque causes an immediate rotation of the vehicle and failure of the manoeuvre.
- We will see that a spinner acquires the so-called "gyroscopic rigidity" and the perturbing torque produces a slight movement of precession and nutation of the spin axis.

Rotational dynamics
Torque-Free rotation
Non-zero torque spins
Constant external torque.
Gravity gradient. Stable orientation.

Spinning body subject to a constant external torque.

- Euler's equations reduce to

$$\begin{aligned} I\dot{\omega}_1 + (I_3 - I)\omega_2\omega_3 &= M_1 \\ I\dot{\omega}_2 + (I - I_3)\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 &= 0 \end{aligned}$$

- We find immediately $\omega_3 = \mathrm{Cst} = n$ and define $\lambda = \frac{I-I_3}{I} n$ y $\mu = \frac{M_1}{I}$. Two equations remain to be solved:

$$\begin{aligned} \dot{\omega}_1 - \lambda\omega_2 &= \mu \\ \dot{\omega}_2 + \lambda\omega_1 &= 0 \end{aligned}$$

- Taking time derivative in the first equation and substituting the second:

$$\ddot{\omega}_1 + \lambda^2\omega_1 = 0$$

- Harmonic oscillator: $\omega_1(t) = A\sin\lambda t + B\cos\lambda t$.

Spinning body subject to a constant external torque.

- Substituting the solution in the 1st equation $\omega_2(t) = A\cos\lambda t - B\sin\lambda t - \frac{\mu}{\lambda}$.
- Replacing initial conditions $\omega_1(0)$ and $\omega_2(0)$ we reach: $B = \omega_1(0)$, $A = \omega_2(0) + \frac{\mu}{\lambda}$. Thus:

$$\begin{aligned}\omega_1 &= \left(\omega_2(0) + \frac{\mu}{\lambda}\right)\sin\lambda t + \omega_1(0)\cos\lambda t = \frac{\mu}{\lambda}\sin\lambda t \\ \omega_2 &= \left(\omega_2(0) + \frac{\mu}{\lambda}\right)\cos\lambda t - \omega_1(0)\sin\lambda t - \frac{\mu}{\lambda} = \frac{\mu}{\lambda}(\cos\lambda t - 1)\end{aligned}$$

where finally we have replaced $\omega_1(0) = \omega_2(0) = 0$.

- Use now Euler angles

$$I \xrightarrow[x^n]{\theta_1} S \xrightarrow[y^S]{\theta_2} S' \xrightarrow[z^{S'}]{\theta_3} BFS$$

- Developing the kinematic equations we stop at:

$$\begin{aligned}\dot\theta_1 &= \frac{\omega_1\cos\theta_3 - \omega_2\sin\theta_3}{\cos\theta_2} \\ \dot\theta_2 &= \omega_1\sin\theta_3 + \omega_2\cos\theta_3 \\ \dot\theta_3 &= \omega_3 + (-\omega_1\cos\theta_3 + \omega_2\sin\theta_3)\tan\theta_2\end{aligned}$$

Spinning body subject to a constant external torque.

- Take zero initial conditions for the angles.
- With the expectation that θ_1 and θ_2 should be rather small whereas θ_3 has to be large (it is the angle of the spin axis) we replace $\cos\theta_2 \approx 1$ y $\tan\theta_2 \approx \theta_2$ (verify later!). Reaching:

$$\begin{aligned}\dot\theta_1 &= \omega_1\cos\theta_3 - \omega_2\sin\theta_3 \\ \dot\theta_2 &= \omega_1\sin\theta_3 + \omega_2\cos\theta_3 \\ \dot\theta_3 &= \omega_3 + \theta_2(-\omega_1\cos\theta_3 + \omega_2\sin\theta_3) = \omega_3 - \theta_2\dot\theta_1\end{aligned}$$

- Assume as well $\omega_3 \gg \theta_2\dot\theta_1$, then we find $\theta_3 = \omega_3 t = nt$.
- The equations for θ_1 y θ_2 are:

$$\begin{aligned}\dot\theta_1 &= \omega_1\cos nt - \omega_2\sin nt \\ \dot\theta_2 &= \omega_1\sin nt + \omega_2\cos nt\end{aligned}$$

- Substituting the values of ω_1 and ω_2 previously found:

$$\begin{aligned}\dot\theta_1 &= \frac{\mu}{\lambda}\sin\lambda t\cos nt - \frac{\mu}{\lambda}(\cos\lambda t - 1)\sin nt = \frac{\mu}{\lambda}(\sin(\lambda - n)t + \sin nt) \\ \dot\theta_2 &= \frac{\mu}{\lambda}\sin\lambda t\sin nt + \frac{\mu}{\lambda}(\cos\lambda t - 1)\cos nt = \frac{\mu}{\lambda}(\cos(\lambda - n)t - \cos nt)\end{aligned}$$

Spinning body subject to a constant external torque.

- By simple integration and using the initial condition we reach

$$\theta_1 = \frac{\mu}{\lambda}\left(\frac{1-\cos(\lambda-n)t}{\lambda-n}+\frac{1-\cos nt}{n}\right)$$
$$\theta_2 = \frac{\mu}{\lambda}\left(\frac{\sin(\lambda-n)t}{\lambda-n}-\frac{\sin nt}{n}\right)$$

- Defining $A_p = \frac{\mu}{\lambda(n-\lambda)}$ y $\omega_p = n-\lambda$, amplitude and frequency of precession, respectively, and $A_n = \frac{\mu}{\lambda n}$ y $\omega_n = n$, amplitude and frequency of nutation, respectively. The solution is then written as:

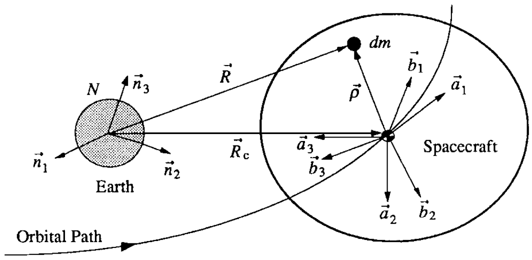
$$\theta_1 = -A_p(1-\cos\omega_p t) + A_n(1-\cos\omega_n t)$$
$$\theta_2 = A_p \sin\omega_p t - A_n \sin\omega_n t$$

- Superposition of two circular movements: epicycloid.
- Amplitudes are given by $A_p = \frac{M_1}{(I-I_3)n^2}\frac{I}{I_3}$ y $A_n = \frac{M_1}{(I-I_3)n^2}$, and the gyroscopic effect increases as n, I_3/I, and the difference $I-I_3$ increases. The amplitudes should be small for the assumptions to be true: large n.

Gravity gradient.

- The most important perturbation torque is gravity gradient, as it is always present in orbit.
- Simplification: consider an asymmetrical spacecraft in circular orbit with radius R around an spherical planets; elliptical orbits and/or deviations from speherical gravity (i.e. the J_2 perturbation) introduce higher-order terms that we do not analyze (they produce the so-called librations: oscillations around the stable orientation).
- Angular velocity is defined as usual in body axes with respect to inertial, but the selected Euler angles are w.r.t. the orbit frame, which is non-inertial. This subtlety has to be taken into account in the analysis.
- The situation is as in the figure of the next slide. N axes are inertial, A axes are from the orbit frame (to be defined) and B the body axes (principal axes of inertia).

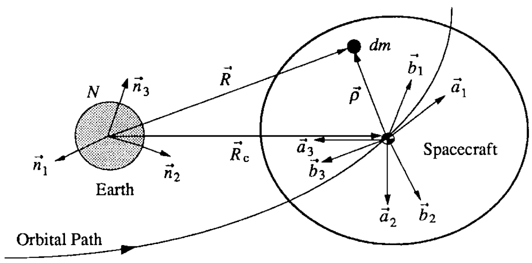
Gravity gradient.



- Orbit frame: centered in the spacecraft. The direction z ($\vec{a}_3$) points towards Earth's center (rotation:yaw). The direction x ($\vec{a}_1$) along the orbital velocity (rotation:roll). The direction y ($\vec{a}_2$) opposite to the orbital angular momentum $\vec{h}$ (orthogonal to the orbital plane, rotation:pitch).
- These axis spin with respect to the inertial frame N about the $-\vec{a}_2$ axis with angular speed $n = \sqrt{\frac{\mu_\oplus}{R^3}}$.
- Thus the relationship between frames is as follows

$$N \xrightarrow[y^n]{-nt} A \xrightarrow[z^A]{\theta_3} S \xrightarrow[y^S]{\theta_2} S' \xrightarrow[x^{S'}]{\theta_1} B$$

Gravity gradient.

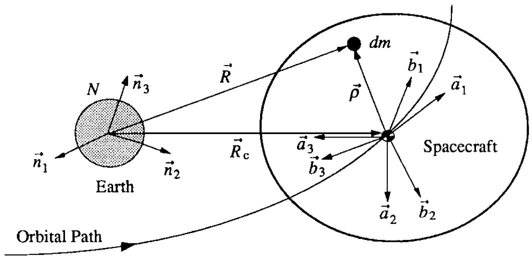


- The matrix C_A^B and its differential kinematic equation is:

$$C_A^B = \begin{bmatrix} c\theta_2 c\theta_3 & c\theta_2 s\theta_3 & -s\theta_2 \\ -c\theta_1 s\theta_3 + s\theta_1 s\theta_2 c\theta_3 & c\theta_1 c\theta_3 + s\theta_1 s\theta_2 s\theta_3 & s\theta_1 c\theta_2 \\ s\theta_1 s\theta_3 + c\theta_1 s\theta_2 c\theta_3 & -s\theta_1 c\theta_3 + c\theta_1 s\theta_2 s\theta_3 & c\theta_1 c\theta_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix} = \frac{1}{c\theta_2} \begin{bmatrix} c\theta_2 & s\theta_2 s\theta_1 & s\theta_2 c\theta_1 \\ 0 & c\theta_1 c\theta_2 & -s\theta_1 c\theta_2 \\ 0 & s\theta_1 & c\theta_1 \end{bmatrix} \vec{\omega}^B_{B/A}$$

Gravity gradient.

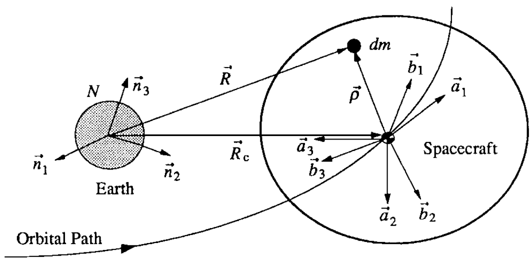


- First let us derive the gravity gradient torque. For each dm of the spacecraft, there is an acting (gravity) force $d\vec{F} = -\frac{\mu\vec{R}}{R^3}dm = -\frac{\mu(\vec{R}_c+\vec{\rho})}{|\vec{R}_c+\vec{\rho}|^3}dm.$
- The moment of the forces is therefore:

$$\vec{M} = \int_V \rho \times d\vec{F} = -\mu \int_V \rho \times \frac{\vec{R}_c + \vec{\rho}}{|\vec{R}_c + \vec{\rho}|^3} dm = -\mu \int_V \frac{\rho \times \vec{R}_c}{|\vec{R}_c + \vec{\rho}|^3} dm$$

Rotational dynamics
Torque-Free rotation
Non-zero torque spins
Constant external torque.
Gravity gradient. Stable orientation.

Gravity gradient.



- Since $|\vec{\rho}| \ll |\vec{R}_c|$, $|\vec{R}_c + \vec{\rho}|^{-3} \approx \frac{1}{R_c^3} - 3\frac{\vec{R}_c \cdot \vec{\rho}}{R_c^5}$. Then:

$$\begin{aligned}\vec{M} &\approx -\frac{\mu}{R_c^3}\int_V \rho \times \vec{R}_c dm + 3\frac{\mu}{R_c^5}\int_V \rho \times \vec{R}_c(\vec{R}_c \cdot \vec{\rho})dm \\ &= 3\frac{\mu}{R_c^5}\int_V \rho \times \vec{R}_c(\vec{R}_c \cdot \vec{\rho})dm = -3\frac{\mu}{R_c^5}\vec{R}_c^{\times}\left(\int_V \vec{\rho}\vec{\rho}^T dm\right)\vec{R}_c \\ &= 3\frac{\mu}{R_c^5}\vec{R}_c^{\times}\mathcal{I}\vec{R}_c - 3\frac{\mu}{R_c^5}\vec{R}_c^{\times}\left(\int_V (|\vec{\rho}|^2)dm\right)\vec{R}_c = 3\frac{\mu}{R_c^5}\vec{R}_c^{\times}\mathcal{I}\vec{R}_c\end{aligned}$$

Gravity gradient.

- Thus $\vec{M} = 3\frac{\mu}{R_c^5}\vec{R}_c^{\times}\mathcal{I}\vec{R}_c$. In the A axes, $\vec{R}_c^A = [0\ 0\ -R_c]^T$. Thus, in the B frame:

$$\vec{R}_c^B = C_A^B\vec{R}_c^A = -R_c\begin{bmatrix} -\mathrm{s}\theta_2 \\ \mathrm{s}\theta_1\mathrm{c}\theta_2 \\ \mathrm{c}\theta_1\mathrm{c}\theta_2 \end{bmatrix}$$

- Thus:

$$\vec{M}^B = 3\frac{\mu}{R_c^3}\begin{bmatrix} 0 & -\mathrm{c}\theta_1\mathrm{c}\theta_2 & \mathrm{s}\theta_1\mathrm{c}\theta_2 \\ \mathrm{c}\theta_1\mathrm{c}\theta_2 & 0 & \mathrm{s}\theta_2 \\ -\mathrm{s}\theta_1\mathrm{c}\theta_2 & -\mathrm{s}\theta_2 & 0 \end{bmatrix}\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{bmatrix} -\mathrm{s}\theta_2 \\ \mathrm{s}\theta_1\mathrm{c}\theta_2 \\ \mathrm{c}\theta_1\mathrm{c}\theta_2 \end{bmatrix}$$

- Operating:

$$\begin{aligned} \vec{M}^B &= 3n^2\begin{bmatrix} 0 & -\mathrm{c}\theta_1\mathrm{c}^2\theta_2 & \mathrm{s}\theta_1\mathrm{c}\theta_2 \\ \mathrm{c}\theta_1\mathrm{c}\theta_2 & 0 & \mathrm{s}\theta_2 \\ -\mathrm{s}\theta_1\mathrm{c}\theta_2 & -\mathrm{s}\theta_2 & 0 \end{bmatrix}\begin{bmatrix} -\mathrm{s}\theta_2 I_1 \\ \mathrm{s}\theta_1\mathrm{c}\theta_2 I_2 \\ \mathrm{c}\theta_1\mathrm{c}\theta_2 I_3 \end{bmatrix} \\ &= 3n^2\begin{bmatrix} -\mathrm{c}\theta_1\mathrm{c}^2\theta_2\mathrm{s}\theta_1(I_2 - I_3) \\ \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_2(I_3 - I_1) \\ \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_2(I_1 - I_2) \end{bmatrix} \end{aligned}$$

Rotational dynamics
Torque-Free rotation
Non-zero torque spins
Constant external torque.
Gravity gradient. Stable orientation.

Gravity gradient.

- Replacing the gravity gradient torque in Euler's equations, we get ODEs for the angular velocity:

$$\begin{aligned} I_1\dot{\omega}_1 &= \left[\omega_2\omega_3 - 3n^2\mathrm{c}\theta_1\mathrm{c}^2\theta_2\mathrm{s}\theta_1\right](I_2 - I_3) \\ I_2\dot{\omega}_2 &= \left[\omega_1\omega_3 + 3n^2\mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_2\right](I_3 - I_1) \\ I_3\dot{\omega}_3 &= \left[\omega_2\omega_1 + 3n^2\mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_2\right](I_1 - I_2) \end{aligned}$$

- On the other hand since $\vec{\omega}_{B/N}^B = \vec{\omega}_{B/A}^B + \vec{\omega}_{A/N}^B = \vec{\omega}_{B/A}^B + C_A^B\vec{\omega}_{A/N}^A$, there follows:

$$\vec{\omega}_{B/A}^B = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} - C_A^B\begin{bmatrix} 0 \\ -n \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + n\begin{bmatrix} \mathrm{c}\theta_2\mathrm{s}\theta_3 \\ \mathrm{c}\theta_1\mathrm{c}\theta_3 + \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{s}\theta_3 \\ -\mathrm{s}\theta_1\mathrm{c}\theta_3 + \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{s}\theta_3 \end{bmatrix}$$

- Then the kinematic ODEs are

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{\mathrm{c}\theta_2}\begin{bmatrix} \mathrm{c}\theta_2 & \mathrm{s}\theta_2\mathrm{s}\theta_1 & \mathrm{s}\theta_2\mathrm{c}\theta_1 \\ 0 & \mathrm{c}\theta_1\mathrm{c}\theta_2 & -\mathrm{s}\theta_1\mathrm{c}\theta_2 \\ 0 & \mathrm{s}\theta_1 & \mathrm{c}\theta_1 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \frac{n}{\mathrm{c}\theta_2}\begin{bmatrix} \mathrm{s}\theta_3 \\ \mathrm{c}\theta_2\mathrm{c}\theta_3 \\ \mathrm{s}\theta_2\mathrm{s}\theta_3 \end{bmatrix}$$

Stable orientation

- System of 6 nonlinear ODEs. Making zero the derivatives we can find the equilibria:

$$
\begin{aligned}
0 &= \left[\omega_2\omega_3 - 3n^2 \mathrm{c}\theta_1 \mathrm{c}^2\theta_2 \mathrm{s}\theta_1\right](I_2 - I_3) \\
0 &= \left[\omega_1\omega_3 + 3n^2 \mathrm{c}\theta_1 \mathrm{c}\theta_2 \mathrm{s}\theta_2\right](I_3 - I_1) \\
0 &= \left[\omega_2\omega_1 + 3n^2 \mathrm{s}\theta_1 \mathrm{c}\theta_2 \mathrm{s}\theta_2\right](I_1 - I_2) \\
\vec{0} &= \frac{1}{\mathrm{c}\theta_2}\begin{bmatrix} \mathrm{c}\theta_2 & \mathrm{s}\theta_2\mathrm{s}\theta_1 & \mathrm{s}\theta_2\mathrm{c}\theta_1 \\ 0 & \mathrm{c}\theta_1\mathrm{c}\theta_2 & -\mathrm{s}\theta_1\mathrm{c}\theta_2 \\ 0 & \mathrm{s}\theta_1 & \mathrm{c}\theta_1 \end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2 \\ \omega_3\end{bmatrix} + \frac{n}{\mathrm{c}\theta_2}\begin{bmatrix}\mathrm{s}\theta_3 \\ \mathrm{c}\theta_2\mathrm{c}\theta_3 \\ \mathrm{s}\theta_2\mathrm{s}\theta_3\end{bmatrix}
\end{aligned}
$$

- One equilibrium is $\omega_1 = \omega_3 = 0$, $\omega_2 = -n$, $\theta_1 = \theta_2 = \theta_3 = 0$,. Warning:there are other possible equilibria (i.e. $\theta_1 = \pi$).
- If we are close to the equilibrium and to analyze its stability, we can consider small angles and linealize the equations, finding

$$
\begin{aligned}
\dot{\omega}_1 &= -\left[n\omega_3 + 3n^2\theta_1\right](I_2 - I_3) \\
\dot{\omega}_2 &= 3n^2\theta_2(I_3 - I_1) \\
\dot{\omega}_3 &= -n\omega_1(I_1 - I_2) \\
\dot{\theta}_1 &= \omega_1 + n\theta_3 \\
\dot{\theta}_2 &= \omega_2 \\
\dot{\theta}_3 &= \omega_3 - n\theta_1
\end{aligned}
$$

Stable orientation

- Taking a derivative in the angle equations

$$
\begin{aligned}
\ddot{\theta}_1 &= \dot{\omega}_1 + n\dot{\theta}_3 \\
\ddot{\theta}_2 &= \dot{\omega}_2 \\
\ddot{\theta}_3 &= \dot{\omega}_3 - n\dot{\theta}_1
\end{aligned}
$$

- Using these equations to eliminate the ω_i's we find

$$
\begin{aligned}
I_1\ddot{\theta}_1 &= -\left[n\dot{\theta}_3 + 4n^2\theta_1\right](I_2 - I_3) + nI_1\dot{\theta}_3 \\
I_2\ddot{\theta}_2 &= 3n^2\theta_2(I_3 - I_1) \\
I_3\ddot{\theta}_3 &= -n(\dot{\theta}_1 - n\theta_3)(I_1 - I_2) - nI_3\dot{\theta}_1
\end{aligned}
$$

- The second equation is stable if $I_3 < I_1$. The first and third are more challenging. Writing the system matrix:

$$
\frac{d}{dt}\begin{bmatrix}\theta_1\\ \theta_3 \\ \dot{\theta}_1 \\ \dot{\theta}_3\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4n^2\frac{I_3-I_2}{I_1} & 0 & 0 & n\frac{I_3-I_2+I_1}{I_1} \\ 0 & n^2\frac{I_1-I_2}{I_3} & n\frac{I_2-I_1-I_3}{I_3} & 0 \end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_3 \\ \dot{\theta}_1 \\ \dot{\theta}_3\end{bmatrix}
$$

- Define $k_1 = \frac{I_2-I_3}{I_1}$ y $k_3 = \frac{I_2-I_1}{I_3}$. Since $I_1 + I_2 > I_3$, $I_2 + I_3 > I_1$, $I_1 + I_3 > I_2$, one gets $k_1, k_3 \in [-1, 1]$.

Stable orientation

- The matrix writes

$$\frac{d}{dt}\begin{bmatrix}\theta_1\\ \theta_3\\ \dot\theta_1\\ \dot\theta_3\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -4n^2k_1 & 0 & 0 & n(1-k_1)\\ 0 & -n^2k_3 & n(k_3-1) & 0\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_3\\ \dot\theta_1\\ \dot\theta_3\end{bmatrix}$$

- Studying the eigenvalues of the matrix, we find the characteristic polynomial:
$\lambda^4 + \lambda^2 n^2(1 + k_1(3 + k_3)) + 4n^4k_1k_3 = 0$, cuya solución es:

$$\lambda = \pm n\sqrt{\frac{-(1 + k_1(3 + k_3)) \pm \sqrt{(1 + k_1(3 + k_3))^2 - 16k_1k_3}}{2}}$$

- Eigenvalues are stable (non-positive real part) if and only if the two options insde the square root are real and negative, this is: $-(1 + k_1(3 + k_3)) \pm \sqrt{(1 + k_1(3 + k_3))^2 - 16k_1k_3} < 0$.
This only happens if:
  - $-(1 + k_1(3 + k_3)) < 0$, this is, $1 + k_1(3 + k_3) > 0$.
  - $\sqrt{(1 + k_1(3 + k_3))^2 - 16k_1k_3}$ is real, this is,$(1 + k_1(3 + k_3))^2 - 16k_1k_3 > 0$.
  - $16k_1k_3 > 0$ (if not there would be a positive number inside the root)

Stable orientation

- Plotting the conditions in a chart:

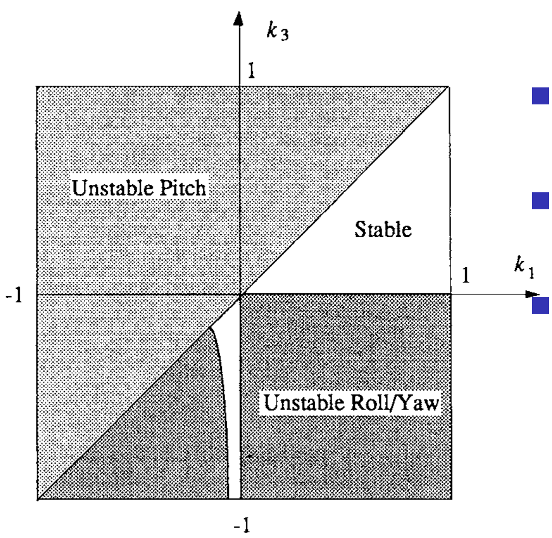


Fig. 6.9 Gravity-gradient stability plot.

- From $16k_1k_3 > 0$, we obtain k_1 and k_3 with the same sign.
- Since $I_3 < I_1$, one gets that $k_1 - k_3 > 0$.
- if $k_1 > k_3 > 0$ we obtain "Lagrange's region " (right-upper triangle).
- There is another region (known as "De Bra-Delp") obtained from $(1 + k_1(3 + k_3))^2 - 16k_1k_3 > 0$. However it is sensitive to energy dissipation, which makes it unstable.

Stable orientation

- In summary:

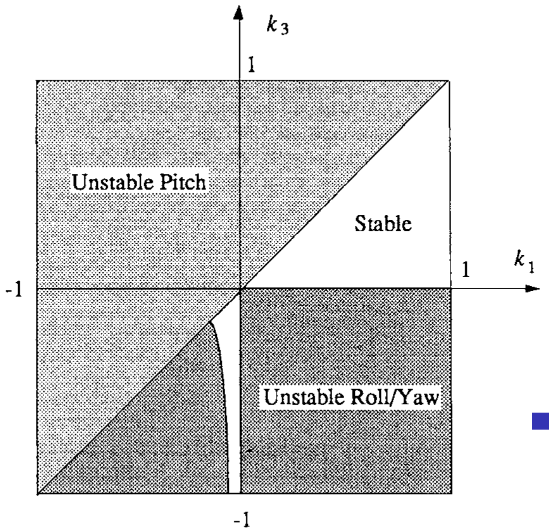


Fig. 6.9 Gravity-gradient stability plot.

- The practical stable zone corresponds to $k_1 > k_3 > 0$, which in turn implies that $I_2 > I_1$ and $I_2 > I_3$. Before we already obtained $I_3 < I_1$. Thus axis 2 (orthogonal to the orbital plane) must be the major axis, axis 3 (pointing to the planet) the minor axis of inertia, and axis 1 (in the direction of orbital velocity) the intermediate.
- Careful: the angles at the equilibrium are 0º but they can also be 180º (the "opposite" attitude is also stable!).
- How many stable equilibria? How many unstable equilibria?

Spacecraft Dynamics

Lesson 6: Attitude estimation. Kalman Filtering.

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June 30, 2022

Attitude estimation

- The (dynamic) estimation of attitude (classicaly known simply as attitude estimation) requires the use of kynematic models, gyro measurements, and Kalman filter, as well as complementary sensors (measuring a direction).
- Gyros measure the angular velocity $\omega^B_{B/I}$ w.r.t. the inertial frame. One can recover the attitude by using this measurement to integrate the kinematic differential equations. Unfortunately, small errors acumulate over time generating a certain drift in the estimation; thus it is always necessary to use additional sensors to improve the measurement.
- To better understand how errors acumulate, one needs to model it as an stochastic (random) process, and use the propagation equations.
- Notation: in this lesson, arrows will not be used for vectors.

Stochastic Processes

- A stochastic process (or stochastic variable) is a random variable $X(t)$ whose distribution evolves (changes) with time. Estimation errors are modelled as this.
- Thus, mean and covariance also change with time: $m(t)$, $\Sigma(t)$.
- For a process, one can define the autocorrelation as $R(t, \tau) = E[X(t)X(\tau)^T]$. Autocorrelation allows to model how the past history of X influences its present value.
- Gaussian process: A Gaussian process verifies $X(t) \sim N_n(m(t), \Sigma(t))$, this is, it is distributed as a multivariate normal whose mean and covariance evolve with time.

Error as an stochastic process
Estimation: Kalman filtering

White noise.

- White noise: It is the process $\nu(t)$ verifying:
  - $E[\nu(t)] = 0$.
  - $E[\nu(t)\nu(t)^T] = Q$.
  - $R(t, \tau) = E[\nu(t)\nu(\tau)^T] = \delta(t - \tau)Q$, where $\delta(x)$ is 1 if $x = 0$ and 0 otherwise.
- The last condition means that the value of white noise at present is independent of its value in any previous instant.
- Gaussian white noise: It is a process verifying the previous conditions and in addition, being Gaussian.
- A good model for sensor errors is $\delta\epsilon(t) = b + D\nu$, where ν is Gaussian white noise. The value of b is the mean of the error (bias), which sometimes is also modelled as a process itself (albeit slowly varying).

Propagation of error. Continuous case

- Consider a differential equation such as

$$\dot{x} = Ax + D\nu,$$

where ν is Gaussian white noise with covariance Q, and the initial condition is also a Gaussian: $x_0 \sim N_n(m_0(t), P_0(t))$. This is called a *stochastic differential equation* (the simplest possible one). Then one has that x is a Gaussian process, $x \sim N_n(m(t), P(t))$, with mean and covariance evolving as follows:

$$\begin{aligned} \dot{m} &= Am, \\ \dot{P} &= AP + PA^T + DQD^T, \\ m(0) &= m_0, \\ P(0) &= P_0 \end{aligned}$$

Error as an stochastic process
Estimation: Kalman filtering

Propagation of error. Discrete case

- Consider a discrete equation of the type

$$x_{k+1} = Ax_k + Db_k,$$

where b_k is Gaussian white noise with covariance Q_k, and the initial condition is also a Gaussian: $x_0 \sim N_n(m_0(t), P_0(t))$This is called a *stochastic discrete process* (the simplest possible one). Then one has that x_k is a Gaussian process, $x_k \sim N_n(m_k(t), P_k(t))$, with mean and covariance evolving as follows:

$$\begin{aligned} m_{k+1} &= Am_k, \\ P_{k+1} &= AP_kA^T + DQ_kD^T, \end{aligned}$$

1-D example: gyro drift

- When one has gyro measurement, one needs to integrate the kinematic differential equations with the measurement.
- To easily grasp the concept of "error as a process", let us analyze the easiest possible case: a single degree of freedom in rotation. Thus, there is a single angle θ, whose kinematic differential equation is

$$\dot{\theta} = \omega$$

- A gyro produces a measurement of ω which we can denote by $\hat{\omega}$; for simplification purposes, assume we have a continuous measurement (it will be fast but not really continuous). In reality, it will not be exactly ω, but it'd rather be corrupted by some noise (which we model as Gaussian white noise, with variance Q related to the quality of the gyro) ν:

$$\hat{\omega} = \omega - \nu$$

Error as an stochastic process
Estimation: Kalman filtering

1-D example: gyro drift

- If one tries to estimate θ (denote the estimation as $\hat{\theta}$) from $\hat{\omega}$ and assuming we know some estimation of its initial value $\hat{\theta}_0$, one would just write:

$$\dot{\hat{\theta}} = \hat{\omega}, \hat{\theta}(0) = \hat{\theta}_0$$

- Thus the estimation error $\delta\theta = \theta - \hat{\theta}$ verifies:

$$\delta\dot{\theta} = \omega - \hat{\omega} = \nu$$

- Assuming some initial error $\delta\theta(0) \approx N(0, P_0)$, one finds by applying the previous theory that the error $\delta\theta(t) \approx N(m(t), P(t))$, with:

$$\dot{m} = 0, m(0) = 0 \longrightarrow m(t) = 0, \quad \dot{P} = Q, P(0) = P_0 \longrightarrow P = P_0 + Qt$$

- Thus, even if the mean of the error is always zero, the variance grows linearly in time and eventually blows up, thus this estimator is useless in the medium-long term (but note error is small in the short term if P_0 was small to begin with).

External measurement

- Now assume one has external measurements of the angle with an additional sensor. When time $t = t_k$ (this is at certain time instants) one gets $\hat{\theta}(t_k)$, which we denote as $\hat{\theta}_k^m$, with some other device (which also should have some associated error, thus $\hat{\theta}_k^m = \theta_k - \epsilon$, where ϵ is white noise with variance R.
- Since time in-between measurements could be large, maybe it is not a good idea to ignore the gyro and say $\hat{\theta}(t) = \hat{\theta}_k^m$ for $t \in [t_k, t_{k+1})$.
- A possible idea is to *reset* the estimator of the previous slide when $t = t_k$, this is, combining the measures as follows:

$$\dot{\hat{\theta}} = \hat{\omega}, \quad \hat{\theta}(t_k) = \hat{\theta}_k^m, \quad t \in [t_k, t_{k+1}),$$

- Thus every new external measurement resets the initial condition of the differential equation and one integrates again.
- It is easy to see that the estimation error now verifies $\delta\theta \approx N(m(t), P(t))$, with $m(t) = 0$ and $\dot{P} = Q$, for $t \in [t_k, t_{k+1})$, with $P(t_k) = R$, thus $P = R + Q(t - t_k)$.

Kalman Filter

- The resetting idea makes the error maximum just before a measurement. The error would be $P = R + Q(t_{k+1} - t_k)$ right at that time instant.
- The problem with resetting is that it neglects the previous estimation from the differential equation, when in-between measurements it does not grow so large (it is short term). The idea of Kalman filtering is to *combine* the estimation from the differential equation (called the "a priori" estimation obtained from a "propagation step") with the external measurement in an "update step" to obtain the "best possible combination" (called the "a posteriori" estimation). The combination is best in the sense that it minimizes the covariance.

Kalman Filter

- Some notation: estimation before the measurement is called a priori and denoted as $\hat{\theta}_k^-$.
- Estimation after the measurement is the a posteri estimation, denoted as $\hat{\theta}_k^+$ and it is computed as:

$$\hat{\theta}_k^+ = \hat{\theta}_k^- + K(\hat{\theta}_k^m - \hat{\theta}_k^-)$$

  where K is the Kalman gain and the parenthesis is the difference between the external measurement and the a priori estimation.
- K is computed to minimize the covariance of the a posteriori error.

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Kalman Filter

- Covariance a priori is called P_k^-.
- Remember the formulas for combination of normals from Lesson 3 (slide 32).
- A posteriori, computing the covariance of θ_k^+:

$$P_k^+ = (1-K)^2 P_k^- + K^2 R$$

- Take derivative w.r.t. K and make it zero to find a minimizer: $0 = -2(1-K)P_k^- + 2KR$, thus $K = \frac{P_k^-}{P_k^- + R}$.
- Then covariance a posteriori becomes with that value of K:

$$P_k^+ = \frac{P_k^- R}{P_k^- + R}$$

- It can be seen that P_k^+ is less than both R and P_k^- (since both are positive numbers): thus one gets to improve estimation by using all the available information in the best way!

Kalman Filter

- Summarizing the algorithm:
- Initialization: For $t_0 = t = 0$ start with $\theta_0^+ = \hat{\theta}_0$ and $P_0^+ = P_0$.
- Propagation: For $t \in [t_k, t_{k+1})$, $k = 0, \dots,$ one integrates from the last a posteriori estimation both the estimation and the covariance of the error

$$\dot{\hat{\theta}} = \hat{\omega}, \quad \hat{\theta}(t_k) = \hat{\theta}_k^+, \qquad \dot{P} = Q, \quad P(t_k) = P_k^+,$$

- Update: When $t = t_{k+1}$ set $\hat{\theta}_{k+1}^- = \hat{\theta}(t_{k+1})$ and $P_{k+1}^- = P(t_{k+1})$, and one gets the external measurement $\hat{\theta}_{k+1}^m$. Apply the KF:

$$\hat{\theta}_{k+1}^+ = \hat{\theta}_{k+1}^- + K(\hat{\theta}_{k+1}^{med} - \hat{\theta}_{k+1}^-),$$

  where $K = \frac{P_{k+1}^-}{P_{k+1}^- + R}$, also $P_{k+1}^+ = \frac{P_{k+1}^- R}{P_{k+1}^- + R}$.
- Increase k and repeat the propagation step.

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Kalman Filter: dependence on process/measurement noise

- If the measurement of the gyro is of very bad quality (Q is very large) then $P_k^- \to \infty$, one can see tha then $P_k^+ \to R$, $K \to 1$, and therefore $\hat{\theta}_k^+ \to \hat{\theta}_k^m$ (this is the resetting method: one takes the external measurement ignoring the result of integrating the differential equation).
- If the external measurement is of very bad quality (R is very large), then $P_k^+ \to P_k^-$, $K \to 0$, and thus $\hat{\theta}_k^+ \to \hat{\theta}_k^-$ (the estimation is just the result of integrating as if there was no external measurement).
- If it happens that $P_k^- \to R$, this is, the a priori estimation and the external measurement have the same level of error, then $P_k^+ \to R/2$, $K \to 1/2$, and then $\hat{\theta}_k^+ \to \frac{\hat{\theta}_k + \hat{\theta}_k^-}{2}$ (one takes the average between the integration step and the external measurement; note that the error is halved).

Kalman Filter: additional considerations

- This is a considerable simplification because only a 1-D linear case has been considered.
- Next the n-D linear case will be studied, the the nonlinear case (addressed by linearization), and finally a special case involving quaternions.
- In any case, conceptually all are the same: one integrates the kinematic differential equation with the gyroscopes and when obtaining an external measurement, the Kalman algorithm is used to weight the a priori estimation and the measurement.
- In aircraft and missiles Kalman Filtering is used to integrate the use of IMUs (gyros+ accelerometers) with external measurements such as GPS.

Kalman Filter for linear systems

- Next the KF will be explained for linear systems which are continuous with discrete measurement.
- All systems are in practice discrete, however, this explanation is simpler conceptually speaking and can be easily implemented in a lab setting.
- In the nomenclature of KF, a system is known as a "process".
- Note that the following development is conceptually very similar to the 1-D example, but more abtruse in terms of notation (and the number of involved matrices).
- KF is used in many engineering contexts (e.g. navigation, orbital mechanics, tracking...). It is a very useful tool to know.

System model (linear case)

- PROCESS: The process is continuous $\dot{x}(t) = A(t)x(t) + B(t)u(t) + D(t)\epsilon(t)$, where x is a Gaussian process of dimension n_x, $A(t)$ is a matrix (that could be time-varying) of dimension $n_x \times n_x$, $\epsilon(t)$ is Gaussian white noise of dimension n_ϵ and convariance $Q(t)$ (process noise), and $D(t)$ is a matrix es una matriz (that could be time-varying) of dimension $n_x \times n_\epsilon$. $u(t)$ if it exists is some input (e.g. gyro measurement) of dimension n_u and $B(t)$ is of dimension $n_x \times n_u$.
- MEASUREMENT: In discrete times $t = t_k$ a measurement z is taken, defined as follows: $z(t_k) = H_k x(t_k) + \nu(t_k)$, where z is of dimension n_z, H_k is a matrix (that could be time-varying) of dimension $n_z \times n_x$, and $\nu(t_k)$ is Gaussian white noise of dimension n_ν and convariance R_k (measurement noise).
- In addition $\nu(t_k)$ and $\epsilon(t)$ should be independent and the initial condition of x is $x(t_0) \sim N_{n_x}(\hat{x}_0, P_0)$.

System model (linear case)

- Summarizing:

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) + D(t)\epsilon(t), \\
z(t_k) &= H_k x(t_k) + \nu(t_k), \\
E[\epsilon(t)] &= E[\nu(t_k)] = 0, \\
E[\epsilon(t)\epsilon^T(\tau)] &= \delta(t-\tau)Q(t), \\
E[\nu(t_k)\nu^T(t_j)] &= \delta_{kj}R_k, \\
E[\epsilon(t)\nu^T(t_j)] &= 0, \\
x(t_0) &\sim N_{n_x}(\hat{x}_0, P_0).
\end{aligned}
$$

- Define the estimation (in t) of $x(t)$ as $\hat{x}(t)$.
- Define the covariance of the estimation error as $P(t) = E[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T]$.
- The goal of KF is, using the above model, and from the measurements $z(t_k)$, obtain the best possible estimation, this is, the value of $\hat{x}(t)$ that minimizes $P(t)$.

KF I

- If there are no measurements one can take $\hat{x}$ as the mean of the process; then, $x(t) \sim N_{n_x}(\hat{x}(t), P_k)$, where:

$$\begin{aligned} \dot{\hat{x}}(t) &= A(t)\hat{x}(t) + B(t)u(t), \\ \dot{P} &= A(t)P + PA^T(t) + D(t)Q(t)D^T(t). \end{aligned}$$

- The idea of the KF is that this is the best we can do until we get a new measurement at $t = t_k$, $z(t_k)$. Denote the estimation until then (the “a priori” estimation) as $\hat{x}^-(t_k)$ and the covariance of the error as P_k^-.
- Now if the estimation and measurement were perfect, one would have $z(t_k) = H_k\hat{x}^-(t_k)$. However, since this is not the case, one updates the estimation (obtaining the “a posteriori” estimation) proportionally to the discrepance between what we expect to measure and what we really measure: $\hat{x}^+(t_k) = \hat{x}^-(t_k) + K_k(z(t_k) - H_k\hat{x}^-(t_k))$.

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KF II

- In $\hat{x}^+(t_k) = \hat{x}^-(t_k) + K_k(z(t_k) - H_k\hat{x}^-(t_k))$ we don't know K_k, which is the Kalman gain. This is determined to guarantee that he covariance of $\hat{x}^+(t_k)$, P_k^+, is as small as possible.
- Compute P_k^+: $P_k^+ = E[(x(t_k) - \hat{x}^+(t_k))(x(t_k) - \hat{x}^+(t_k))^T]$, and replacing $\hat{x}^+(t_k)$:

$$\begin{aligned} P_k^+ &= E\left[\left(x(t_k) - \hat{x}^+(t_k)\right)\left(x(t_k) - \hat{x}^+(t_k)\right)^T\right] \\ &= E\left[\left(x(t_k) - \hat{x}^-(t_k) - K_k(z(t_k) - H_k\hat{x}^-(t_k)\right)\right. \\ &\quad \times \left.\left(x(t_k) - \hat{x}^-(t_k) - K_k(z(t_k) - H_k\hat{x}^-(t_k))\right)^T\right] \end{aligned}$$

- Substituting $z(t_k) = H_kx(t_k) + \nu(t_k)$:

$$\begin{aligned} P_k^+ &= E\left[\left(x(t_k) - \hat{x}^-(t_k) - K_k(H_kx(t_k) + \nu(t_k) - H_k\hat{x}^-(t_k)\right)\right. \\ &\quad \times \left.\left(x(t_k) - \hat{x}^-(t_k) - K_k(H_kx(t_k) + \nu(t_k) - H_k\hat{x}^-(t_k))\right)^T\right] \end{aligned}$$

KF III

- Simplifying terms:

$$\begin{aligned} P_k^+ &= E\left[\left((I - K_k H_k)(x(t_k) - \hat{x}^-) - K_k \nu(t_k)\right) \right. \\ & \quad \left. \times \left((I - K_k H_k)(x(t_k) - \hat{x}^-) - K_k \nu(t_k)\right)^T\right] \\ &= (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k R_k K_k^T \end{aligned}$$

- One needs to find K_k to minimize the previous expression. However one cannot "minimize a matrix" (what does that even mean?). However, since the diagonal of the covariance matrix is the individual variances, one idea is to minimize the trace of the matrix.
- The following mathematical relations help a lot:

$$\frac{\partial \mathrm{Tr}[\mathrm{ABA^T}]}{\partial A} = 2BA^T, \quad \frac{\partial \mathrm{Tr}[\mathrm{AB}]}{\partial A} = B$$

KF III

- Using these relations:

$$\mathrm{Tr}[P_k^+] = \mathrm{Tr}[K_k(R_k + H_k P_k^- H_k^T) K_k^T] - 2\mathrm{Tr}[K_k H_k P_k^-]$$

- Thus:

$$\frac{\partial \mathrm{Tr}[P_k^+]}{\partial K_k} = 2(R_k + H_k P_k^- H_k^T) K_k^T - 2 H_k P_k^-$$

- Equating to zero:

$$K_k^T = (R_k + H_k P_k^- H_k^T)^{-1} H_k P_k^-$$

- Therefore we find an expression for the optimal Kalman gain

$$K_k = P_k^- H_k^T (R_k + H_k P_k^- H_k^T)^{-1}$$

- And substituting in P_k^+ to find the minimum we get

$$P_k^+ = (I - K_k H_k) P_k^-$$

KF algorithm

- Summarizing the algorithm:
  1. (Initialization): In $t = t_k$, we start from $\hat{x}^+(t_k)$ and $P^+(t_k)$. If $k = 0$ we take $\hat{x}^+(t_0) = \hat{x}_0$ y $P_0^+ = P_0$.
  2. (Propagation): For $t \in (t_k, t_{k+1})$, use the process model:

$$\begin{aligned}\dot{\hat{x}} &= A(t)\hat{x} + B(t)u(t), \quad \hat{x}(t_k) = \hat{x}^+(t_k)\\ \dot{P} &= A(t)P + PA^T(t) + D(t)Q(t)D^T(t), \quad P(t_k) = P^+(t_k)\end{aligned}$$

  3. (Update): In $t = t_{k+1}$ we get $z(t_{k+1})$, call $\hat{x}^-(t_{k+1}) = \hat{x}(t_{k+1})$ and $P^-(t_{k+1}) = P(t_{k+1})$. Compute the Kalman gain: $K_{k+1} = P_{k+1}^- H_{k+1}^T \left(H_{k+1}P_{k+1}^- H_{k+1}^T + R_{k+1}\right)^{-1}$. With $z(t_{k+1})$ compute the a posteriori estimation:

$$\begin{aligned}\hat{x}^+(t_{k+1}) &= \hat{x}^-(t_{k+1}) + K_{k+1}(z(t_{k+1}) - H_{k+1}\hat{x}^-(t_{k+1})),\\ P_{k+1}^+ &= (I - K_{k+1}H_{k+1})P_{k+1}^-.\end{aligned}$$

  4. Iterate for the next value of k.

About measurements

- Note: Measurements may change in the different t_k's (more or less measurements).
- This is reflected in changes in H_k (it can even change dimension).

Kalman Filter for nonlinear systems

- Next the EKF will be explained for nonlinear systems which are continuous with discrete measurement.
- The main tool is to linearize *around the estimation*.
- Unfortunately convergence is not guaranteed.
- If the initial estimation is good, the errors are not too large, and the measurements are of decent quality, it should work. However it is very dependent on the quality of the matrices Q and R.

System model (nonlinear case)

- The model is more general:

$$\begin{aligned}
\dot{x}(t) &= f(x,u,t) + D(t)\epsilon(t),\\
z_k &= h(x_k,t_k) + \nu(t_k),\\
E[\epsilon(t)] &= E[\nu(t_k)] = 0,\\
E[\epsilon(t)\epsilon^T(\tau)] &= \delta(t-\tau)Q(t),\\
E[\nu(t_k)\nu^T(t_j)] &= \delta_{kj}R_k,\\
E[\epsilon(t)\nu^T(t_j)] &= 0,\\
x(t_0) &\sim N_{n_x}(\hat{x}_0, P_0).
\end{aligned}$$

- Define the matrices and vectors: $F(\hat{x}(t),t) = \left.\frac{\partial f(x,u,t)}{\partial x}\right|_{x=\hat{x},u}$, $\delta z_k = z_k - h(\hat{x}_k,t_k)$, $H_k(\hat{x}_k) = \left.\frac{\partial h(x,t_k)}{\partial x}\right|_{x=\hat{x}_k}$.

EKF algorithm

- The EKF is as follows:
  1. (Initialization): In $t = t_k$, we start from $\hat{x}^+(t_k)$ and $P^+(t_k)$. If $k = 0$ we take $\hat{x}^+(t_0) = \hat{x}_0$ y $P_0^+ = P_0$.
  2. (Propagation): For $t \in (t_k, t_{k+1})$, use the (nonlinear) process model:
     $$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u, t), \quad \hat{x}(t_k) = \hat{x}^+(t_k) \\ \dot{P} &= F(\hat{x}(t), t)P + PF^T(\hat{x}(t), t) + D(t)Q(t)D^T(t), \quad P(t_k) = P^+(t_k) \end{aligned}$$
  3. (Update): In $t = t_{k+1}$ we get $z(t_{k+1})$, call $\hat{x}^-(t_{k+1}) = \hat{x}(t_{k+1})$ and $P^-(t_{k+1}) = P(t_{k+1})$. Compute $\delta z_{k+1} = z_{k+1} - h(\hat{x}^-_{k+1}, t_{k+1})$ and $H_{k+1} = H_k(\hat{x}^-_{k+1}, t_{k+1})$. Compute the Kalman gain: $K_{k+1} = P^-_{k+1} H^T_{k+1} \left(H_{k+1} P^-_{k+1} H^T_{k+1} + R_{k+1} \right)^{-1}$. Then:
     $$\begin{aligned} \hat{x}^+(t_{k+1}) &= \hat{x}^-(t_{k+1}) + K_{k+1}\delta z_{k+1}, \\ P^+_{k+1} &= (I - K_{k+1}H_{k+1})P^-_{k+1}. \end{aligned}$$
  4. Iterate for the next value of k.

Multiplicative Extended Kalman Filter (MEKF)

- This is specific for attitude estimation.
- The EKF can be altered to take into account that the quaternions cannot be linearized in the standard way, but rather using the quaternion error (in a multiplicative way). Then one gets the MEKF.
  1. Assume one has gyros in the 3 axis, so that angular velocity $\hat{\omega}^B_{B/N}$ is estimated, with white noise error of covariance Q. This is assumed as continouous.
  2. At instants t_k one gets measurements of n directions in body axes $\hat{v}_i^B$, so that $v_i^B = C_N^B v_i^N$ and $v_i^B = \hat{v}_i^B + \epsilon_i$ for $i = 1, \ldots, n$. ϵ_i is Gaussian white noise with covariance R_i.
- With only measurements one could use TRIAD or the q algorithm.
- With only gyros the estimation would be $\dot{\hat{q}} = \frac{1}{2} q \star q_{\hat{\omega}}$.

Multiplicative Extended Kalman Filter (MEKF)

- To linearize kinematics remember the quaternion error $q = \hat{q} \star \delta q$, with

$$\delta q(a) = \frac{1}{\sqrt{4 + \|a\|^2}} \begin{bmatrix} 2 \\ a \end{bmatrix}, \quad \dot{a} \approx \nu + a \times \hat{\omega} = -\hat{\omega}^{\times} a + \nu.$$

- Thus one can study the covariance of the vector a which represents the error:

$$\dot{P} = -\hat{\omega}^{\times} P + P\hat{\omega}^{\times} + Q, \quad P(0) = P_0$$

Multiplicative Extended Kalman Filter (MEKF)

- From the estimated quaternion $\hat{q}$ one can get $\hat{C}_N^B(\hat{q})$ (Euler-Rodrigues).
- Call δz_i the discrepance between measurement and expected measurement: $\delta z_i = \hat{v}_i^B - \hat{C}_N^B(\hat{q}) v_i^N$. If everything was perfect then $\delta z_i = 0$.
- Measurement is not pefect: $\hat{v}_i^B = v_i^B - \epsilon_i$.
- Estimation is not perfect: $\hat{C}_N^B = C_N^{\hat{B}} = C_B^{\hat{B}} C_N^B$.
- Thus $\delta z_i = v_i^B - C_B^{\hat{B}} v_i^B - \epsilon_i$.
- Remember that from the relationship between the error quaternion and the small angles DCM: $C_B^{\hat{B}} = I - a^{\times}$, thus $\delta z_i = -a^{\times} v_i^B - \epsilon_i = (v_i^B)^{\times} a - \epsilon_i$.
- Thus we have n measurements of error in the form $\delta z_i = H_i a - \epsilon_i$, where $H_i \approx (\hat{v}_i^B)^{\times}$. (NOTE: take only two rows to avoid invertibility issues). The covariance of the measurement is R_i.

Multiplicative Extended Kalman Filter (MEKF)

- Use the a priori $(-)$ and a posteriori $(+)$ notation. From integration we had $\hat{q}^-$ with error a^- whose covariance is P^-.
- With the measurements available from

$$\delta z = \begin{bmatrix} \delta z_1 \\ \vdots \\ \delta z_n \end{bmatrix}, H = \begin{bmatrix} H_1 \\ \vdots \\ H_n \end{bmatrix}, R = \begin{bmatrix} R_1 & & \\ & \ddots & \\ & & R_n \end{bmatrix}$$

- Using the measurements $a^+ = a^- + K(z - Ha^-)$, but since the mean of the error is zero, $a^- = 0$, thus: $a^+ = K\delta z$, where K is the Kalman gain, computed as $K = P^- H^T (HP^- H^T + R)^{-1}$. Covariance is updated as $P^+ = P^- - KHP^-$.
- With a^+ update $\hat{q}$:$\hat{q}^+ = \hat{q}^- \star \delta q = \hat{q}^- \star \begin{bmatrix} 2 \\ a^+ \end{bmatrix} \frac{1}{\sqrt{4 + \|a^+\|^2}}$
- This procedure is iterated.

Multiplicative Extended Kalman Filter (MEKF)

- Summary. Initial data: $\hat{q}_0$, P_0, Q, R_i. One considers $\hat{\omega}$ continuous. Ocassionally, one gets measurements and thus can compute $\delta z_i = \hat{v}_i^B - \hat{C}_N^B(\hat{q}) v_i^N$.
  1. Initialize and compute $\hat{q}$ and P:

$$\begin{aligned} \dot{\hat{q}} &= \frac{1}{2} q \star q_{\hat{\omega}}, \quad q(0) = q_0, \\ \dot{P} &= -\hat{\omega}^\times P + P\hat{\omega}^\times + Q, \quad P(0) = P_0 \end{aligned}$$

  2. At time $t = t_k$ one gets measurements, call $\hat{q}^- = \hat{q}(t_k)$ and $P^- = P(t_k)$. Compute δz, H, R. Compute $K = P^- H^T (HP^- H^T + R)^{-1}$. Compute $a^+ = K\delta z$. Update $\hat{q}^+ = \hat{q}^- \star \delta q = \hat{q}^- \star \begin{bmatrix} 2 \\ a^+ \end{bmatrix} \frac{1}{\sqrt{4 + \|a^+\|^2}}$, $\quad P^+ = P^- - KHP^-$.
  3. Keep integrating the equations from the a posteri estimations until more measurements arrive:

$$\begin{aligned} \dot{\hat{q}} &= \frac{1}{2} q \star q_{\hat{\omega}}, \quad q(t_k) = q^+, \\ \dot{P} &= -\hat{\omega}^\times P + P\hat{\omega}^\times + Q, \quad P(t_k) = P^+ \end{aligned}$$

  4. When new measurements arrive, go back to 2.

Multiplicative Extended Kalman Filter (MEKF)

- Additional ideas:
  - Don't forget to renormalize $\hat{q}(t)$ if modulus goes away from unity.
  - The covariance matrix $P(t)$ must be symmetric. One can "symmetrize" by forcing $P = 1/2(P + P^T)$, or compute only a triangular matrix and impose the rest is the transpose.
  - The Kalman gain is optimal only for the linearized system. If estimation has large errors, the filter may diverge.
  - One can and should include gyro bias in the estimation.
  - In practice it is not so easy to obtain Q and R so some simulation/experiments are required.
- Other filtering algorithms exist. MEKF is "simple" and flexible but not necessarily the best (this is a research field).
- In a lab we will test the MEKF with a cell phone.

Spacecraft Dynamics

Lesson 7: Passive attitude control systems.

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1 de julio de 2022

Attitude control

- The attitude control subsystem of satellites can be divided, in general, in two families:
  - Spin-stabilized satellites: using the gyroscopic effect/major axis rule to mantain an inertial direction (which would be the major axis). Cheap and simple but only the major axis can be stabilized (unless wheels are used).
  - Three-axis stabilized satellites: they use some kind of active control to maintain the attitude with some orientation w.r.t. some reference frame.
- Satellites can potentially use the two types of control, depending on the phase of the mission (for instance interplanetary probes).
- Another possible method is the use of gravity gradient, which does not require control (in principle), but it is not very accurate.

Attitude control

- Another classification of attitude control methods is in two kinds: active and passive.
  - We interpret active in the sense of requiring additional use of energy and some command logic (requiring some computational power).
  - Whereas a passive control system does rely on some natural/physical effect to achieve stabilization (e.g. the major axis rule).
- Nevertheless these two classes sometimes overlap as for instance to start the rotation of a spin-stabilized satellite (a passive kind of stabilization) some kind of command and energy contribution is required.
- Thus all satellites in the end should have some kind of active system.

Control of a spin stabilized satellite

- By the major axis rule, we know that a satellite spinning along its major axis is stable; in addition, we know that its response to external perturbations is a small nutation/precession movement that would end up dissipating.
- A spin-stabilized stallite can have a rather simple control system, with the following goals:
  1. Initiate or increase rotation.
  2. Increase the stability of the satellite.
  3. Modify the direction of the rotation axis.
  4. Slow down or completely stop.
- The first goal is trivial with thrusters or even considering the initial spin due to the launch.
- The second goal can be achieved with nutation dampers that increase energy dissipation and thus strengthen the major axis rule (see Lesson 5 and 8).
- In the rest of the lesson we study goals 3 and 4.

Modifying the direction of the rotation axis

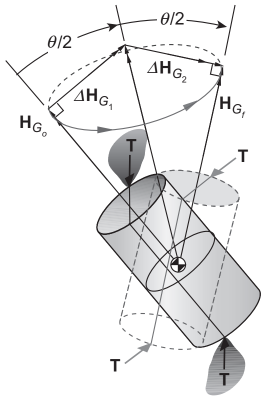
- A simple way to modify the direction of a rotation axis is to stop the rotation, modify the axis, and then start spinning again. However, this procedure would be expensive and slow. Another procedure, known as the "coning" manoeuvre, is explained next.
- To simplify, consider an axisymmetrical spacecraft ($I_1 = I_2 = I < I_3$) and consider we can perform impulsive manoeuvres that instantaneously modify the angular momentum, namely, apply an impulse $\Delta\Gamma$ by using thrusters.
- Let us consider the vehicle rotating only along axis 3 (major axis) with angular velocity n, so that $\vec{\omega}$ and $\vec{\Gamma}$ are aligned.
- Remember (Lesson 5) when we studied the gyroscopic effect, if we apply a perpendicular torque to the axis 3, we get a precession and nutation movement of the body axis 3.

Modifying the direction of the rotation axis

- To simplify, consider that we can directly apply an impulse in $\vec{\Gamma}$, so that $\vec{\Gamma}_f = \vec{\Gamma}_i + \Delta\vec{\Gamma}$. After that, the movement is free again.
- In Lesson 5 we studied that the free movement of an axisymmetrical satellite rotating around its simmetry axis was a precession movement with fixed nutation, so that $\vec{\omega}$ rotates describing a cone around the angular momentum $\vec{\Gamma}$.
- Thus, with this hypothesis of instantaneous change of angular momentum, we simplify the nutation which also changes instantaneously and stays constant, so we can use the exact solution of the free movement of an axisymmetrical body.

Coning

- Consider that we want to displace the rotation axis an angle θ.
- Apply a $\Delta\vec{\Gamma}$ so that $\vec{\Gamma}$ has an angle of $\theta/2$ with the angular velocity. This causes that the speed describes a cone around the new $\vec{\Gamma}$ with angle $\theta/2$, and when it has gone 180° around the cone it has rotated a total angle θ w.r.t. its former position. Then apply a $\Delta\vec{\Gamma}$ such that the final $\vec{\Gamma}$ is again coincident with the angular speed. Note that in the figure, $\mathbf{H_G} = \vec{\Gamma}$.

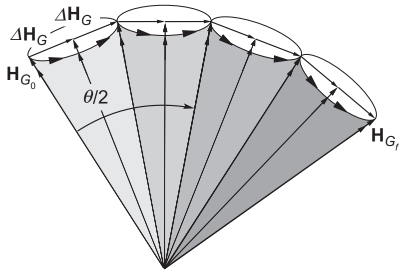


Coning

- From the figure: $\Delta\Gamma_1 = \Delta\Gamma_2 = \Gamma \tan\theta/2$, so the total $\Delta\Gamma_{coning} = 2\Gamma\tan\theta/2$. The final angular momentum es equal to the initial one (but in the intermediate position it is slightly larger: $\frac{\Gamma}{\cos\theta/2}$).
- The time one takes to perform the manoeuvre is π divided by the precession angular speed: $t = \frac{\pi}{\dot{\phi}}$.
- From Lesson 5 (free movement of axisymmetrical spacecraft) $\dot{\phi} = \frac{I_3 n}{I\cos\theta/2} = \frac{\Gamma}{I\cos\theta/2}$, thus $t = \frac{\pi I\cos\theta/2}{\Gamma}$.
- During that time, the body would rotate w.r.t. its symmetry axis (Lesson 5), an angle $\psi = t\lambda = \frac{\pi I\cos\theta/2}{\Gamma}\frac{n(I-I_3)}{I} = \frac{\pi(I-I_3)\cos\theta/2}{I_3}$.
- In general this angle is not 180 degrees (unless $\frac{(I-I_3)\cos\theta/2}{I_3} = 1$) thus one has to use a different set of thrusters to get to the final position.

Multiple coning

- An idea to reduce the fuel consumption (and break down large angles of rotation of the spin axis) is dividing the coning manoeuvre into m smaller manoeuvres, as seen in the figure.



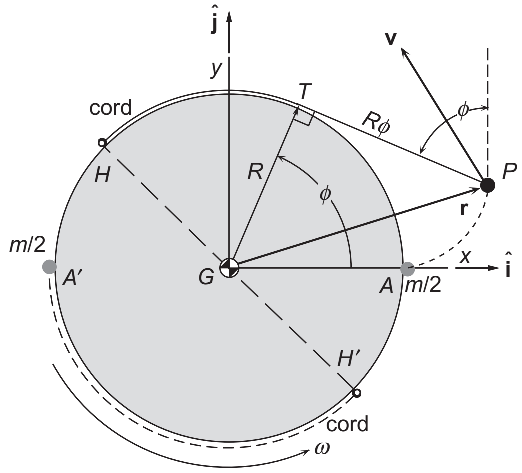
- In each manoeuvre one needs to displace Γ by an angle $\theta/2m$ and wait 180 degrees.
- The total manoeuvre is $\Delta\Gamma_{coning} = 2m\Gamma\tan\theta/2m$ (if m is large this tends to $\theta\Gamma$, and thus this is worthy for large angles).
- The total time of manoeuvre is $t = \frac{m\pi I\cos\theta/2m}{\Gamma}$ (if m is large this goes to infinity, so there is a tradeoff).

Slowing the rotation: yo-yo device

- A yo-yo device is a single-use device that can be used to totally or partially stop the satellite's spin. The mechanism consists on two symmetrical masses fixed to the vehicle by a joint that can be released. The masses are also attached to a wire that is winded around the vehicle with a single point of union, in a plane perpendicular to the rotation that has to be stopped.
- To slow down or stop the rotation, one releases the masses. The start to get away from the vehicle and the wire starts to unwind until the stress reaches the point at which the wire is fixed to the vehicle. Then the wire is also released. If the length of the wire is well designed, then the vehicle has stopped.
- Assumptions: Masses are considered points with mass $m/2$, the wire is massless and not flexible, axisymmetrical vehicle of radius R initially spinning around its symmetry axis with speed n_0.

Yo-yo device

- Initial kinetic energy is $T_0 = \frac{1}{2}\left(I_3 n_0^2 + mR^2 n_0^2\right)$. Initial angular momentum is $\Gamma_0 = I_3 n_0 + mR^2 n_0$. Defining $K = 1 + \frac{I_3}{mR^2}$, we can write $T_0 = \frac{1}{2} mR^2 K n_0^2$ and $\Gamma_0 = mR^2 K n_0$.
- At a given instant the situation is as in the figure:



Yo-yo device

- In the figure, the angle has already been unwound by an angle ϕ, and the vector $\vec{r}$ is the position vector of one of the masses (since they are symmetrical it is enough to study one of them). Given the assumptions, the wire should be tangent at the point T. Use body axes $\vec{i}$ and $\vec{j}$ as in the figure.
- In this frame, $\vec{r}$ is written as $\vec{r} = \vec{GT} + \vec{TP} = R(\cos\phi\vec{i} + \operatorname{sen}\phi\vec{j}) + R\phi(\operatorname{sen}\phi\vec{i} - \cos\phi\vec{j})$.
- To find the kinetic energy and the angular momentum we need the inertial speed. One has:

$$\vec{v} = \dot{\vec{r}}|_{IN} = \dot{\vec{r}}|_{ROT} + \vec{\omega} \times \vec{r}$$

where $\vec{\omega} = n\vec{k}$.

- Now, $\dot{\vec{r}}|_{ROT} = \dot{\phi}R\phi(\cos\phi\vec{i} + \operatorname{sen}\phi\vec{j})$ y $\vec{\omega} \times \vec{r} = nR(\cos\phi\vec{j} - \operatorname{sen}\phi\vec{i}) + nR\phi(\operatorname{sen}\phi\vec{j} + \cos\phi\vec{i})$.

Yo-yo device

- Therefore
  $\vec{v} = R\left((\dot\phi + n)\phi\cos\phi - n\,\mathrm{sen}\,\phi\right)\vec{i} + R\left((\dot\phi + n)\phi\,\mathrm{sen}\,\phi + n\cos\phi\right)\vec{j}.$
- Computing the norm of the speed:
  $v = R\sqrt{\left((\dot\phi + n)\phi\cos\phi - n\,\mathrm{sen}\,\phi\right)^2 + \left((\dot\phi + n)\phi\,\mathrm{sen}\,\phi + n\cos\phi\right)^2}.$
- Therefore: $v = R\sqrt{(\dot\phi + n)^2\phi^2 + n^2}$.
- Thus, $T = \frac{1}{2}\left(I_3 n^2 + mR^2((\dot\phi + n)^2\phi^2 + n^2)\right)$ and using K,
  $T = \frac{mR^2}{2}\left(Kn^2 + (\dot\phi + n)^2\phi^2\right)$.
- On the other hand the angular momentum of the masses is $\Gamma_m = |\vec{r} \times m\vec{v}|$. Computing the product we get $\Gamma_m = mR^2(n + (n + \dot\phi)\phi^2)$.
- Therefore
  $\Gamma = I_3 n + mR^2(n + (n + \dot\phi)\phi^2) = mR^2(Kn + (n + \dot\phi)\phi^2)$.

Yo-yo device

- By conservation of kinetic energy and angular momentum $T = T_0$, $\Gamma = \Gamma_0$, thus reaching two equations

$$K(n_0^2 - n^2) = (\dot\phi + n)^2\phi^2, \quad K(n_0 - n) = (n + \dot\phi)\phi^2$$

- Dividing the first equation by the second, we find $n_0 + n = n + \dot\phi$, thus $\dot\phi = n_0$, this is, the unwinding rate of the wire is equal to the initial angular velocity of the vehicle. Substituting this value in the second equation and solving for ϕ, one can find the angle of unwound wire as a function of the instantaneous angular velocity:

$$\phi = \sqrt{K\frac{n_0 - n}{n_0 + n}}$$

- If one wants that at the end $n = 0$, replacing this value, we find $\phi = \sqrt{K}$, and since the length of wire is $l = R\phi$, we find $l = R\sqrt{K}$, which does not depend on the initial speed!
- One can find an adequate length of wire for any value $n \in (-n_0, n_0)$.

Spacecraft Dynamics

Lesson 8: Active Attitude Control

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1 de julio de 2022

Active control systems

- Passive control systems can allow for some perturbation rejection and give stability enough for some applications.
- However, particularly at the beginning of a mission, all spacecraft need to perform:
  - Slew maneuvers
  - Adjustments of spin speed
  - Stationkeeping maneuvers
- Thus, in many cases, one needs an active control systems (active in the sense of requiring additional energy to work as well as some kind of logic).
- In missions requiring high accuracies, that active control system will be the primary system. Then, one speaks about three-axis stabilized attitude control.
- In other cases, it may be a secondary system, which only requires occasional use.

Actuators

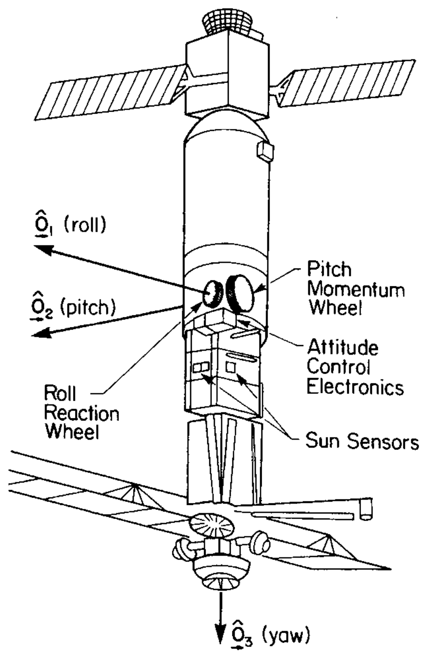
- Before explaining the algorithms for attitude control, it is important to quickly review the actuators that are used to modify the attitude of a spacecraft (through some term in Euler's equations). The different types of actuators are:
  - Thrusters: based on expelling mass. Since mass is finite these devices have limited use. Known as Reaction Control Systems.
  - Reaction wheels and inertia wheels, with changing angular speeds, as seen in Lesson 5.
  - Control Moment Gyroscopes (CMG): they are as inertia wheels (a disc-like device spinning at large speeds), which, instead of modifying their angular speeds, tilt their axis of rotation through motorized gimbals, thus quickly modifying their angular momentum.
  - Magnetorquers, which use the magnetic field to produce a torque.
  - Structural elements for passive control: booms, yo-yo devices, nutation dampers... not covered here.
- It is normal to have several kind of actuators in a spacecraft for redundancy and given that they have different properties.

Three-axis stabilized attitude control

- Satellites with three-axis stabilized attitude control can have any kind of pointing (inertial, orbital, some ground target...)
- Objectives may be two: either to keep the satellite (in the presence of perturbations) in a fixed attitude (a simple regulation/stabilization problem) or to perform a slew maneuver (which maybe to track a target or just modifying the attitude).
- There are two main families of actuators to achieve these goals: reaction/inertia wheels /CMGs (also known as momentum exchange systems) and RCS. Magnetorquers can also partially perform this but it is a bit more difficult due to a direction without actuation: we will not consider them.
- We will start with the first goal, since the second is more difficult, for both reaction/inertia wheels and RCS.
- How to perform slew maneuvers will also be consider but only for reaction/inertia wheels.

Momentum exchange systems

- For the highest degree of precision in attitude, manoeuvrability and stabilization, and for any orientation independent of the inertia tensor, one can use momentum exchange systems which use reaction wheels, inertia wheels and/or CMGs, based on conservation of angular momentum.
- Nevertheless these are expensive system, with low tolerance to failures, and require an auxiliary system (thruster or magnetorquers) to unload momentum and thus avoid saturation.



Spacraft with three reaction wheels

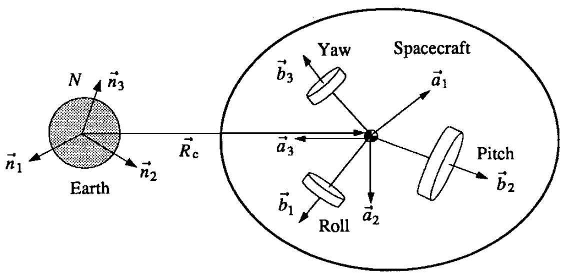


Fig. 6.10 Gyrostat in a circular orbit.

- Assume the situation in the figure, with three reaction wheels in the three principal axes:

$$\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 + \dot{h}_1 + \omega_2 h_3 - \omega_3 h_2 &= M_1 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 + \dot{h}_2 + \omega_3 h_1 - \omega_1 h_3 &= M_2 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 + \dot{h}_3 - \omega_2 h_1 + \omega_1 h_2 &= M_3
\end{aligned}$$

- The angular momentum of wheels is denoted as $h_i = \omega_{R_i} I_{R_i}$. These are control variables!

Spacraft with three reaction wheels

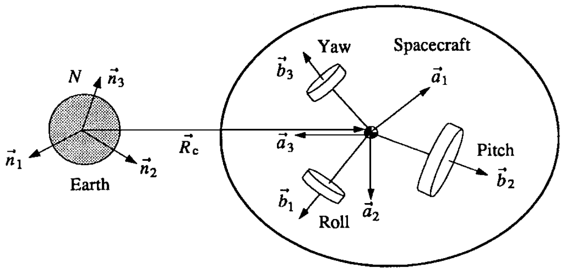


Fig. 6.10 Gyrostat in a circular orbit.

- Remember also from Lesson 5 that once we know the speed we need for the wheels, it can be achieved by using the wheels' internal electrical motors.
- The model from Lesson 5 was:

$$
\begin{aligned}
I_{R1}\dot{\omega}_1 + \dot{h}_1 &= J_1 \\
I_{R2}\dot{\omega}_2 + \dot{h}_2 &= J_3 \\
I_{R3}\dot{\omega}_3 + \dot{h}_3 &= J_3
\end{aligned}
$$

where J_i is the torque of the electrical motors. This is in the end what we can really actuate directly.

Spacecraft with three reaction wheels

- Let us now use a nomenclature in which we denote the effect of the wheels with the letter u by following the classical control nomenclature:

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= u_1 + M_1 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= u_2 + M_2 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 &= u_3 + M_3
\end{aligned}
$$

where

$$
\begin{aligned}
u_1 &= -\dot{h}_1 - \omega_2 h_3 + \omega_3 h_2 \\
u_2 &= -\dot{h}_2 - \omega_3 h_1 + \omega_1 h_3 \\
u_3 &= -\dot{h}_3 - \omega_1 h_2 + \omega_2 h_1
\end{aligned}
$$

This is, $\vec{u} = -\dot{\vec{h}} + \vec{h}^{\times}\vec{\omega}$

- In addition we have the kinematic differential equation

$$\dot{q} = \frac{1}{2} q \star q_{\vec{\omega}}$$

Regulation: Stabilizing a given attitude

- For regulation of a fixed attitude, the problem is stabilizing the values $q(t) = q_{ref}$ and $\omega(t) = 0$. In addition, we assume that we initially start close to that value of the state.
- Thus, we linearize Euler's equations around $\omega(t) = 0$. Ignoring perturbing torques (Question: what could we try to do to mitigate perturbing torques?):

$$\frac{d}{dt}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\end{bmatrix} + \begin{bmatrix}1/I_1&0&0\\0&1/I_2&0\\0&0&1/I_3\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

where $\vec{u} = -\dot{\vec{h}} + \vec{h}^{\times}\delta\vec{\omega}$

- Notice that if we find $\vec{u}$ solving the control problem, we could find the corresponding values of $\vec{h}$ by solving the differential equation (however: physical limitations, such as saturations or rate limits could pose a problem).

Stabilization

- On the other hand, the attitude quaternion should be close to the reference attitude (if we start close to the attitude q_{ref}).
- By following Lesson 2, then we can write $q = q_{ref} \star \delta q$, where q_{ref} is the desired attitude and δq the attitude quaternion:

$$\delta q(\vec{a}) = \frac{1}{\sqrt{4 + \|\vec{a}\|^2}}\begin{bmatrix}2\\ \vec{a}\end{bmatrix}$$

- From Lesson 4 the relationship between $\vec{a}$ and the angular velocity is $\dot{\vec{a}} \approx \delta\vec{\omega} + \vec{a}\times\vec{\omega}_{ref}$, since $\vec{\omega}_{ref} = \vec{0} \to \dot{\vec{a}} \approx \delta\vec{\omega}$.
- Thus:

$$\frac{d}{dt}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\end{bmatrix}$$

- Combining the equations for the error in angular velocity and attitude we find a full description of the error of the system, in the next slide.

Stabilization

- System description:

$$\frac{d}{dt}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\\ a_1\\ a_2\\ a_3\end{bmatrix} = \begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{bmatrix}\begin{bmatrix}\delta\omega_1\\ \delta\omega_2\\ \delta\omega_3\\ a_1\\ a_2\\ a_3\end{bmatrix} + \begin{bmatrix}1/I_1&0&0\\0&1/I_2&0\\0&0&1/I_3\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

- Call $\vec{x}$ to the variables describing the state, this is a classical way to write a linear system

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

- We can use "our favorite linear method" to find a (linear) control law $\vec{u} = K\vec{x}$, which then later one needs to transform in required velocities for the wheels by solving the angular speed that relates $\vec{u}$ with the angular momentum of the wheels, and then later transform that into commands for the wheels' motors.
- A possible method is LQR (linear quadratic regulator) with "infinite horizon". Another is pole placement.

The LQR method

- Given

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

find a control law $\vec{u}(t)$ (with feedback: $\vec{u} = K\vec{x}$) minimizing:

$$J = \int_0^\infty (\vec{x}^T(t)Q\vec{x}(t) + \vec{u}^T(t)R\vec{u}(t))dt$$

- Problem posed and solved first by Rudolph Kalman!
- Assumptions: Q, R symmetrical and $Q > 0$ (definite semidefinite positive, which is equivalent to all eigenvalues positive) and $R \geq 0$ (semidefinite positive, which is equivalent to all eigenvalues non-negative).
- Additional assumption: The system is "controlable". Meaning that "is is possible to solve the problem" (it is easy to solve control problems that cannot be solved. For instance $\dot{x}_1 = u_1,\ \dot{x}_2 = x_2$.) Mathematically a problem is controllable if $\mathcal{C} = [B\ AB\ A^2B\ A^{n-1}B]$ is full row rank, where n is the number of states. Is this verified in our case?

The LQR method

- The control law that solves the problem is

$$\vec{u} = K\vec{x}$$

where the gain K is found as follows

1. Find the matrix P that solve the so-called "algebraic Riccati equation":

$$Q + A^T P + PA - PBR^{-1}B^T P = 0$$

for instance with the Matlab command "are" (which requires the Control Systems Toolbox) `P=are(A,B*inv(R)*B',Q);`

2. The gain is then $K = -R^{-1}B^T P$

- The Riccati equation is solvable only if the system is controllable.
- Optimal control should guarantee a good behavior of the system, but does not take into account the actuator's saturation or other nonlinear behavior. The choice of Q and R greatly influences the quality of the controller (more conservative or more aggresive).

Active control systems
Momentum exchange systems
Reaction Control Systems

The LQR method

- To implement a control law

$$\vec{u} = K\vec{x}$$

let us first remember the definition of $\vec{x}$.

- As $\vec{\omega}_{ref} = \vec{0}$, the first three components are the real value of angular speed.
- The next three components are $\vec{a}$, from which one extracts the quaternion error. It is easy to see that

$$\vec{a} = 2\frac{\delta\vec{q}}{\delta q_0}$$

which comes from $\delta q = q^*_{ref} \star q(t)$.

- Once the control $\vec{u}$ is computed, one needs to solve $\dot{\vec{h}} = -\vec{u} + \vec{h}^\times \delta\vec{\omega}$ to find out how to solve the angular momentum of the wheels.

Slew maneuvers and tracking

- We have studied in Lessons 2 and 4 how to compute a given angular velocity to maneuver from a given attitude to another.
- Remember that, given q_i and q_f and a certain time T it was required to find $q_R = q_i^* \star q_f$, extract Euler's axis $\vec{e}$ and angle θ, and then $\vec{\omega} = \vec{e}\omega(t)$, where ω needs to verify $\int_0^T \omega(\tau)d\tau$.
- In addition, we can impose additional conditions such as starting and finishing at rest, for instance by imposing a shape to $\omega(t)$ of the form $\omega(t) = At(t-T)$ (Exercise: find A). Other conditions could be imposed.
- Once we find the required angular velocity, if we substitute it in Euler's equation we can find the control. This is sometimes called "open loop control" or feedforward control, and does not use feedback.

Active control systems | Momentum exchange systems
Reaction Control Systems

Slew maneuvers and tracking

- If we call the found angular velocity $\vec{\omega}_{ref}(t)$, and the quaternion generated by that angular speed the reference quaternion $q_{ref}(t)$, we can also find a "reference control" (feedforward control) $\vec{u}_{ref}$ as:

$$
\begin{aligned}
u_{ref1} &= I_1\dot{\omega}_{ref1} + (I_3 - I_2)\omega_{ref2}\omega_{ref3} \\
u_{ref2} &= I_2\dot{\omega}_{ref2} + (I_1 - I_3)\omega_{ref3}\omega_{ref1} \\
u_{ref3} &= I_3\dot{\omega}_{ref3} + (I_2 - I_1)\omega_{ref1}\omega_{ref2}
\end{aligned}
$$

- As before from this $\vec{u}_{ref}$ we can find the required speed of the wheels and from that speed of the wheels, the internal electrical motors' torque that would be needed to perform the maneuver.
- What would happen if we try just to apply this feedforward control without any feedback mechanism?
- The problem of following the reference profile $\vec{\omega}_{ref}(t)$,$q_{ref}(t)$ is sometimes called the tracking problem.

Tracking

- One possible idea to solve tracking is as follows: linearize around the reference profile. Compute an *additional* feedback controller around the reference profile that is added to the feedforward control (so we have feedforward+feedback) so we close the loop and guarantee stability (at least with respect to small errors and perturbations) so that the system is kept on the desired reference profile.
- Thus let $\delta\vec{\omega} = \vec{\omega} - \vec{\omega}_{ref}$, $\delta\vec{u} = \vec{u} - \vec{u}_{ref}$, and use the quaternion error as previously defined. The linearized equations are:

$$\begin{aligned} I_1\delta\dot{\omega}_1 + (I_3 - I_2)(\omega_{ref2}\delta\omega_3 + \delta\omega_2\omega_{ref3}) &= \delta u_1 + M_1 \\ I_2\delta\dot{\omega}_2 + (I_1 - I_3)(\omega_{ref3}\delta\omega_1 + \delta\omega_3\omega_{ref1}) &= \delta u_2 + M_2 \\ I_3\delta\dot{\omega}_3 + (I_2 - I_1)(\omega_{ref1}\delta\omega_2 + \delta\omega_1\omega_{ref2}) &= \delta u_3 + M_3 \end{aligned}$$

and for the attitude error:

$$\dot{\vec{a}} \approx \delta\vec{\omega} - \vec{\omega}_{ref}^{\times}\vec{a}$$

Tracking

- System description ignoring perturbing torques:

$$\frac{d}{dt}\begin{bmatrix} \delta\omega_1 \\ \delta\omega_2 \\ \delta\omega_3 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{I_2-I_3}{I_1}\omega_{ref3} & \frac{I_2-I_3}{I_1}\omega_{ref2} & 0 & 0 & 0 \\ \frac{I_3-I_1}{I_2}\omega_{ref3} & 0 & \frac{I_3-I_1}{I_2}\omega_{ref1} & 0 & 0 & 0 \\ \frac{I_1-I_2}{I_3}\omega_{ref2} & \frac{I_1-I_2}{I_3}\omega_{ref1} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \omega_{ref3} & -\omega_{ref2} \\ 0 & 1 & 0 & -\omega_{ref3} & 0 & \omega_{ref1} \\ 0 & 0 & 1 & \omega_{ref2} & -\omega_{ref1} & 0 \end{bmatrix} \begin{bmatrix} \delta\omega_1 \\ \delta\omega_2 \\ \delta\omega_3 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$+ \begin{bmatrix} 1/I_1 & 0 & 0 \\ 0 & 1/I_2 & 0 \\ 0 & 0 & 1/I_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{bmatrix}$$

- Classical description as before

$$\dot{\vec{x}} = A(t)\vec{x} + B(t)\delta\vec{u}$$

- Now A and B are time-varying: cannot use the LQR method as before.
- We need more advanced methods, such as LQR (linear quadratic regulator) with "finite horizon".

Tracking with finite horizon LQR

- Given

$$\dot{\vec{x}} = A(t)\vec{x} + B(t)\delta\vec{u}$$

  find $\delta\vec{u}(t)$ with feedback ($\delta\vec{u}(t) = K(t)\vec{x}$) minimizing

$$J = \int_0^T (\vec{x}^T(t)Q(t)\vec{x}(t) + \delta\vec{u}^T(t)R(t)\delta\vec{u}(t))dt + \vec{x}^T(T)Q_{end}\vec{x}(T)$$

- Assumptions: Q, R, Q_{end} symmetric and $Q_{end}, Q > 0, R \geq 0$.
- Since it is a finite horizon controller, the controllability hypothesis is not required, but there could be problems if there is a loss of controllability of the system.

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Tracking with finite horizon LQR

- The control law that minimizes J is as follows:

$$\delta\vec{u} = K(t)\vec{x}$$

  where the gain $K(t)$ is found as follows:
  1. Find $P(t)$ that solved the so-called "Riccati differential equation":

$$-\dot{P} = A^T P + PA - PBR^{-1}B^T P + Q, \quad P(T) = Q_{end}$$

     for instance using ode45 in Matlab.
  2. The gain is then $K(t) = -R^{-1}B^T P(t)$
- Riccati's differential equation is always solvable! However, it cannot be solved in real time, because it needs to be solved backwards in time (there is a final condition instead of an initial condition). Thus one solves it in advance and stores the values of $K(t)$.
- As before: Choices of Q and R (also Q_{end}) determines the quality of the controller (more conservative or more aggresive).

Tracking with finite horizon LQR

- To implement the control law

$$\delta\vec{u} = K(t)\vec{x}$$

one needs to remember the definition of $\vec{x}$.
- As $\vec{\omega}_{ref} \neq \vec{0}$, the first three components are $\vec{\omega} - \vec{\omega}_{ref}$.
- The second three components correspond to $\vec{a}$, that need to be extracted from the quaternion error. Remember that

$$\vec{a} = 2\frac{\delta\vec{q}}{\delta q_0}$$

for which we need to compute $\delta q = q^*_{ref} \star q(t)$.
- The final control is $\vec{u} = \vec{u}_{ref} + \delta\vec{u}$.
- Remember that once $\vec{u}$ is known, at each instant is required to solve $\dot{\vec{h}} = -\vec{u} + \vec{h}^\times\delta\vec{\omega}$ to know how to modify the angular momentum of the wheels and therefore their internal torque J_i.

Nonlinear control

- "Nonlinear control" comprises a wide range of techniques that do not require the use of linearization.
- Consider the following problem. Starting from $\vec{\omega}(0)$ and $q(0)$ we want to reach the identity attitude at rest. It is enough for us if the system "tends" to that state, this is, our goal is that $\vec{\omega}(t) \to \vec{0}$ y $q_0(t) \to 1$, $\vec{q}(t) \to \vec{0}$ when $t \to \infty$.
- This is, we make "asymptotically stable" the equilibrium $\vec{\omega} = \vec{0}$, $q_0 = 1$, $\vec{q} = \vec{0}$.
- If this is true, for any initial condition, then one says that the equilibrium is globally asymptotically stable.
- Notice that the target attitude could be any, just by making a rotation of the inertial frame as $q' = q^*_{ref} \star q$.
- We solve this problem with the so-called "Lyapunov function technique".

Nonlinear control

- Let us start by remembering than since we don't linearize, now our system is the original one, writing as before the control terms in the equations.
- First, the angular velocity equations:

$$\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{u_1}{I_1} \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \frac{u_2}{I_2} \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{u_3}{I_3}
\end{aligned}$$

Nonlinear control: Lyapunov functions

- Can we find u_1, u_2 and u_3 such that the equilibirum $\vec{\omega} = \vec{0}$ is globally asymptotically stable?
- The technique of Lyapunov functions is as follows. Let V be a regular function (continuous, differentiable) that depends on the state (in this case, the angular velocity and quaternions) such that :
  - It is always positive for any value of the states, except when the state is zero; and for zero, it is zero (this is, positive definite).
  - The time derivative of V is definite negative (this is, negative for any value of the state except zero).
- Then it follows that the origin (zero value of the state) is asymptotically stable (this method can be understood by looking at the level curves of V).
- If in addition the limit of V when the state goes to infinity also tends to infinity, the result is global.

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Nonlinear control: Lyapunov functions

- Let us see how this works out for our first case with only angular velocity. Consider:

$$V = I_1 \frac{\omega_1^2}{2k} + I_2 \frac{\omega_2^2}{2k} + I_3 \frac{\omega_3^2}{2k}$$

- We see that the first conditions is fulfilled if k is a positive constant (we will define it later).
- Taking derivative:

$$V_t = I_1 \frac{\omega_1 \dot{\omega}_1}{k} + I_2 \frac{\omega_2 \dot{\omega}_2}{k} + I_3 \frac{\omega_3 \dot{\omega}_3}{k}$$

- Substituting the derivatives:

$$V_t = \frac{\omega_1((I_2 - I_3)\omega_2\omega_3 + u_1)}{k} + \frac{\omega_2((I_3 - I_1)\omega_3\omega_1 + u_2)}{k} + \frac{\omega_3((I_1 - I_2)\omega_1\omega_2 + u_3)}{k}$$

Nonlinear control: finding the control

- Simplifying

$$V_t = \frac{\omega_1 u_1}{k} + \frac{\omega_2 u_2}{k} + \frac{\omega_3 u_3}{k}$$

- Let us choose now: $u_1 = -c_1\omega_1$, $u_2 = -c_2\omega_2$, $u_3 = -c_3\omega_3$, where c_i is a positive constant. Replacing this in V_t:

$$V_t = -\frac{c_1\omega_1^2 + c_2\omega_2^2 + c_3\omega_3^2}{k}$$

- Thus by the technique of Lyapunov, it is proven that $\vec{\omega} = 0$ is globally asymptotically stable. Note that the value of C_i and k does not matter as long as they are positive, but the value of C_i will influence the performance of the control law.

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Nonlinear control: including quaternions

- Let us consider now the full system including the quaternions

$$\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{u_1}{I_1} \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \frac{u_2}{I_2} \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{u_3}{I_3} \\
\dot{q}_0 &= -\frac{1}{2}(q_1\omega_1 + q_2\omega_2 + q_3\omega_3) \\
\dot{q}_1 &= \frac{1}{2}(q_0\omega_1 - q_3\omega_2 + q_2\omega_3) \\
\dot{q}_2 &= \frac{1}{2}(q_3\omega_1 + q_0\omega_2 - q_1\omega_3) \\
\dot{q}_3 &= \frac{1}{2}(-q_2\omega_1 + q_1\omega_2 + q_0\omega_3)
\end{aligned}$$

Nonlinear control: La Salle's Theorem

- Can we find values of u_1, u_2 and u_3 guaranteeing that the equilibrium $\vec{\omega} = \vec{q} = \vec{0}$, $q_0 = 1$ is asymptotically stable?
- Unfortunately Lyapunov is not enough!
- We also need "La Salle's Theorem":
  - Let V be a Lyapunov function such that its derivative is **semidefinite negative** (this is negative or zero). Let us call E the set of states verifying $\dot{V} = 0$.
  - Let M be the largest **invariant set** of the system contained in E.
- Then the state goes to M when time goes to infinity.
- What is the invariant set of a system? Is a set such that if the initial condition starts in the set, the state stays in the set for all t.

Nonlinear control: finding the control (again)

- Use the Lyapunov function

$$V = I_1 \frac{\omega_1^2}{2k} + I_2 \frac{\omega_2^2}{2k} + I_3 \frac{\omega_3^2}{2k} + (q_0 - 1)^2 + q_1^2 + q_2^2 + q_3^2$$

- We see that the first condition of being a Lyapunov function is verified (q_0 has been displaced so that $q_0 = 1$ is at the origin).
- Taking a derivative:

$$V_t = I_1 \frac{\omega_1 \dot{\omega}_1}{k} + I_2 \frac{\omega_2 \dot{\omega}_2}{k} + I_3 \frac{\omega_3 \dot{\omega}_3}{k} + 2(q_0 - 1)\dot{q}_0 + 2q_1\dot{q}_1 + 2q_2\dot{q}_2 + 2q_3\dot{q}_3$$

- Substituting:

$$\begin{aligned} V_t = & \frac{\omega_1((I_2 - I_3)\omega_2\omega_3 + u_1)}{k} + \frac{\omega_2((I_3 - I_1)\omega_3\omega_1 + u_2)}{k} + \frac{\omega_3((I_1 - I_2)\omega_1\omega_2 + u_3)}{k} \\ & -(q_0 - 1)(q_1\omega_1 + q_2\omega_2 + q_3\omega_3) + q_1(q_0\omega_1 - q_3\omega_2 + q_2\omega_3) \\ & + q_2(q_3\omega_1 + q_0\omega_2 - q_1\omega_3) + q_3(-q_2\omega_1 + q_1\omega_2 + q_0\omega_3) \end{aligned}$$

Nonlinear control: finding the control (again)

- Simplifying

$$V_t = \frac{\omega_1 u_1}{k} + \frac{\omega_2 u_2}{k} + \frac{\omega_3 u_3}{k} + (q_1\omega_1 + q_2\omega_2 + q_3\omega_3)$$

- Let us choose now: $u_1 = -(kq_1 + c_1\omega_1)$, $u_2 = -(kq_2 + c_2\omega_2)$, $u_3 = -(kq_3 + c_3\omega_3)$, where c_i is a positive constant. Substituting:

$$\begin{aligned} V_t &= -\frac{\omega_1(kq_1 + c_1\omega_1)}{k} - \frac{\omega_2(kq_2 + c_2\omega_2)}{k} - \frac{\omega_3(kq_3 + c_3\omega_3)}{k} \\ &\quad + (q_1\omega_1 + q_2\omega_2 + q_3\omega_3) \\ &= -\frac{c_1\omega_1^2 + c_2\omega_2^2 + c_3\omega_3^2}{k} \end{aligned}$$

- We cannot apply Lyapunov directly, we need La Salle!
- First of all, the set E is just $\omega_1 = \omega_2 = \omega_3 = 0$ for all t.

Finding the invariant set M

- Replace $\omega_1 = \omega_2 = \omega_3 = 0$ in the syste for all t (in particular this implies that the derivatives are zero):

$$\begin{aligned} 0 &= 0 + u_1 \\ 0 &= 0 + u_2 \\ 0 &= 0 + u_3 \\ \dot{q}_0 &= 0 \\ \dot{q}_1 &= 0 \\ \dot{q}_2 &= 0 \\ \dot{q}_3 &= 0 \end{aligned}$$

- Thus the invariant set verifies $u_1 = u_2 = u_3 = 0$, and q constant.
- Since $u_1 = -(kq_1 + c_1\omega_1)$, $u_2 = -(kq_2 + c_2\omega_2)$, $u_3 = -(kq_3 + c_3\omega_3)$, we obtain $q_1 = q_2 = q_3 = 0$.

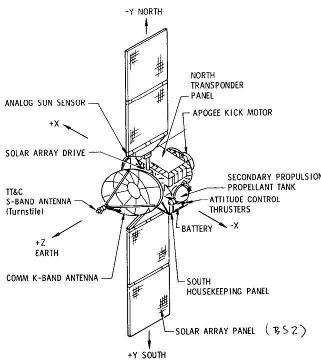
Final stability result. Winding phenomenon.

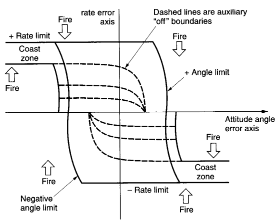
- Finally, since the quaternion must be unity, we get $q_0 = \pm 1$. Since $q_0 = 1$ is the origin of the Lyapunov function, it becomes stable (in fact $q_0 = -1$ becomes unstable; which is a problem since it is the same point, this is called the winding phenomenon).
- If one uses negative k in the control law then it can be similarly shown that $q_0 = -1$ becomes stable and $q_0 = 1$ unstable. This can be verified by switching the Lyapunov function to

$$V = -I_1 \frac{\omega_1^2}{2k} - I_2 \frac{\omega_2^2}{2k} - I_3 \frac{\omega_3^2}{2k} + (q_0 + 1)^2 + q_1^2 + q_2^2 + q_3^2$$

- If one fixes $k = k_0 \cdot \mathrm{sgn}(q_0)$ then one stabilizes the "closest" equilibirum.
- Very interestingly: in the control law there are no inertias in the formulas, thus we don't need knowledge of them. This is an universal control law. However one needs to know the state ($\vec{\omega}$ and q) to be able to apply the control law.

Reaction Control Systems (RCS)





- In situations that require high/fast manoeuvrability one can use a Reaction Control Systems or RCS, using a set of thruster distributed on the vehicle to quickly and efficiently modify attitude.
- The so-called "propulsion logic" establishes when thrusters are fired and if a small tolerance of attitude/angular velocity can be accepted.
- Normally it is a combination of "dead zones" (no actuations) and hysteresis (to avoid the repetitive firing of thrusters exhausting all fuel).
- Thrusters usually are actuators "all or nothing", thus always acting in saturation.
- This means that RCS are intrinsically nonlinear, but discontinuous as well.

Reaction Control Systems

- For RCS, we can model the effect of the thrusters as torques in Euler's Equation.
- We are only going to consider the regulation problem (stabilization of an attitude to which we are already close). Linearizing and taking Euler angles in the sequence 1-2-3 with small angles, and combining the linearized kinematic and dynamics, the system to be controlled becomes:

$$\begin{aligned} I_1\ddot{\theta}_1 &\approx u_1, \\ I_2\ddot{\theta}_2 &\approx u_2, \\ I_3\ddot{\theta}_3 &\approx u_3, \end{aligned}$$

- Next we design u_1, u_2 and u_3 to stabilize the system; each axis is independent of one another. Classical methods of control (or Lyapunov) cannot be used for thrusters since they cannot give a variable value (a control law such as $u = Kx$ does not work). This is the only options are $u = 0, u_{MAX}, u_{MIN}$, where u_{MIN} should be negative (we can assume to simplify $u_{MIN} = -u_{MAX}$). We will use more explicit/geometrical ideas.

Control with thrusters

- Consider a single axis, then $\ddot{\alpha} = u$ (where u is redefined by dividing by inertia), with initial conditions $\dot{\alpha}_0$ and α_0. Integrating the differential equation:

$$\dot{\alpha} - \dot{\alpha}_0 = tu, \quad \alpha - \alpha_0 - t\dot{\alpha}_0 = \frac{t^2}{2}u$$

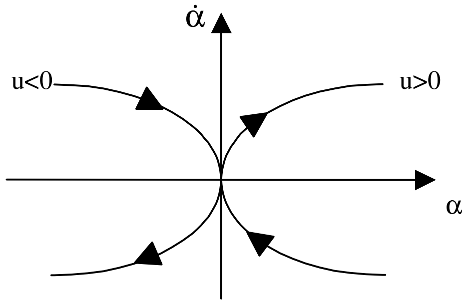
- If one removes time from the system:

$$\alpha - \alpha_0 = \frac{\dot{\alpha}_0(\dot{\alpha} - \dot{\alpha}_0)}{u} + \frac{(\dot{\alpha} - \dot{\alpha}_0)^2}{2u}$$

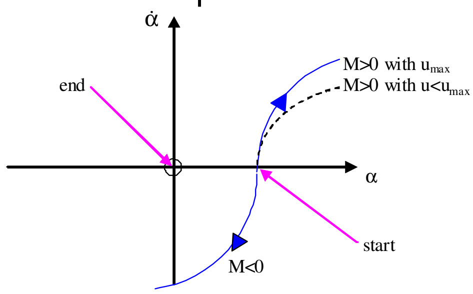
- This is the equation of a parabola in the phase plane (θ-$\dot{\theta}$), whose shape will depend from initial conditions and the choices of control ($u = 0, u_{MAX}, -u_{MAX}$). If $u = 0$ time cannot be removed and the system's behavior is reduced to moving along the segment $\alpha - \alpha_0 = t\dot{\alpha}_0$.

Control with thrusters

- Example of parabolas with zero initial condition (arrows indicate how the system behaves):



- To move we need to use the parabolas:



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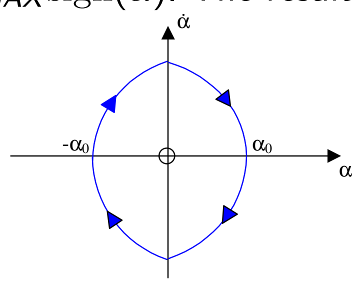
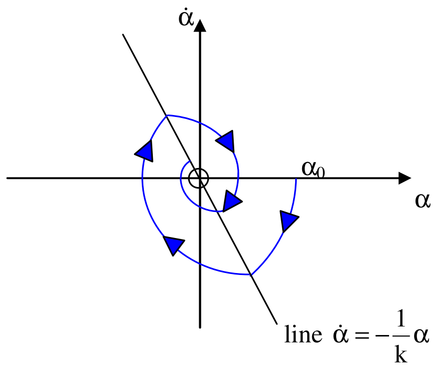
Control with thrusters

- First idea: $u = -u_{MAX}\mathrm{sign}(\alpha)$. The result is a limit cycle:

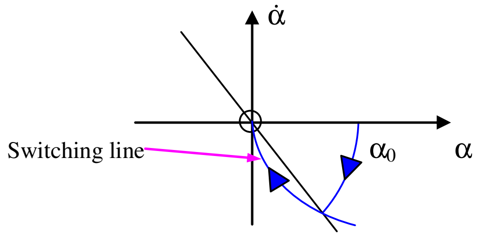


- To avoid oscillation: $u = -u_{MAX}\mathrm{sign}(\alpha + k\dot{\alpha})$, with $k > 0$. The result:

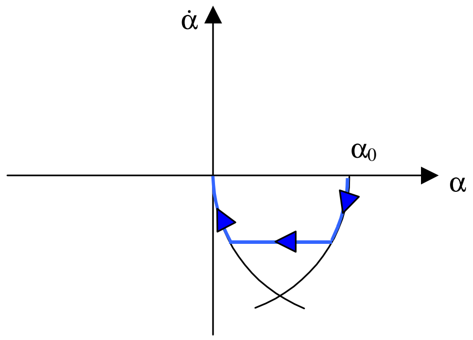


Control with thrusters

- To arrive in a finite time: $u = -u_{MAX}\text{sign}(\alpha - \frac{1}{2u_{MAX}}\dot{\alpha}|\dot{\alpha}|)$ (exercise). The result:



- If one fixes a minimum time and wants to minimize fuel (exercise):



Control with thrusters: additional considerations

- The procedure just explained cannot be applied if one cannot neglect nonlinearities (gyroscopic couplings make necessary the use of all the axis simultaneously). Then one needs to use the full theory of optimal control.
- In practice it is enough that the solutions converge close enough to the origin (to avoid switching on the thurster too often). This requires the use of dead zones and hysteresis.